Exam of Lie Algebras and Lie Groups

DMIST - 22 Feb. 2011

1. Let S be a complex $n \times n$ matrix and define

$$\mathsf{gl}_S(n,\mathbb{C}) := \{x \in \mathsf{gl}(n,\mathbb{C}) : x^t S + Sx = 0\}$$

   (a) Show that $\mathsf{gl}_S(n,\mathbb{C})$ is a Lie subalgebra of $\mathsf{gl}(n,\mathbb{C})$

   (b) Let I_m be the identity $m \times m$ matrix and let $n = 2m$. With $S = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$, prove that $\mathsf{gl}_S(n,\mathbb{C})$ is isomorphic to the even orthogonal Lie algebra $\mathsf{so}(2m,\mathbb{C}) := \{x \in \mathsf{gl}(n,\mathbb{C}) : x^t + x = 0\}$.

2. Let $L = \mathsf{sl}(2,\mathbb{C})$ and M be a finite dimensional L-module, and let e, f, h be the usual basis of $\mathsf{sl}(2,\mathbb{C})$. Define a linear map $c : M \to M$ by $c(v) = \left(ef + fe + \frac{1}{2}h^2\right) \cdot v$, for all $v \in M$.

   (a) Show that c is a homomorphism of L-modules.

   (b) Let V_λ be the irreducible L-module of dimension $\lambda + 1$. Show that the action of c on V_λ is given by $\frac{1}{2}\lambda(\lambda+2)I_{\lambda+1}$ [Hint: use Schur's lemma, and compute c on a highest weight vector]

3. Let $\Phi \subset E$ be a root system and consider its dual root system $\Phi^\vee = \{\alpha^\vee = 2\frac{\alpha}{\|\alpha\|^2} : \alpha \in \Phi\}$.

   (a) Show that the Cartan matrix of $\Phi^\vee$ is the transpose of that of Φ

   (b) Show that the root systems of B_l and C_l are dual of each other.

4. Show that the exponential map from $\mathsf{gl}(n,\mathbb{C})$ to $GL(n,\mathbb{C})$ is surjective.

5. Let N be a normal subgroup of a Lie group G.

   (a) If N is discrete, prove that N is contained in the center of G. [Hint: consider the map $G \to N$ sending g to gng^{-1} for some fixed $n \in N$].

   (b) Show that, for any homomorphism $\phi : G \to H$ with finite fibers (ie, $\phi^{-1}(h)$ is finite for every $h \in H$), the kernel of ϕ is an abelian subgroup of G.

6. Let V be an irreducible representation of a compact Lie group G, and $\rho : G \to GL(V)$ be the corresponding homomorphism. Show that every linear transformation $F : V \to V$ has the form $F(v) = \sum_{j=1}^n c_j \rho(g_j)(v)$ for some $n \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $g_j \in G$ $(j = 1, \ldots, n)$. [Hint: First check that the representation of $G \times G$ on $End(V) = V \otimes V^*$ is irreducible.]