Talks – VBAC 2003

Takeshi Abe: *An explicit basis of the space of anticanonical divisors of a moduli space of parabolic vector bundles of half weight on $\mathbb{P}^1$.*

The dimension of the vector space of global sections of the anticanonical divisor of the moduli space of parabolic vector bundles on an algebraic curve is computed by Verlinde formula. When the curve is a projective line and the parabolic weight is half, the parabolic vector bundle corresponds to a vector bundle with the involution on a hyperelliptic curve. Using this correspondence, we find an explicit basis of the above vector space.

Rui Albuquerque: *On the Twistor Theory of Symplectic Manifold.*

We study the complex geometry of the twistor space of a symplectic manifold M, ie the bundle of complex structures, with fibre the Siegel domain, together with its tautological complex structure induced from a connection on M. Hoping to find some new answers on problems about symplectic connections we found new proofs of known results. The question of holomorphic completeness of the space is solved, the canonical metric is considered and the condition to have kahlerian twistor spaces is shown - contrary to the integrability equation, this stronger property requires a flat connection. Finally, the induced action of $\mathrm{Symp}(M,\omega)$ on the twistor space has some particular features.

Jørgen Andersen: *Asymptotic faithfulness of the quantum $SU(n)$ representations of the mapping class groups.*

Igor Artamkin: *Vector bundles on large limit curves.*

Large limit curves were introduced by A.N. Tyurin in his last works. Moduli of vector bundles on ll-curves were studied due to many connections with theory of spin networks and complex gauge theory. Author's interest to the subject was inspired by collaboration with A.N. Tyurin on this field during summer and autumn 2002. Further development of this topic will be presented.

V. Balaji: *Moduli of principal bundles and ramified principal bundles.*

Usha N. Bhosle: *Maximal subsheaves of torsion free sheaves.*

Let Y be a reduced irreducible projective curve of arithmetic genus $g \geq 2$ with at most ordinary nodes as singularities. For a subsheaf F of rank r', degree d' of a torsionfree sheaf E of rank r, degree d, let $s(E,F) = r'd - rd'$. Define $s_{r'}(E) = \min s(E,F)$, where the minimum is taken over all subsheaves of E of rank r'. For a fixed $r', s_{r'}$ defines a stratification of the moduli space $U(r,d)$ of stable torsionfree sheaves of rank r, degree d by locally closed subsets $U_{r',s}$. We study the nonemptiness and dimensions of the strata. We show that the general element in each nonempty stratum is a vector bundle and it has only finitely many (resp. unique) maximum subsheaves of rank r' for $s \leq r'(r-r')(g-1)$ (resp. $s < r'(r-r')(g-1)$). We prove that the tensor product of two general stable vector bundles on an irreducible nodal curve Y is nonspecial.

Steven Bradlow: *Augmented holomorphic bundles: theory and practice.*

It's an old story that holomorphic bundles have interesting moduli spaces. More recently a new chapter has been added in which the bundles are augmented by the specification of additional data such as holomorphic sections. We will describe some general features of such augmented holomorphic bundles and their moduli spaces, and will discuss some specific problems where moduli spaces of this sort play a useful role. In particular, we will discuss applications to questions in higher rank Brill-Noether theory and to representation theory for the fundamental group of a Riemann surface. In both cases the relevant augmented bundle moduli spaces occur in discrete families. We will show how the utility of these families derives from their birational geometry.

Alexander Braverman: *Uhlenbeck spaces via Drinfeld compactification.*

For an almost simple simply connected group G over $\mathbb{C}$ let $\mathcal{A}^a$ ($a \in \mathbb{N}$) denote the moduli space of G-bundles on $\mathbb{P}^2$ trivialized at the "infinite" $\mathbb{P}^1 \subset \mathbb{P}^2$ and having second Chern class equal to a. Using the theory of instantons Uhlenbeck and Donaldson have shown that $\mathcal{A}^a$ is an open subset of a larger (singular) topological space $\mathcal{U}^a$ (*the Uhlenbeck space*). For $G = SL(n)$ Donaldson has defined a scheme structure on $\mathcal{U}^a$ (using the so called *ADHM construction*). We are going to explain a different purely algebraic construction of $\mathcal{U}^a$ which works for arbitrary G. The construction is based on a certain compactification of the moduli space of parabolic G-bundles on any *smooth complete curve* defined by Drinfeld (which here in fact must be applied to the affine Kac-Moody group corresponding to G and not to G itself). This construction allows to compute (both local and global) intersection cohomology of $\mathcal{U}^a$. If time permits we shall mention why the latter result might be useful for explicit calculations of the Seiberg-Witten prepotential generalizing the recent work of N. Nekrasov (where it is done for $G = SL(n)$).

(Joint work with M. Finkelberg and D. Gaitsgory)

Andrei Caldararu: *On elliptic Calabi-Yau threefolds with 1-, 2- and 4-sections.*

We analyze a certain example of Vafa and Witten from the point of view of relative moduli spaces of sheaves. The main result is the construction of a family of smooth Calabi-Yau 3-folds, mirror to their original family. Our construction gives an explanation for their puzzle, which was caused by the fact that the classical constructions from physics do not yield a smooth model for this family.

Ana-Maria Castravet: *Rational families of vector bundles on curves.*

We find and describe the irreducible components of the space of rational curves on moduli spaces M of rank 2, stable vector bundles, with fixed determinant of odd degree, on curves C of genus $g \geq 2$. We prove that the maximally rationally connected quotient of such a component is either the Jacobian of the curve C, or a direct sum of two copies of the Jacobian. We show that moduli spaces of rational curves on M are in one-to-one correspondence with moduli of rank 2 vector bundles on the surface $\mathbf{P}^1 \times C$.

Gavril Farkas: *K3 sections, rank two bundles and effective divisors on the moduli space of curves.*

A fundamental problem in the geometry of the moduli space of curves is to understand the cone of effective divisors on M_g which essentially describes all rational maps from M_g to other varieties. The shape of this cone is predicted by the Harris-Morrison Slope Conjecture, which singles out the classical Brill-Noether divisors as those having minimal slope.

We describe ways to construct new divisors on M_g (fundmamentally different than the Brill-Noether ones) which provide counterexamples to the Slope Conjecture for certain genera.

These divisors on the moduli space consist of curves carrying certain rank two vector bundles with canonical determinant. The connection between the Slope Conjecture and the moduli space of K3 surfaces will also be explained. This is joint work with M. Popa.

Elizabeth Gasparim: *Moduli of bundles on surfaces and birational tranformations.*

We show that a birational transformation on a surface implies in a change on the homology of the moduli of bundles over the surface.

Francesca Gavioli: *Theta functions on the moduli space of parabolic bundles.*

Let X be a smooth, irreducible projective curve of genus $g \geq 2$ over the field of complex numbers and I a finite subset of points of X. Let $\mathcal{M}^{par}$ denote the moduli space of parabolic vector bundles of rank r, trivial determinant and fixed parabolic structure at I. There is a natural ample line bundle $\mathcal{L}^{par}$ on $\mathcal{M}^{par}$, which is the analogue of the determinant bundle $\mathcal{D}$ on the moduli space $\mathcal{M}$, of vector bundles on X of fixed rank and determinant. Let $|I|$ denote the cardinality of the parabolic subset; the aim of the talk is to prove the following

Theorem
Let ℓ be an integer such that $\ell \geq [\frac{r^2}{4}]$. Then the invertible bundle $\mathcal{L}^{par \otimes \ell}$ is globally generated.

We are actually going to prove that, for ℓ given by this bound, parabolic theta functions generate $\mathcal{L}^{par \otimes \ell}$.

In order to get this result, we construct a parabolic analogue of Grothendieck's scheme of quotients, which parametrizes quotient bundles of a parabolic vector bundle E_*, of fixed parabolic Hilbert polynomial. We prove an estimate for its dimension, which extends the result of Popa and Roth on the dimension of the Quot scheme. This estimate allows us to determine a bound, which is independent of the fixed (quasi)parabolic structure.

William Goldman: *The Complex-Symplectic Geometry of $SL(2, \mathbf{C})$-Moduli Spaces over Riemann surfaces.*

The $SL(2, \mathbf{C})$-character variety parametrizes gauge-equivalence classes of flat connections over a surface S. It enjoys a natural complex-symplectic structure invariant under the mapping class group of S. Furthermore it is a model space for the deformation space of complex-projective structures on S. Analogous to the Fenchel-Nielsen flows on Teichmueller space are complex twist flows on the moduli space of flat $SL(2, \mathbf{C})$-connections on a Riemann surface. The twist flows relate both to the action of the mapping class group on the character variety, and extend to complex-symplectic flows on the deformation space of complex projective structures.

A.L. Gorodentsev: *ALAG (Abelian Lagrangian Algebraic Geometry)*

This is a joint work with A.N.Tyurin done during 2000-2002, the main goal of which is to link the Lagrangian and Algebraic geometries via the geometric quantization. Given a symplectic manifold M equipped with a geometric prequantization data (L, a) (Hermitean line bundle plus $U(1)$-connection) and compartible $\mathrm{Mp}^{\mathbb{C}}$-structure, one can define an integrable Kähler structure on the space $\mathcal{B}$ of half-weighted Bohr-Zommerfeld Lagrangian cycles $S \to M$, which is a smooth submanifold of finite codimension $\dim H^1(S, \mathbb{C})$ in the (infinite dimensional) manifold of all half weighted Lagrangian cycles $S \to M$. This infinite dimensional Kähler manifold $\mathcal{B}$ has canonical holomorphic prequantization equipment (Barry bundle) and plays a central role in geometrical understanding and comparing different types of geometric quantization, what will be illustrated by several examples: given a real polarization of M, then the corresponding Lagrangean fibration contains a finite number of Bohr-Sommerfeld cycles, which give a natural basis in the theta-functions spaces; given a complex polarization of M, then there is an universal holomorphic locally epimorphic Bortwick-Paul-Uribe map from $\mathcal{B}$ onto the holomorphic spaces of wave functions, which induces all their good properties such as independence on a choice of a complex structure on M...; e.t.c.

I'll try to explaine the last ideas of A.N.Tyurin in this direction (at least as far as I'm understanding them) and outline his great role in the current development of this huge branch of mathematical physics.

Tomas L. Gomez: *Moduli space of principal sheaves over projective varieties.*

Let X be a smooth complex projective variety, and let G be a complex reductive group. We construct the moduli space of semistable principal G-bundles on X. This moduli space is compactified by considering *principal G-sheaves.* These are triples (P, E, ψ), where E is a torsion free sheaf on X, P is a principal G-bundle on the open set U where E is locally free and ψ is an isomorphism between $E|_U$ and the vector bundle associated to P by the adjoint representation.

We say it is (semi)stable if all filtrations $E_\bullet$ of E as sheaf of (Killing) orthogonal algebras, i.e. filtrations with $E_i^\perp = E_{-i-1}$ and $[E_i, E_j] \subset E_{i+j}{}^{\vee\vee}$, have

$$\sum (P_{E_i}\mathrm{rk}E - P_E\mathrm{rk}E_i)\,(\preceq)\,0,$$

where P_{E_i} is the Hilbert polynomial of E_i. After fixing the Chern classes of E and of the line bundles associated to the principal bundle P and characters of G, we obtain a projective moduli space of semistable principal G-sheaves. We prove that, in case $\dim X = 1$, our notion of (semi)stability is equivalent to Ramanathan's notion.

(Joint work with I. Sols. Preprint math.AG/0206277)

Samuel Grushevsky: *Odd theta characteristics: embedding M_g and A_g.*

The classical problem of recovering a plane quartic from the set of its bitangents, first studied by Aronhold in 1860s, was solved by Caporaso and Sernesi in 2000. The problem generalizes to the question of recovering a canonical genus g curve in $\mathbf{P}^{g-1}$ from the set of hyperplanes tangent to it at $g-1$ points. Caporaso and Sernesi have showed that generically this is also possible.

In this talk we reformulate this question in terms of derivatives of classical odd theta functions and extend the construction to principally polarized abelian varieties. We then show that an abelian variety can be recovered uniquely from the images of the two-torsion points of its theta divisor under the Gauss map, i.e. that these images define an embedding of the (level cover of) the moduli space of abelian varieties into a certain Grassmannian. This provides a generalization and strengthening of the result of Caporaso and Sernesi for curves.

This is joint work with Riccardo Salvati Manni.

Alexander Gurevich: *p-adic period map on the Lubin-Tate moduli space.*

Let G be a one-dimensional formal group over F_p of height h. According to Lubin and Tate, the set M of *-isomorphism classes of deformations of G over Z_p is in one-to-one correspondence with the direct product of $h-1$ copies of the maximal ideal of Z_p, what provides a parameterization of M. On the other hand, one can naturally define an action of the automorphism group of G on M. This group can be considered as a factor group of $Z_p[[D]]$, and that gives another parameterization of M. We construct a p-adic period map from M to a group of formal power series supplied with an $Z_p[[D]]$-action and prove that this map is $Z_p[[D]]$-invariant. Thus the p-adic period map iduces an isomorphism between M and the set of cosets of some subgroup of $Z_p[[D]]$ in $Z_p[[D]]$ with the $Z_p[[D]]$-action given by multiplication. Moreover, it allows us to obtain an explicit formula which connects two different parameterizations on M. Namely, it expresses the coordinates of a point on M coming from $Z_p[[D]]$-action through the Lubin-Tate coordinates. This construction generalizes the p-adic period map of Gross and Hopkins, who considered deformations of the reduction of the Artin-Hasse formal group and and established their p-adic period map with the aid of rigidified extensions of formal groups and equivariant vector bundles on the Lubin-Tate moduli space.

George Hitching: *Moduli of principal* $\mathrm{Sp}_2(\mathbb{C})$*-bundles over algebraic curves.*

Let X be a complex algebraic curve of genus $g \geq 2$. The moduli space $\mathcal{M}_X(\mathrm{Sp}_2(\mathbb{C}))$ of principal $\mathrm{Sp}_2(\mathbb{C})$-bundles on X can be identied, via the standard represention of $\mathrm{Sp}_2(\mathbb{C})$ on $\mathbb{C}^4$, with the moduli space of holomorphic vector bundles of rank 4 on X which carry a symplectic form. Using this identication, we describe the semistable boundary and singular locus of $\mathcal{M}_X(\mathrm{Sp}_2(\mathbb{C}))$. We then describe a method for building symplectic bundles of rank $2n$ from given vector bundles of rank n and describe in detail a family of such bundles in the case where X has genus 2.

Donghoon Hyeon: *On bundles over a twisted curve.*

A twisted curve is a nodal curve with a nontrivial stack (or orbifold) structure at its nodes. I will describe basic results on vector bundles over a twisted curve and their direct images, and discuss how they can be used to compactify the moduli space of bundles over a nodal curve.

Lisa Jeffrey : *Representations of fundamental groups of nonorientable* 2*-manifolds.*

If Σ is a Riemann surface, let $M(\Sigma, G)$ be the moduli space of conjugacy classes of representations of the fundamental group of Σ in a compact Lie group G. In his paper “Quantum gauge theories in two dimensions” (Commun. Math. Phys. **141** (1991) 153-209) Witten defined a volume on this space using Reidemeister-Ray-Singer torsion, and proved this volume is equal to the symplectic volume.

Witten also defined a volume on the corresponding moduli space $M(\Sigma', G)$ when Σ' is a nonorientable 2-manifold. The latter space does not admit a symplectic structure. We show that in this case Witten's volume can be obtained from the Riemannian volume associated to a choice of Riemannian metric on Σ'.

(Joint work with Nan-Kuo Ho)

Ludmil Katzarkov: *Homological Mirror Symmetry and Fukaya Seidel categories.*

In this talk we will outline a proof of Homological Mirror symmetry for weighted projective planes and other surfaces. Homological Mirror symmetry for high genus curves and some connections with moduli spaces of vector bundles will be discussed as well.

Ivan Kausz: *A canonical decomposition of generalised theta functions on the moduli stack of Gieseker vector bundles.*

In this talk I present a new geometric approach to the factorisation rule for generalised theta functions.

Let X be an irreducible projective nodal curve with one singularity and let Y be its normalisation. Recently I have constructed the moduli stack $GVB(X)$ of rank r Gieseker vector bundles on X and have shown that its normalisation is a locally trivial fibration over the moduli stack $VB(Y)$ of vector bundles on Y, where the fibre is a canonical compactification of Gl_r. In this talk I indicate how to prove a canonical dirict sum decomposition of the space of global sections of a power of the canonical determinant line bundle on $GVB(X)$ where the summands are spaces of global sections of certain line bundles on the moduli stack of parabolic bundles on the two-pointed curve Y.

Besides being conceptually clearer than the proof of the analogous result (in the context of torsion free sheaves) by Narasimhan, Ramadas and Sun, it also yields a somewhat stronger result, since the decomposition proved by those authors is not canonical.

Frances Kirwan: *Relations in the cohomology of moduli spaces of bundles.*

The cohomology ring of the moduli space of stable holomorphic bundles of rank n and degree d over a Riemann surface of genus $g > 1$ has a standard set of generators when n and d are coprime. When n is 2 the relations between these generators are well understood, and in particular a conjecture of Mumford, that a certain set of relations is a complete set, is known to be true. In this talk, based on joint work with Richard Earl, I will describe generalisations of Mumford's relations to the case when $n > 2$, and also for moduli spaces of parabolic bundles, which form complete sets of relations.

Adrian Langer: *Moduli spaces of sheaves and bounds on the cohomology groups.*

I would like to talk about my recent results concerning bounds on the cohomology groups on higher dimensional varieties of any characteristic. In particular, my results imply the best possible results on vanishing of H^1 on surfaces giving the first effective results on irreducibility of moduli spaces of vector bundles for large discriminant. They can also be used for construction of moduli spaces of sheaves in mixed characteristic via Simpson's method avoiding quite complicated Maruyama's construction and giving more information.

Vikram Mehta: *Low-height Representations and Semistable Bundles in char p, a survey.*

This is a survey talk relating low-height rpresentations in char p to semistability. This is applied to Behrend's conjecture, Luna's etale slice theorem and moduli spaces of G-bundles in char p.

Ignasi Mundet i Riera: *A compactification of the universal Jacobian and symplectic invariants.*

We describe a compactification of the universal Jacobian which fibres over the Deligne-Mumford moduli space of stable curves. As an application of our construction we define Hamiltonian Gromov-Witten invariants coupled to gravity.

Vicente Munoz: *Topology of the moduli space of parabolic Higgs bundles and parabolic triples.*

We study the topology of the moduli space of parabolic Higgs bundles of rank 3 and compute its Betti numbers. For this it is necessary to study the moduli space of parabolic pairs of rank 2 (alternatively, parabolic triples with one parabolic line bundle and one rank 2 parabolic bundle). There is a phenomenon of flips analogous to the one studied by Thaddeus in the non-parabolic situation.

(Joint work with O. Garcia-Prada and P. Gothen)

V.V. Nikulin: *Tyurin's invariants of 4-dimensional pseudo-Riemannian manifolds.*

In 1981 Tyurin introduced local invariants of 4-dimensional Riemannian (over $\mathbf{C}$) manifolds, which is now called Tyurin's invariant. Let $m \in M$ be a point of a 4-dimensional pseudo-Riemannian manifold M, and T_m the 4-dimensional tangent space of $m \in M$. Tyurin considered the surface X_m which is intersection of three quadrics in $P^5 = P(\Lambda^2(T_m \otimes \mathbf{C}))$ defined by the quadratic forms $\Lambda^2 G$ (G is the metric), Vol (Volume in T_m) and R, Riemann tensor in $\Lambda^2 T_m$. The surface X_m is a K3 surface or its degeneration. Thus, we get a map from M to moduli of K3 surfaces (Tyurin's map).

In 1983 - 1986 I studied the corresponding real Tyurin invariants for Lorentzian and other types of 4-dimensional metrics and showed that they divide the 4-dimensional space-time of the general relativity, where we live, in different types. This is similar to subdivision points of Riemannian manifolds by positive, negative and zero curvature. Thus Tyurin's invariant has a very important Physical sense.

Bruno De Oliveira *Moduli spaces of vector bundles on universal covers and the Shafarevich conjecture.*

The results of this talk were obtained in collaboration with F. Bogomolov. The Shafarevich conjecture states: the universal cover $\tilde{X}$ of a projective variety X is holomorphic convex. If X has an infinite fundamental group the conjecture implies, in particular, that $\tilde{X}$ has non-constant holomorphic functions. Let $\rho : \tilde{X} \to X$ be the covering map and $\rho^* : \mathrm{Vect}(X) \to \mathrm{Vect}(\tilde{X})$ be the pullback map. Behind all known methods to produce holomorphic functions on $\tilde{X}$ there is an identification $\rho^* V = \rho^* V'$ for some special distinct bundles V and V' on X. This is present in the work of Gromov, Essidieux, Katzarkov, Pantev and Ramachandran. In this talk we will prove another result that relates the identification of bundles on the universal cover $\tilde{X}$ with the existence of holomorphic functions on $\tilde{X}$. Let X be a projective manifold with Pic(X) = $\mathbb{Z}$. Assume the universal cover $\tilde{X}$ has no non-constant holomorphic functions. Then for any stable bundle V on X the map ρ^* induces an embedding of the space of 1st order deformations of V into the space of 1st order deformations of $\rho^* V$.

S. Ramanan: *Hecke, Higgs and Hitchin.*

Miles Reid: *Homage to Andrei Tyurin.*

Topics in algebraic curves, vector bundles, moduli spaces and theoretical physics, from the life of Andrei Tyurin.

Mohamed Saidi: *On The Fundamental group of complete algebraic curves in positive characteritic.*

We use Raynaud's Theory of Theta divisors in positive characteristic, in order to prove some results concerning the variation of the structure of the fundamental group of projective curves in positive characteristic.

Giovanna Scataglini: *Remarks on stable vector bundles with "special" line subbundles.*

Let C be a smooth, non hyperelliptic curve of genus 3 and let $\mathcal{SU}_C(2)$ be the moduli space of rank 2 semi-stable vector bundles over C with trivial determinant. We consider stable vector bundles $E \in \mathcal{SU}_C(2)$ with "many" global sections, i.e., such that $h^0(C, E) > 1$ and describe their maximal line subbundles. We then show how this contradicts a claim made by Narasimhan and Ramanan during the course of their classical proof that $\mathcal{SU}_C(2)$ embeds in the projective space $|2\Theta|$. We emphasise how this fact does not affect their main ideas as the necessary corrections only help to highlight the strong relation between Narasimhan and Ramanan's proof and the particular geometry of the genus 3 case.

Stefan Schröer: *Vector bundles and Azumaya algebras on singular surfaces.*

Using vector bundles on formal curves, I solve some problems about vector bundles on singular surfaces. Suppose X is a proper algebraic surface, not necessarily projective.

The first result is concerned with the resolution property: If X is normal, then each coherent $\mathcal{O}_X$-module is the quotient of a locally free $\mathcal{O}_X$-module. The second result has to do with Grothendieck's question on existence of Azumaya algebras: If X is normal, then each torsion étale cohomology class in $H^2(X, \mathbb{G}_m)$ comes from an Azumaya $\mathcal{O}_X$-algebra. The last result is about the existence of nontrivial vector bundles on surfaces with arbitrary singularities: There are vector bundles $\mathcal{E}$ of rank two with given determinant and $c_2(\mathcal{E}) \gg 0$.

In each case, the idea is to solve the problem on a modification of X, and then improve the solution on the formal exceptional curve by studying vector bundles on such formal curves.

References

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Michael Thaddeus: *Stable Maps to a Loop Group.*

Let X be a compact complex manifold, ΩK the space of based loops on a compact Lie group. Atiyah pointed out that the space of based holomorphic maps $\phi : X \to \Omega K$ is finite-dimensional. When X is a curve (say the projective line) this suggests the possibility of compactifying the space and evaluating Gromov-Witten invariants of the loop group. We explain how to do this. Surprisingly, the moduli space is not smooth, but it can be canonically deformed to a smooth space. We will outline a few simple applications, proving, for example, the associativity of the quantum cohomology.

Nikolai Tyurin: *Local invariant of riemannian metrics in dimension 3 and 4.*

More than 20 years ago A. Tyurin introduced some special invariant for riemannian metrics over smooth 4 - dimensional real compact manifolds. For a given metric one has a special map which includes our based manifold to the moduli space of polarized K3 - surfaces which should be called now Tyurin's indicatrix map. Almost 20 years later the construction was partially adopted to the 3 dimensional case by the same author and it leads to some interesting questions in the theory of algebraic Calabi - Yau threefolds.