Lie Algebras

2nd Problem List

Due November, 3rd

1. Consider the Lie algebra $\mathsf{sl}(2, \mathbb{C})$ and its standard basis:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

   Determine the dual basis with respect to the Killing form.

2. Let $L = \mathsf{gl}(n, \mathbb{C})$. Show that the Killing form of L is given by

$$\kappa(x, y) = 2n\mathsf{tr}(xy) - 2(\mathsf{tr}x)(\mathsf{tr}y)$$

   and compute its radical.

3. Show that, if L is solvable, every irreducible representation of L is one-dimensional.

4. Let $V = L$ be the representation space of the adjoint representation of the Lie algebra L. Prove that the submodules of V are precisely the ideals of L.

5. Let V be an L-module. Show that V is completely reducible (that is, a direct sum of irreducible submodules) if and only if every L-submodule W of V has a complement (ie, another L-submodule W' such that $W \oplus W' \cong V$ as L-modules).

6. Let L be a simple Lie algebra. Show that any two symmetric, bilinear, associative (w.r.t. the Lie bracket) and non-degenerate forms in L are multiple of each other. (so, in a sense, the Killing form is unique) [Hint: Use Schur's lemma]

7. Let V and W be L-modules. Show that $Hom(V, W)$, the vector space of all linear maps from V into W, has the structure of an L-module with the action given by $(x \cdot f)(v) := x \cdot f(v) - f(x \cdot v)$, for all $v \in V$, $f \in Hom(V, W)$ and $x \in L$.

8. Consider the inclusion $\mathsf{sl}(2,\mathbb{C}) \subset \mathsf{sl}(3,\mathbb{C})$ defined by taking the 2×2 matrices in the left upper corner, and the action of $\mathsf{sl}(2,\mathbb{C})$ in $M \equiv \mathsf{sl}(3,\mathbb{C})$ given by the adjoint representation. Show that the decomposition of M into irreducible $\mathsf{sl}(2,\mathbb{C})$-modules is

$$M = V(0) \oplus V(1) \oplus V(1) \oplus V(2),$$

where $V(m)$ denotes the unique irreducible $m+1$ dimensional representation of $\mathsf{sl}(2,\mathbb{C})$.

9. Let $P_d \subset \mathbb{C}[x,y]$ be the vector space of degree d homogeneous polynomials in two variables. Show that, for the usual basis of $\mathsf{sl}(2,\mathbb{C})$, and for any polynomial $p = p(x,y) \in P_d$, the formulas:

$$\begin{aligned} e \cdot p &= x\frac{\partial p}{\partial y} \\ f \cdot p &= y\frac{\partial p}{\partial x} \\ h \cdot p &= x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y}, \end{aligned}$$

turn P_d into an $\mathsf{sl}(2,\mathbb{C})$ module, which is irreducible.

10. Let L be the Heisenberg algebra over $\mathbb{C}$ with basis x, y, z such that $[x,y] = z$ and z is central. Show that L does not have any faithful finite-dimensional irreducible representation.

11. Let $L = \mathsf{sl}(n,\mathbb{C})$. Show that the set of diagonal matrices in L is a Cartan subalgebra of dimension $n-1$.

12. Show that any Cartan subalgebra of $\mathsf{sl}(2,\mathbb{C})$ has dimension 1.

13. If L is semisimple and H is a Cartan subalgebra, prove that H is self-normalizing, that is $N_L(H) = H$.

14. For the algebras $\mathsf{sl}(n,\mathbb{C})$ determine explicitly the root strings and the Cartan integers, and verify that whenever α and β are non-proportional,

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \{-1,0,1\}.$$

15. Show that there are no semisimple Lie algebras of dimensions 2, 4, 5 or 7.

16. Consider the set of diagonal matrices $D_4 \subset \mathsf{gl}(4,\mathbb{C})$ and the Lie algebra $\mathsf{so}(4,\mathbb{C}) = \{x \in \mathsf{gl}(4,\mathbb{C}) : x^t S + Sx = 0\}$, where x^t is the transpose of x, $S = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$ and I_2 is the 2×2 identity matrix. Show that D_4 is a Cartan subalgebra of $\mathsf{so}(4,\mathbb{C})$, and that the Cartan decomposition provides an isomorphism $\mathsf{so}(4,\mathbb{C}) \cong \mathsf{sl}(2,\mathbb{C}) \oplus \mathsf{sl}(2,\mathbb{C})$.