

Lie Groups

1st Problem List

Due December, 3rd

1. Show that $SO(n)$ is a connected Lie group.
2. Show that $U(n) \subset SO(2n)$ and that $GL(n, \mathbb{C}) \subset GL^+(2n, \mathbb{R})$ (here, \mathbb{C}^n is viewed as a real vector space) and $GL^+(2n, \mathbb{R})$ is the subset of $GL(2n, \mathbb{R})$ of positive determinant. Show that $GL(n, \mathbb{C}) \cap SO(2n) = U(n)$.
3. Let $U \subset GL(n, \mathbb{R})$ be the group of upper triangular matrices with positive entries in the diagonal. Show that multiplication of matrices $D \times O(n) \rightarrow GL(n, \mathbb{R})$ is a diffeomorphism and deduce that $GL(n, \mathbb{R})$ is a manifold diffeomorphic to $O(n) \times \mathbb{R}^{\frac{1}{2}n(n+1)}$ [Hint: use the Gram-Schmidt orthogonalization process].
4. Let $\mathbb{C}\mathbb{P}^1$ be the one dimensional complex projective space (the space of all one-dimensional vector subspaces in \mathbb{C}^2). Show that the Möbius transformations given by $T([z : w]) = [az+bw : cz+dw]$; $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(n, \mathbb{C})$ are well defined as maps $T : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ (where $(z, w) \neq (0, 0)$), and prove that the group of all Möbius transformations (where the product is given by composition) is a Lie group isomorphic to $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/(\pm I)$, where I is the identity matrix.
5. Show that the Lie algebra of $PGL(n, \mathbb{C}) = GL(n, \mathbb{C})/\mathbb{C}^*$ (where \mathbb{C}^* , the center of $GL(n, \mathbb{C})$, are the non-zero scalar matrices) is isomorphic to $\mathfrak{sl}(n, \mathbb{C})$.
6. Show that every one parameter subgroup $\lambda : \mathbb{R} \rightarrow GL(n, \mathbb{C})$ is of the form $\lambda(t) = \exp(tX) := \sum_{k=0}^{\infty} \frac{t^k}{k!} X^k$ for some matrix $X \in Mat_{n \times n}(\mathbb{C})$; show that the image $\lambda(\mathbb{R}) \subset GL(n, \mathbb{C})$ may be closed or not.
7. Show that the adjoint representation defines a surjective homomorphism from $SU(2)$ to $SO(3)$ with kernel $\{\pm I\}$.
8. Show that a bijective homomorphism between Lie groups is an isomorphism.

9. Let $G \subset GL(n, \mathbb{R})$ be a linear Lie group. Show that the Lie algebra LG can be identified with a linear subspace $LG \subset Mat_{n \times n}(\mathbb{R})$ in such a way that the exponential map $exp : LG \rightarrow G$ coincides with the matrix exponential $exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$, for all $X \in LG$.
10. Show that the exponential map for the group $SL(2, \mathbb{R})$ is not surjective. Determine the image of this map and the possible values of $tr(exp(A))$ for matrices $A \in \mathfrak{sl}(2, \mathbb{R})$.
11. Show that the exponential map is surjective for the group $GL(n, \mathbb{C})$ [Hint: Consider first the case of an element $g \in GL(n, \mathbb{C})$ in Jordan canonical form].
12. Show that if H is an open subgroup of a connected Lie group G , then $H = G$.
13. Show that a subgroup H of a Lie group G is discrete if and only if there is a neighborhood of $e \in G$ that contains no other elements of H .
14. Show that a compact abelian Lie group is isomorphic to the product of a torus and a finite abelian group.
15. Show that any complex Lie group (manifold with a complex holomorphic atlas) which is compact and connected is abelian [Hint: show that the adjoint representation is trivial, using the fact that holomorphic functions on compact complex manifolds are constant].
16. Let $H, K \subset G$ be Lie subgroups. Show that there is a G -equivariant diffeomorphism of homogeneous spaces $G/H \xrightarrow{\cong} G/K$ if and only if H and K are conjugate in G .