Lie Groups

2nd Problem List
Due December, 22th

1. Show that $\mathbb{R}^2$ equipped with the multiplication $(a_1, b_1) \cdot (a_2, b_2) = (a_1 + e^{b_1} a_2, b_1 + b_2)$ is a Lie group, and there there is a left invariant integral which is not right invariant and is not conjugation invariant.

2. Let $G = \mathbb{R} \setminus \{0\}$ be the real multiplicative group. Show that the invariant integral is given by $\int_G f(g)dg = \int_G f(t)\frac{dt}{|t|}$, where dt is the usual Lebesgue measure on the real line. More generally, show that the invariant integral on $G = GL(n, \mathbb{R})$ is given by:
$$\int_G f(g)dg = \int_G f(x)\frac{dx}{|\det x|^n},$$
where dx is the Lebesgue measure on the real vector space $\mathbb{R}^{n^2} \cong Mat_{n\times n}(\mathbb{R})$ where G is naturally embeded.

3. Let α, β be the one-parameter subgrups of $SO(3)$ given by $\alpha(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\beta(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$. Prove that the map $\gamma : (S^1)^3 \to SO(3)$ defined by $\gamma(\varphi, \vartheta, \psi) = \alpha(\varphi)\beta(\vartheta)\alpha(\psi)$ is surjective, where $\varphi, \vartheta, \psi \in [0, 2\pi]$ and that $\gamma * dg = \pm\frac{\sin\vartheta}{8\pi^2}\, d\varphi \wedge d\vartheta \wedge d\psi$. Conclude that the left invariant integral in $SO(3)$ is given by $f \mapsto \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}(f \circ \gamma)\ \sin\vartheta\ \ d\varphi d\vartheta d\psi$, for a function $f : SO(3) \to \mathbb{R}$.

4. Show that the representation $V = \mathbb{C}^2$ of the abelian group $(\mathbb{R}, +)$ given by $t \mapsto \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ does not have an invariant (Hermitian) inner product, and that V is not a direct sum of irreducible submodules. Determine all irreducible subrepresentations of V.

5. Determine all the irreducible subrepresentations of the representation $S^1 \to O(2) \subset U(2) \subset GL(2, \mathbb{C})$ given by $e^{it} \mapsto \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = R(t)$. Find a unitary matrix A such that $e^{it} \mapsto AR(t)A^{-1}$ is a direct sum of two one-dimensional representations.

6. Let V be a representation of a Lie group G. If V has a G-invariant Hermitian inner product show that the associated matrix representation $\rho : G \to GL(V)$ has values in $U(n) \subset GL(n, \mathbb{C})$.

7. If H is an abelian subgroup of $U(n)$, show that H is conjugated to a subgroup of the group of diagonal matrices in $U(n)$.

8. Let V and W be two equivalent unitary representations of a compact Lie group G. Show that they are isometric, ie, there exists an isomorphism $\phi : V \to W$ such that $\langle \phi(v_1), \phi(v_2) \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in V$ ($\langle , \rangle$ denotes both inner products in V and in W).

9. Use an invariant (Hermitian) inner product to show that, for any representation V of a compact Lie group G, there is an isomorphism $\overline{V} \cong V^*$.

10. Let V be an irreducible representation of a compact Lie group G, with character $\chi = \chi_V$. Prove that $\chi(x)\chi(y) = (\dim V) \int_G \chi(gxg^{-1}y)\ dg$

11. Show that the space of conjugacy classes in $G = SU(2)$ (the space of orbits of the conjugation action $G \times G \to G$ which is given by $(g, x) \mapsto gxg^{-1}$) is homeomorphic to a closed interval in $\mathbb{R}$.

12. Let $V_n \subset \mathbb{C}[z_1, z_2]$ be the vector space of homogeneous polynomials of degree n in two variables, which is naturally a representation of $SL(2, \mathbb{C})$. Show the $SU(2)$-invariance of the inner product in V_n defined by $\langle a, b \rangle := \sum_{k=0}^{n} k!(n-k)! a_k \bar{b}_k$, where $a(z_1, z_2) = \sum_{k=0}^{n} a_k P_k$, $b(z_1, z_2) = \sum_{k=0}^{n} b_k P_k$, and $P_k(z_1, z_2) = z_1^k z_2^{n-k}$, $0 \leq k \leq n$, denotes a natural basis of V_n.

13. Let G be a compact matrix group and $f \in C(G)$, a complex-valued function on G. Define the left and right actions of G on $C(G)$ by $(L_g f)(h) := f(g^{-1}h)$ and $(R_g f)(h) := f(hg)$. Show that the following are equibvalent: (a) The left (resp. right) translates of f span a finite-dimensional space. (b) f is the matrix coefficient of a finite-dimensional representation of G. (c) f is a polynomial function on $G \subset U(n)$.

14. Let G be a compact matrix group, and χ the character of a representation $\rho : G \to GL(V)$ of G. Prove that $|\chi(g)| \leq \chi(e)$ with equality if and only if $\rho(g)$ is a scalar.

15. Let G be a finite group, and $|G|$ denote its cardinality. Show that: (a) the number of irreducible characters of G equals the number of conjugacy classes of G; (b) The characters of the left and right regular representations of G are both given by $\chi(e) = 1/|G|$ and $\chi(g) = 0$ for $g \neq e$.

16. Let G be a compact Lie group. Show that $\mathcal{T}(G, \mathbb{R})$, the algebra of real representative functions on G, is dense in the algebra of continuous functions $C(G, \mathbb{R})$.

17. Show that any compact Lie group G has a faithful embedding in $U(n)$ for some natural number n.

18. Suppose that all irreducible representations of a compact Lie group G are one-dimensional. Show that G is abelian.