Lie Algebras

3rd Problem List

Due November, 19th

In these problems, Φ is a (abstract) root system in a real Euclidean space E, with inner product $(,)$ and W will denote its Weyl group.

1. Let α and β be elements of Φ that generate a subspace E' (of dimension 2) of E. Show that both $\Phi \cap E'$ and $\Phi \cap (\mathbb{Z}\alpha + \mathbb{Z}\beta)$ are root systems of E', which in general, do not coincide.

2. If Δ is a basis of Φ and $\alpha_1, ..., \alpha_m$ are distinct elements of Δ, show that the set $\Phi \cap (\mathbb{Z}\alpha_1 + ... + \mathbb{Z}\alpha_m)$ is a root system of rank m in the space generated by $\alpha_1, ..., \alpha_m$.

3. Let $E = \mathbb{R}^3$, with orthonormal basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) : 1 \leq i, j \leq 3\}$. Show that Φ is a root system, and that W is isomorphic to S_3, the symmetric group of permutations of 3 elements.

4. Show that the only root systems of rank 2 are the ones described in Humphreys, p. 44 ($A_1 \times A_1$, A_2, $B_2 = C_2$ or G_2).

5. Let $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$ where α is a root of Φ. Show that the set $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ defines a root system in E, whose Weyl group is naturally isomorphic to W and that $< \alpha^\vee, \beta^\vee > = < \alpha, \beta >$. Sketch $\Phi^\vee$ when Φ is one of the root systems of Problem 4.

6. Show that the Weyl group of a root system is isomorphic to the direct product of the Weyl groups of its irreducible components.

7. Show that, given two non-proportional roots $\alpha, \beta \in \Phi$, the order of the element $\sigma_\alpha \sigma_\beta \in W$ can only be 2, 3, 4 or 6.

8. Show that the Weyl group of a root system of rank 2 is a dihedral group (the symmetry group of a regular poligon of n sides).

9. Show that the sign function $sn : W \to \{\pm 1\}$ is a group homomorphism. By definition, $sn(\sigma) = (-1)^{l(\sigma)}$ where $l(\sigma)$ is the length of $\sigma \in W$ (the smallest number of factors in the product $\sigma = \prod_i \sigma_{\alpha_i}$, where all the roots α_i are simple).

10. Given $\alpha \in E$ let $\Phi^+(\alpha) := \{\gamma \in E : (\gamma, \alpha) > 0\}$ denote the positive half-space relative to α. Show that the intersection of the positive half-spaces relative to elements of a basis $\Delta = \{\alpha_1, ..., \alpha_n\}$ of Φ is non-empty, that is $C(\Delta) \equiv \Phi^+(\alpha_1) \cap \cdots \cap \Phi^+(\alpha_n) \neq \emptyset$.

11. Show that if λ is in the set $C(\Delta)$ defined above, and $\sigma\lambda = \lambda$ for some $\sigma \in W$, then $\sigma = 1$.

12. Compute the determinant of the Cartan matrices of all root systems of type A_n.

13. Show that a root system is irreducible if and only if its Dynkin diagram is connected.

14. Show that the inclusion of a Dynkin diagram in another induces an inclusion of one root system into the other.

15. Show that the root systems of types B_n and C_n are dual to one another in the sense of Problem 5.