Lie Algebras

1st Problem List

Due October, 6

1. Let L be the vector space $\mathbb{R}^3$ with the commutator given by the exterior product between 2 vectors:
$$[x, y] = x \times y, \qquad x, y \in \mathbb{R}^3.$$
Verify that L is a Lie algebra and determine its structure constants relative to the canonical basis of $\mathbb{R}^3$.

2. Let L be a Lie algebra. Show that the bracket is associative, ie $[x, [y, z]] = [[x, y], z]$ for all $x, y, z \in L$, if and only $[a, b] \in Z(L)$ for all $a, b \in L$.

3. Let (x, h, y) be the ordered basis of $\mathsf{sl}(2, \mathbb{C})$ given by
$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
Compute the matrices of ad_x, ad_h and ad_y with respect to this basis.

4. Let $x \in \mathsf{gl}(n, \mathbb{C})$ be an endomorphism with n distinct eigenvalues $a_1, a_2, ..., a_n \in \mathbb{C}$. Show that the eigenvalues of ad_x are precisely the n^2 scalars $a_i - a_j$, $1 \leq i, j \leq n$.

5. Let L be the 2 dimensional Lie algebra given by
$$[x, y] = x$$
where (x, y) is a basis of L. Determine a linear Lie algebra (subalgebra of $\mathsf{gl}(V)$, for some vector space V) isomorphic to L. [Hint: consider the adjoint representation].

6. Recall that the symplectic Lie algebra $\mathsf{sp}(2n, \mathbb{C})$ is the set of $2n \times 2n$ complex matrices A such that $JA + A^t J = 0$, where t denotes transposition and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ is the standard symplectic matrix. Compute the dimension of $\mathsf{sp}(2n, \mathbb{C})$ as a complex vector space.

7. Let $\mathsf{s}(n, \mathbb{C})$ be the subset of scalar matrices (scalar multiples of the identity matrix) inside $\mathsf{gl}(n, \mathbb{C})$. Show that $\mathsf{gl}(n, \mathbb{C}) = \mathsf{sl}(n, \mathbb{C}) + \mathsf{s}(n, \mathbb{C})$ and that $[\mathsf{gl}(n, \mathbb{C}), \mathsf{s}(n, \mathbb{C})] = 0$.

8. Show that $[\mathsf{gl}(n,\mathbb{C}),\mathsf{gl}(n,\mathbb{C})] = [\mathsf{sl}(n,\mathbb{C}),\mathsf{sl}(n,\mathbb{C})] = \mathsf{sl}(n,\mathbb{C})$.

9. Show that $\mathsf{t}(n,\mathbb{C})$ and $\mathsf{d}(n,\mathbb{C})$ coincide with their normalizer inside $\mathsf{gl}(n,\mathbb{C})$ and that $N_{\mathsf{gl}(n,\mathbb{C})}(\mathsf{n}(n,\mathbb{C})) = \mathsf{t}(n,\mathbb{C})$.

10. Let L be a Lie algebra. Show that the set $\{ad_x : x \in L\}$ forms an ideal of the Lie algebra of all derivations $Der(L) \subset \mathsf{gl}(L)$. This ideal is called the ideal of inner derivations.

11. Prove that, for any Lie algebra L, the quotient algebra $L/Z(L)$ is always a linear Lie algebra.

12. Show that $\mathsf{sl}(2,\mathbb{C})$ is simple. More generally, show that if L is any three dimensional Lie algebra isomorphic to its derived algebra $L = [L, L]$, then L is simple [observe that any homomorphic image of L also equals his derived algebra].

13. Let L be a Lie algebra. Prove that all terms in the derived series and in the lower central series of L are indeed ideals of L.

14. Show that any nilpotent Lie algebra has a codimension 1 ideal.

15. Show that L is a solvable Lie algebra if and only if there exists a chain of subalgebras
$$L = L_0 \supset L_1 \supset ... \supset L_n = 0$$
such that L_{j+1} is an ideal of L_j and all quotients L_j/L_{j+1} are abelian.

16. Prove that L is solvable if and only if $ad(L)$ is solvable.

17. Let V be a finite dimentional vector space and $L = \mathsf{sl}(V)$. Show that $Rad(L) = Z(L)$ and conclude that L is semisimple.

18. If $x, y \in End(V)$ commute show that $(x + y)_s = x_s + y_s$ and that $(x + y)_n = x_n + y_n$ [Hint: show first that if x and y are semisimple (resp. nilpotent) then $x+y$ is also semisimple (resp. nilpotent)]. Show that this does not hold in general when x and y do not commute.

19. Let V be a finite dimensional vector space and L a solvable subalgebra of $\mathsf{gl}(V)$. Prove that $Tr(xy) = 0$ holds for all $x \in [L, L]$ and $y \in L$.

References

[EW] K. Erdmann and M. Wildon, *Introduction to Lie Algebras,* Springer-Verlag SUMS, 2006.

[H] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag GTM 9, 1972.