Index formula for convolution type operators with data functions in $\mathrm{alg}(SO, PC)$[☆]

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Abstract

We establish an index formula for the Fredholm convolution type operators $A = \sum_{k=1}^{m} a_k W^0(b_k)$ acting on the space $L^2(\mathbb{R})$, where a_k, b_k belong to the C^*-algebra $\mathrm{alg}(SO, PC)$ of piecewise continuous functions on $\mathbb{R}$ that admit finite sets of discontinuities and slowly oscillate at $\pm\infty$, first in the case where all a_k or all b_k are continuous on $\mathbb{R}$ and slowly oscillating at $\pm\infty$, and then assuming that $a_k, b_k \in \mathrm{alg}(SO, PC)$ satisfy an extra Fredholm type condition. The study is based on a number of reductions to operators of the same form with smaller classes of data functions a_k, b_k, which include applying a technique of separation of discontinuities and eventually lead to the so-called truncated operators $A_r = \sum_{k=1}^{m} a_{k,r} W^0(b_{k,r})$ for sufficiently large $r > 0$, where the functions $a_{k,r}, b_{k,r} \in PC$ are obtained from $a_k, b_k \in \mathrm{alg}(SO, PC)$ by extending their values at $\pm r$ to all $\pm t \geq r$, respectively. We prove that $\mathrm{ind}\, A = \lim_{r\to\infty} \mathrm{ind}\, A_r$ although $A = \text{s-}\lim_{r\to\infty} A_r$ only.

Keywords: Convolution type operator; Piecewise continuous and slowly oscillating data functions; Truncated operator; C^*-algebra; Symbol;

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1. Introduction

Let $C(\mathbb{R})$, $C(\overline{\mathbb{R}})$ and $C(\dot{\mathbb{R}})$ denote the spaces of continuous functions on $\mathbb{R} = (-\infty, +\infty)$, $\overline{\mathbb{R}} = [-\infty, +\infty]$ and $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, respectively, and let PC stand for the set of all functions $f : \dot{\mathbb{R}} \to \mathbb{C}$ which possess finite one-sided limits at every point $x \in \dot{\mathbb{R}}$.

For a continuous function $f : \mathbb{R} \to \mathbb{C}$ and a set $I \subset \mathbb{R}$, let

$$\operatorname{osc}(f, I) := \sup\{|f(t) - f(s)| : \ t, s \in I\}.$$

Following [19], we denote by SO the set of *slowly oscillating* functions,

$$SO := \Big\{ f \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) : \lim_{x \to +\infty} \operatorname{osc}(f, [-2x, -x] \cup [x, 2x]) = 0 \Big\}.$$

Clearly, SO is a C^*-subalgebra of $L^\infty(\mathbb{R})$ and $C(\dot{\mathbb{R}}) \subset SO$.

We denote by $\operatorname{alg}(SO, PC)$ the smallest C^*-subalgebra of $L^\infty(\mathbb{R})$ that contains SO and PC, and let $\operatorname{alg}(SO, C(\overline{\mathbb{R}}))$ stand for the C^*-subalgebra of $L^\infty(\mathbb{R})$ generated by SO and $C(\overline{\mathbb{R}})$.

In the present paper we deal with an index formula for the operators of the form

$$A = \sum_{k=1}^{m} a_k W^0(b_k) \tag{1.1}$$

that are Fredholm on the space $L^2(\mathbb{R})$, where $a_k W^0(b_k)$ are products of multiplication operators $a_k I$ and convolution operators $W^0(b_k) = \mathcal{F}^{-1} b_k \mathcal{F}$, the functions a_k, b_k are in $\operatorname{alg}(SO, PC)$, and $\mathcal{F}$ is the Fourier transform, $(\mathcal{F}f)(x) = \int_{\mathbb{R}} e^{ixt} f(t)dt$.

Let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators acting on a Banach space X. An operator $A \in \mathcal{B}(X)$ is said to be *Fredholm* if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite-dimensional (see, e.g., [12]). If A is a Fredholm operator, then the number $\operatorname{ind} A = \dim\ker A - \dim\ker A^*$ is called the *index* of A.

The Fredholm theory for the Banach algebra $\mathcal{A}_{PC}$ generated by all operators of the form (1.1) with piecewise continuous functions a_k, b_k on the spaces $L^p(\mathbb{R})$ with $p \in (1, \infty)$, which contains a Fredholm criterion and an index formula for considered operators, was constructed by R.V. Duduchava (see [8], [9] and the references therein). The Fredholm theory for this algebra unites the Fredholm theories for the Banach algebra of singular integral operators with piecewise continuous coefficients (see [11], [22], [7], [4]) and for the Banach algebras of the Wiener-Hopf and paired convolution operators with piecewise continuous presymbols (see [10], [7], [6]).

The Fredholm theory for the C^*-algebra of pseudodifferential operators on $L^2(\mathbb{R})$ with slowly varying symbols, which contains the C^*-algebra $\mathcal{A}_{[SO,C(\overline{\mathbb{R}})]}$ generated by all operators of the form (1.1) with data functions $a_k, b_k \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$, was constructed in [20, Chapter 6] and [21, Section 3] (also see [18, Theorem 7.2]).

A Fredholm criterion for the operators in the Banach algebra $\mathcal{A}_{[SO,PC]}$ generated by the operators of the form (1.1) with $a_k, b_k \in \mathrm{alg}(SO, PC)$ on the spaces $L^p(\mathbb{R})$ $(1 < p < \infty)$, where b_k in addition are Fourier multipliers on $L^p(\mathbb{R})$, was obtained in [1]–[2]. Later on these results were generalized to Banach algebras generated by all operators of the form (1.1) on weighted Lebesgue spaces $L^p(\mathbb{R}, w)$ with Muckenhoupt weights w and data functions a_k, b_k admitting piecewise slowly oscillating discontinuities at arbitrary points of $\dot{\mathbb{R}}$ (see [13]–[15]).

To establish an index formula for the Fredholm operator (1.1) with data functions $a_k, b_k \in \mathrm{alg}(SO, PC)$ on the space $L^2(\mathbb{R})$, we assume that these functions have finite sets of discontinuities and either

(C1) *all* a_k *or all* b_k *are in* $\mathrm{alg}(SO, C(\overline{\mathbb{R}}))$, or

(C2) *there exist functions* $\widetilde{a}_k, \widetilde{b}_k \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ *such that*

$$\lim_{t\to\pm\infty} (\widetilde{a}_k(t) - a_k(t)) = \lim_{x\to\pm\infty} (\widetilde{b}_k(x) - b_k(x)) = 0, \tag{1.2}$$

$$\liminf_{t^2+x^2\to\infty} \left| \sum_{k=1}^{m} \widetilde{a}_k(t)\widetilde{b}_k(x) \right| > 0. \tag{1.3}$$

Condition (1.3) is equivalent to the Fredholmness of the operator $\sum_{k=1}^{m} \widetilde{a}_k W^0(\widetilde{b}_k)$ allowing us to reduce the computation of the index in the case (C2) to the case (C1). To provide an index formula in the case (C1) with all $b_k \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$, we proceed in two main steps, doing reductions to smaller classes of coefficients:

(1) *First decomposition*: reduction of the Fredholm operator A given by (1.1) to Fredholm operators $A_{t_0}^{\pm}$ and $A_{t_0}^{\diamond}$ of the form (1.1) with functions

$b_k \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ and coefficients $a_k^\pm$ and $a_k^\diamond$, respectively, where $a_k^\pm$ are continuous on $\mathbb{R}$ with a limit at $\mp\infty$ and the slowly oscillating behavior of a_k at $\pm\infty$, and $a_k^\diamond \in PC$ have one-sided limits at $\pm\infty$ and are continuous on $\mathbb{R} \setminus \{\tau_1, \dots, \tau_n\}$. Thus, $A_{t_0}^\pm \in \mathcal{A}_{[SO,C(\overline{\mathbb{R}})]}$ and their index formulas follow from [21] or [18], so we are left with computing the index of $A_{t_0}^\diamond$.

(2) *Separation of discontinuities*: we separate discontinuities of $a_k^\diamond \in PC$ by reducing to the case of only one discontinuity point of the functions a_k on $\mathbb{R}$. Using the index formula from [17] for Wiener-Hopf type operators and approximation and stability arguments, we obtain $\mathrm{ind}\, A_{t_0}^\diamond$.

The case (C1) with all $a_k \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ is reduced to the case of all $b_k \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ by applying the map $A \mapsto (\mathcal{F}A\mathcal{F}^{-1})^*$, while the index for $A \in \mathcal{A}_{[SO,C(\overline{\mathbb{R}})]}$ follows from [21] or [18].

In the case (C2), using conditions (1.2)–(1.3), we construct *another decomposition* of the Fredholm operator A that reduces A to Fredholm operators $A_{\mathbb{R},\infty}$, $A_{\infty,\infty}$, $A_{\infty,\mathbb{R}}$ of the form (1.1), where at least all a_k or all b_k are in $\mathrm{alg}(SO, C(\overline{\mathbb{R}}))$. Therefore indices of the operators $A_{\mathbb{R},\infty}$, $A_{\infty,\mathbb{R}}$ and $A_{\infty,\infty}$ are calculated as in the case (C1). Collecting these indices, we obtain $\mathrm{ind}\, A$.

The crucial point is noticing that the index of each operator mentioned above can be reduced to the computation of indices of the same form operators with data functions $a_k, b_k \in PC$. To this end, for every function $c \in \mathrm{alg}(SO, PC)$ with finite sets of discontinuities and every sufficiently large $r > 0$, we define the function $c_r \in PC$ by

$$c_r(t) := \begin{cases} c(-r) & \text{if } \ t < -r, \\ c(t) & \text{if } \ |t| \le r, \\ c(r) & \text{if } \ t > r, \end{cases} \tag{1.4}$$

which we call the truncated function for c. Then with the operator A of the form (1.1) we associate the family of truncated operators

$$A_r := \sum_{k=1}^{m} a_{k,r} W^0(b_{k,r}) \quad (r > 0), \tag{1.5}$$

where $a_{k,r}, b_{k,r} \in PC$ are truncated functions for $a_k, b_k \in \mathrm{alg}(SO, PC)$. It is easily seen that $A = \text{s-lim}_{r\to\infty} A_r$.

The main result of the paper reads as follows.

Theorem 1.1. *If the operator A given by* (1.1) *is Fredholm on the space $L^2(\mathbb{R})$, where the functions $a_k, b_k \in \operatorname{alg}(SO, PC)$ admit finite sets of discontinuities and either all a_k or all b_k are in $\operatorname{alg}(SO, C(\overline{\mathbb{R}}))$, or a_k and b_k satisfy conditions* (1.2)–(1.3)*, then there is an $r_0 > 0$ such that for all $r > r_0$ the truncated operators A_r given by* (1.5) *are Fredholm on the space $L^2(\mathbb{R})$, and*

$$\begin{aligned}
\operatorname{ind} A = \lim_{r\to\infty} \operatorname{ind} A_r = -\lim_{r\to\infty} \frac{1}{2\pi} \Bigg(& \sum_{s=1}^{n+1} \left\{ \arg \frac{\sigma(t,-r)}{\sigma(t,r)} \right\}_{t\in[\tau_{s-1}+0,\tau_s-0]} \\
& + \sum_{s=1}^{n} \left\{ \arg \left(\frac{\sigma(\tau_s+0,-r)}{\sigma(\tau_s+0,r)}\,\mu + \frac{\sigma(\tau_s-0,-r)}{\sigma(\tau_s-0,r)}(1-\mu) \right) \right\}_{\mu\in[0,1]} \\
& + \sum_{j=1}^{l} \left\{ \arg \left(\frac{\sigma(r,y_j+0)}{\sigma(-r,y_j+0)}\,\mu + \frac{\sigma(r,y_j-0)}{\sigma(-r,y_j-0)}(1-\mu) \right) \right\}_{\mu\in[0,1]} \\
& + \sum_{j=1}^{l+1} \left\{ \arg \frac{\sigma(r,x)}{\sigma(-r,x)} \right\}_{x\in[y_{j-1}+0,y_j-0]} \Bigg),
\end{aligned} \tag{1.6}$$

where

$$\sigma(t,x) := \sum_{k=1}^{m} a_k(t) b_k(x) \quad \text{for} \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1.7}$$

$\tau_1 < \ldots < \tau_n$ and $y_1 < \ldots < y_l$ are all possible discontinuity points on $\mathbb{R}$ for the data functions a_k and b_k $(k = 1, 2, \ldots, m)$, respectively, and $\tau_0 = y_0 = -r$, $\tau_{n+1} = y_{l+1} = r$.

The paper is organized as follows. In Sections 2 and 3 we review the main properties of the function spaces needed and Fredholm criteria for operators $A \in \mathcal{A}_{[SO,PC]}$ in terms of their Fredholm symbols. We also give Duduchava's index formula for the truncated operators in $\mathcal{A}_{PC}$ related to operators (1.1) (Theorem 3.4). In Section 4, we give an index formula for the Wiener-Hopf operators with symbols in $\operatorname{alg}(SO, C(\overline{\mathbb{R}}))$, which follows from [17, Theorem 6.5] (see Theorem 4.2).

Sections 5–8 are related to proving the index formula for the operator A in the case (C1). In Section 5 we make a first reduction (see Theorem 5.2) of the Fredholm operator (1.1) with data functions $a_k \in \operatorname{alg}(SO, PC)$ and $b_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$ to Fredholm operators $A_{t_0}^{\pm}$ and $A_{t_0}^{\diamond}$ of the form (1.1) with the same b_k, where $A_{t_0}^{\pm} \in \mathcal{A}_{[SO,C(\overline{\mathbb{R}})]}$ and their indices are given by Theorem 5.3, and $A_{t_0}^{\diamond}$ have coefficients $a_k^{\diamond} \in PC$.

To obtain the index of the Fredholm operator $A_{t_0}^{\diamond}$ with data functions $a_k^{\diamond} \in PC$ and $b_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$, we first in Section 6 separate discontinuities of $a_k^{\diamond} \in PC$ by reducing the index study of the operator $A_{t_0}^{\diamond}$ to the case of an only one discontinuity point of functions a_k on $\mathbb{R}$. This result is applied in Section 7 to the operator $A_{t_0}^{\diamond}$ with piecewise constant coefficients a_k, which gives the index formula for the operator $A_{t_0}^{\diamond}$ with general $a_k^{\diamond} \in PC$ and $b_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$ by using Theorem 4.2 and approximation and stability arguments for the Fredholm index. In particular, it follows that the indices of the operators $A_{t_0}^{\diamond}$ and $A_{t_0}^{\pm}$ are given by limits of the indices of truncated operators.

Finally, in Section 8, we combine the index formulas from Sections 5 and 7 to obtain first the index of the operator A of the form (1.1) with $a_k \in \operatorname{alg}(SO, PC)$ and $b_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$, and then the index of the operator A with $a_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$ and $b_k \in \operatorname{alg}(SO, PC)$ by making use of the transform $A \mapsto (\mathcal{F} A \mathcal{F}^{-1})^*$.

In Section 9, under conditions (1.2)–(1.3), we make a reduction of the operator A with data functions $a_k, b_k \in \operatorname{alg}(SO, PC)$ to Fredholm operators $A_{\mathbb{R},\infty}$, $A_{\infty,\infty}$, $A_{\infty,\mathbb{R}}$ of the form (1.1), which fall into the case (C1) (see Lemma 9.2). Applying Lemma 9.2 and Theorem 8.1, we establish the index formula indicated in Theorem 1.1.

2. The maximal ideal space of the C^*-algebra $\operatorname{alg}(SO, C(\overline{\mathbb{R}}))$

Let $M(SO)$ denote the maximal ideal space of SO, that is, the space of all characters of slowly oscillating functions on $\mathbb{R}$. Identifying the points $t \in \dot{\mathbb{R}}$ with the evaluation functionals at t, one can define the fiber of $M(SO)$ over $t \in \dot{\mathbb{R}}$ by

$$M_t(SO) = \{\xi \in M(SO) : \ \xi|_{C(\dot{\mathbb{R}})} = t\}.$$

If $t \in \mathbb{R}$, the fiber $M_t(SO)$ consists of the only evaluation functional at t, and thus

$$M(SO) = \bigcup_{t \in \dot{\mathbb{R}}} M_t(SO) = \mathbb{R} \cup M_\infty(SO).$$

The fiber $M_\infty(SO)$ is characterized by the following proposition which can be proved by analogy with [7, Proposition 3.29] (cf. [5, Proposition 4.1]).

Proposition 2.1. [3, Proposition 5] *The fiber $M_\infty(SO)$ has the form $M_\infty(SO) = (\operatorname{clos}_{SO^*} \mathbb{R}) \setminus \mathbb{R}$ where $\operatorname{clos}_{SO^*} \mathbb{R}$ is the weak-star closure of $\mathbb{R}$ in SO^*, the dual space of SO.*

The fiber $M_\infty(SO)$ is related to the partial limits of functions $a \in SO$ at infinity as follows (see [5, Corollary 4.3] and [1, Corollary 3.3]).

Proposition 2.2. *If $\{a_k\}_{k=1}^\infty$ is a countable subset of SO and $\xi \in M_\infty(SO)$, then there exists a sequence $\{g_n\} \subset \mathbb{R}_+$ such that $g_n \to \infty$ as $n \to \infty$, and*

$$\lim_{n\to\infty} a_k(g_n t) = \xi(a_k) \quad \text{for all } \ t \in \mathbb{R} \setminus \{0\} \quad \text{and all } \ k \in \mathbb{N}. \tag{2.1}$$

Conversely, if $\{g_n\} \subset \mathbb{R}_+$ is a sequence such that $g_n \to \infty$ as $n \to \infty$ and the limits $\lim\limits_{n\to\infty} a_k(g_n)$ exist for all k, then there is a $\xi \in M_\infty(SO)$ such that (2.1) *holds.*

Let $M(\mathrm{alg}(SO, PC))$ and $M(\mathrm{alg}(SO, C(\overline{\mathbb{R}})))$ denote the maximal ideal spaces of the unital commutative C^*-algebras $\mathrm{alg}(SO, PC)$ and $\mathrm{alg}(SO, C(\overline{\mathbb{R}}))$, respectively, and let $M_\infty(\mathrm{alg}(SO, PC))$ and $M_\infty(\mathrm{alg}(SO, C(\overline{\mathbb{R}})))$ stand for the fibers of these maximal ideal spaces over the point ∞.

According to [19] we have the following.

Proposition 2.3. *The fibers $M_\infty(\mathrm{alg}(SO, PC))$ and $M_\infty(\mathrm{alg}(SO, C(\overline{\mathbb{R}})))$ coincide, they are homeomorphic to the set $M_\infty(SO) \times \{\pm\infty\}$, and these homeomorphisms are given, respectively, by the restriction maps*

$$\beta \mapsto (\beta|_{SO}, \, \beta|_{PC}) \quad \text{and} \quad \beta \mapsto (\beta|_{SO}, \, \beta|_{C(\overline{\mathbb{R}})}).$$

As $M_\infty(PC) = \{\pm\infty\}$, from Proposition 2.3 it follows that for every $\xi \in M_\infty(SO)$ there is a homomorphism

$$\alpha_\xi : \ \mathrm{alg}(SO, PC) \to PC|_{M_\infty(PC)}, \quad \varphi \mapsto (\alpha_\xi \, \varphi)(\pm\infty) := (\xi, \pm\infty)\varphi. \tag{2.2}$$

The maximal ideal space of $\mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ can be written in the form

$$M(\mathrm{alg}(SO, C(\overline{\mathbb{R}}))) = M_{-\infty}(SO) \cup \mathbb{R} \cup M_{+\infty}(SO),$$

where the fibers of $M(\mathrm{alg}(SO, C(\overline{\mathbb{R}})))$ over $\pm\infty$ are given by $M_{\pm\infty}(SO) = M_\infty(SO)$.

3. Fredholmness of the operator A

In this section we introduce the Fredholm symbol and recall the Fredholm criterion from [1]–[2] for the operator A acting on the space $L^2(\mathbb{R})$ and given by (1.1) with $a_k, b_k \in \mathrm{alg}(SO, PC)$.

With each operator (1.1) we associate the functions $c_\pm, d_\pm : \mathbb{R}\times\mathbb{R}\to\mathbb{C}$ given by

$$c_\pm(t,x) := \sum_{k=1}^{m} a_k(t\pm 0)b_k(x+0), \quad d_\pm(t,x) := \sum_{k=1}^{m} a_k(t\pm 0)b_k(x-0). \tag{3.1}$$

Setting $c(\xi^\pm) := c(\xi\pm 0) := (\xi,\mp\infty)c$ by (2.2) for every $c\in \mathrm{alg}(SO,PC)$ and every $\xi\in M_\infty(SO)$, we extend the functions $c_\pm, d_\pm$ to the whole set $(\mathbb{R}\cup M_\infty(SO))\times(\mathbb{R}\cup M_\infty(SO))$. We also consider the set

$$\mathcal{M} = \mathcal{M}_{\mathbb{R},\infty}\cup\mathcal{M}_{\infty,\mathbb{R}}\cup\mathcal{M}_{\infty,\infty}, \tag{3.2}$$

where

$$\begin{aligned} \mathcal{M}_{\mathbb{R},\infty} &= \mathbb{R}\times M_\infty(SO)\times[0,1],\\ \mathcal{M}_{\infty,\mathbb{R}} &= M_\infty(SO)\times\mathbb{R}\times[0,1],\\ \mathcal{M}_{\infty,\infty} &= M_\infty(SO)\times M_\infty(SO)\times\{0,1\}. \end{aligned} \tag{3.3}$$

With each operator A of the form (1.1) we associate the matrix function $\mathcal{A}:\mathcal{M}\to\mathbb{C}^{2\times 2}$ given by

$$\mathcal{A}(t,x,\mu) = \begin{bmatrix} c_+(t,x)\mu + d_+(t,x)(1-\mu) & (c_+(t,x)-d_+(t,x))\nu(\mu)\\ (c_-(t,x)-d_-(t,x))\nu(\mu) & c_-(t,x)(1-\mu)+d_-(t,x)\mu \end{bmatrix} \tag{3.4}$$

for all $(t,x,\mu)\in\mathcal{M}$, where $\nu(\mu)=\sqrt{\mu(1-\mu)}$. Then for all such operators A we define the map

$$A\mapsto \Phi(A) := \mathcal{A}(\cdot,\cdot,\cdot)\in B(\mathcal{M},\mathbb{C}^{2\times 2}), \tag{3.5}$$

where $\mathcal{A}(\cdot,\cdot,\cdot)$ is given by (3.4), and $B(\mathcal{M},\mathbb{C}^{2\times 2})$ is the C^*-algebra of all bounded $\mathbb{C}^{2\times 2}$-valued functions defined on $\mathcal{M}$ and equipped with the norm $\|\mathcal{A}\| := \sup_{(t,x,\mu)\in\mathcal{M}}\|\mathcal{A}(t,x,\mu)\|_{sp}$ (here $\|\cdot\|_{sp}$ is the spectral matrix norm).

Let $\mathcal{B}(H)$ be the C^*-algebra of all bounded linear operators acting on a Hilbert space H, and let $\mathcal{A}_{[SO,PC]}$ denote the C^*-subalgebra of $\mathcal{B}(L^2(\mathbb{R}))$ generated by all multiplication operators aI with $a\in\mathrm{alg}(SO,PC)$ and by all convolution operators $W^0(b)$ with $b\in\mathrm{alg}(SO,PC)$.

Theorem 3.1. *The map Φ given for operators* (1.1) *by* (3.4)–(3.5) *extends to a C^*-algebra homomorphism*

$$\Phi:\mathcal{A}_{[SO,PC]}\to B(\mathcal{M},\mathbb{C}^{2\times 2}), \quad A\mapsto\mathcal{A}(\cdot,\cdot,\cdot),$$

whose kernel coincides with the ideal $\mathcal{K}$ of all compact operators on $L^2(\mathbb{R})$. An operator $A \in \mathcal{A}_{[SO,PC]}$ is Fredholm on the space $L^2(\mathbb{R})$ if and only if

$$\det \mathcal{A}(t,x,\mu) \neq 0 \quad \textit{for all} \quad (t,x,\mu) \in \mathcal{M}.$$

Proof. This theorem follows from [2, Theorems 6.3 and 6.5], where the Banach algebra $\mathcal{A}_{[SO,PC]} \subset \mathcal{B}(L^p(\mathbb{R}))$ is studied for $p \in (1,\infty)$ and the Banach algebra homomorphism Φ given by (3.1) possesses the property $\mathcal{K} \subset \ker \Phi$. But, actually, $\mathcal{K} = \ker \Phi$ for $p = 2$ (see, e.g., [16, Theorem 3.3]). $\square$

Thus, the map Φ defines the Fredholm symbol $\Phi(A)$ for all operators $A \in \mathcal{A}_{[SO,PC]}$. If $\mu \in \{0,1\}$, then $[\Phi(A)](\cdot,\cdot,\mu)$ is a diagonal matrix function. For the operator A given by (1.1), this matrix function is given by

$$\mathcal{A}(t,x,0) = \begin{bmatrix} d_+(t,x) & 0 \\ 0 & c_-(t,x) \end{bmatrix}, \quad \mathcal{A}(t,x,1) = \begin{bmatrix} c_+(t,x) & 0 \\ 0 & d_-(t,x) \end{bmatrix} \tag{3.6}$$

for all $(t,x) \in \mathfrak{M}$, where

$$\mathfrak{M} := (\mathbb{R} \times M_\infty(SO)) \cup (M_\infty(SO) \times \mathbb{R}) \cup (M_\infty(SO) \times M_\infty(SO)). \tag{3.7}$$

Corollary 3.2. *An operator $A \in \mathcal{A}_{[SO,PC]}$ of the form* (1.1) *is Fredholm on the space $L^2(\mathbb{R})$ if and only if the functions $c_\pm$ and $d_\pm$ given by* (3.1) *are invertible and for all $(t,x,\mu) \in \mathcal{M}$,*

$$\left(\frac{c_+(t,x)}{d_+(t,x)}\, \mu + \frac{c_-(t,x)}{d_-(t,x)}\, (1-\mu) \right) \neq 0$$

or, equivalently,

$$\left(\frac{c_+(t,x)}{c_-(t,x)}\, \mu + \frac{d_+(t,x)}{d_-(t,x)}\, (1-\mu) \right) \neq 0.$$

Proof. By Theorem 3.1 and (3.6), the Fredholmness of A implies that

$$c_\pm(t,x) \neq 0, \quad d_\pm(t,x) \neq 0 \quad \text{for all} \quad (t,x) \in \mathfrak{M}. \tag{3.8}$$

Assuming (3.8), we have

$$\begin{aligned} \det \mathcal{A}(t,x,\mu) &= c_+(t,x) d_-(t,x) \mu + c_-(t,x) d_+(t,x)(1-\mu) \\ &= \left(\frac{c_+(t,x)}{d_+(t,x)}\, \mu + \frac{c_-(t,x)}{d_-(t,x)}\, (1-\mu) \right) (d_-(t,x) d_+(t,x)) \\ &= \left(\frac{c_+(t,x)}{c_-(t,x)}\, \mu + \frac{d_+(t,x)}{d_-(t,x)}\, (1-\mu) \right) (c_-(t,x) d_-(t,x)), \end{aligned} \tag{3.9}$$

which completes the proof by Theorem 3.1. $\square$

Theorem 3.1 in view of the stability of matrix invertibility implies the following assertion.

Lemma 3.3. *If the operator $A \in \mathcal{A}_{[SO,PC]}$ given by* (1.1) *is Fredholm on the space $L^2(\mathbb{R})$ and the functions $a_k, b_k \in \mathrm{alg}(SO, PC)$ admit finite sets of discontinuities, then there is an $r_0 > 0$ such that for all $r > r_0$ the truncated operators $A_r \in \mathcal{A}_{PC}$ given by* (1.5) *are also Fredholm on the space $L^2(\mathbb{R})$.*

The index formula for the operators $A_r \in \mathcal{A}_{PC}$ was obtained by R.V. Duduchava (see, e.g., [9]). Applying [9, Theorem 7.7], we get the following.

Theorem 3.4. *If the operator $A_r \in \mathcal{A}_{PC}$ given by* (1.5) *is Fredholm on the space $L^2(\mathbb{R})$, then*

$$\begin{aligned}
\mathrm{ind}\, A_r = -\frac{1}{2\pi}\Bigg(&\sum_{s=1}^{n+1}\left\{\arg\frac{\sigma(t,-r)}{\sigma(t,r)}\right\}_{t\in[\tau_{s-1}+0,\tau_s-0]}\\
&+\sum_{s=1}^{n}\left\{\arg\left(\frac{\sigma(\tau_s+0,-r)}{\sigma(\tau_s+0,r)}\,\mu+\frac{\sigma(\tau_s-0,-r)}{\sigma(\tau_s-0,r)}(1-\mu)\right)\right\}_{\mu\in[0,1]}\\
&+\sum_{j=1}^{l}\left\{\arg\left(\frac{\sigma(r,y_j+0)}{\sigma(-r,y_j+0)}\,\mu+\frac{\sigma(r,y_j-0)}{\sigma(-r,y_j-0)}(1-\mu)\right)\right\}_{\mu\in[0,1]}\\
&+\sum_{j=1}^{l+1}\left\{\arg\frac{\sigma(r,x)}{\sigma(-r,x)}\right\}_{x\in[y_{j-1}+0,y_j-0]}\Bigg),
\end{aligned}\tag{3.10}$$

where σ is defined in (1.7), *$\tau_1 < \ldots < \tau_n$ and $y_1 < \ldots < y_l$ are all possible discontinuity points on $\mathbb{R}$ for the data functions a_k and b_k $(k = 1, 2, \ldots, m)$, respectively, $r > \max\{-\tau_1, \tau_n, -y_1, y_l\}$ and $\tau_0 = y_0 = -r$, $\tau_{n+1} = y_{l+1} = r$.*

Thus, to prove Theorem 1.1, it remains to establish the crucial formula

$$\mathrm{ind}\, A = \lim_{r\to\infty} \mathrm{ind}\, A_r, \tag{3.11}$$

where the truncated operators A_r are given by (1.5). The proof of (3.11) is sufficiently difficult and is related to a separation of discontinuities of functions $a_k, b_k \in \mathrm{alg}(SO, PC)$ based on some appropriate decompositions. The remaining part of the paper is devoted to the proof of equality (3.11).

4. Wiener-Hopf type operators

We now consider the Wiener-Hopf type operators given by

$$B = \chi_\tau^+ W^0(b) + \chi_\tau^- I \tag{4.1}$$

with $b \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$, where χ_τ^+ and χ_τ^- are the characteristic functions of the intervals $[\tau, +\infty)$ and $(-\infty, \tau]$, respectively, and $\tau \in \mathbb{R}$ is fixed.

Applying Theorem 3.1 or [17, Corollary 6.3], we obtain the following.

Proposition 4.1. *If $b \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$, then the Wiener-Hopf type operator (4.1) is Fredholm on the space $L^2(\mathbb{R})$ if and only if $b(x) \neq 0$ for all $x \in \mathbb{R}$ and $b(\eta^+)\mu + b(\eta^-)(1-\mu) \neq 0$ for all $\eta \in M_\infty(SO)$ and all $\mu \in [0,1]$.*

Proof. By (3.1) and (3.9), the determinant of the Fredholm symbol $\mathcal{B} = \Phi(B)$ of the operator (4.1) is given for $(t, \eta, \mu) \in \mathcal{M}_{\mathbb{R},\infty}$ by

$$\det \mathcal{B}(t, \eta, \mu) = \begin{cases} b(\eta^+)b(\eta^-), & \text{if } \ t > \tau, \\ b(\eta^+)\mu + b(\eta^-)(1-\mu), & \text{if } \ t = \tau, \\ 1 & \text{if } \ t < \tau, \end{cases}$$

while for $(\xi, x, \mu) \in \mathcal{M}_{\infty,\mathbb{R}}$ and $(\xi, \eta, \mu) \in \mathcal{M}_{\infty,\infty}$ we get

$$\det \mathcal{B}(\xi, x, \mu) = b(x), \quad \det \mathcal{B}(\xi, \eta, 0) = b(\eta^+), \quad \det \mathcal{B}(\xi, \eta, 1) = b(\eta^-).$$

Applying now Theorem 3.1, we complete the proof. □

Theorem 4.2. *Let $b \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ and the operator $B = \chi_\tau^+ W^0(b) + \chi_\tau^- I$ be Fredholm on the space $L^2(\mathbb{R})$. Then for every sufficiently large $r > 0$ the truncated operator $B_r = \chi_\tau^+ W^0(b_r) + \chi_\tau^- I$, with b_r given by (1.4), is also Fredholm on the space $L^2(\mathbb{R})$, and*

$$\begin{aligned} \mathrm{ind}\, B &= \lim_{r\to\infty} \mathrm{ind}\, B_r \\ &= -\frac{1}{2\pi} \lim_{r\to\infty} \Big(\{\arg b(x)\}_{x\in[-r,r]} + \{\arg[b(-r)\mu + b(r)(1-\mu)]\}_{\mu\in[0,1]} \Big). \end{aligned} \tag{4.2}$$

Proof. By Proposition 4.1, the function b is invertible in $\mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ and

$$b(\eta^+)\mu + b(\eta^-)(1-\mu) \neq 0 \quad \text{for all } \ \eta \in M_\infty(SO) \ \text{ and all } \ \mu \in [0,1].$$

Then there exists a point $x_0 > |\tau|$ such that

$$b(-x)\mu + b(x)(1-\mu) \neq 0 \quad \text{for all} \quad |x| \geq x_0 \quad \text{and all} \quad \mu \in [0,1].$$

Hence, by Proposition 4.1, the truncated operators B_r are Fredholm on $L^2(\mathbb{R})$ for all $r > x_0$. Further, from [17, Theorem 6.5] it follows that

$$\operatorname{ind} B = \lim_{r\to\infty} \operatorname{ind} B_r. \tag{4.3}$$

It remains to apply [6, Theorem 2.20], which gives

$$\operatorname{ind} B_r = -\frac{1}{2\pi}\Big(\{\arg b(x)\}_{x\in[-r,r]} + \{\arg[b(-r)\mu + b(r)(1-\mu)]\}_{\mu\in[0,1]}\Big). \tag{4.4}$$

Combining (4.3) and (4.4), we obtain (4.2). □

5. First decomposition

We now proceed to obtain the index formula for the Fredholm operator (1.1) on the space $L^2(\mathbb{R})$ in the case (C1), that is, assuming that all a_k or all b_k are in $\operatorname{alg}(SO, C(\overline{\mathbb{R}}))$. We start with reviewing the index formula when all a_k and b_k are in $\operatorname{alg}(SO, C(\overline{\mathbb{R}}))$, and check that (3.11) holds in this case.

Theorem 5.1. *If $a_k, b_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$ for all $k = 1, 2, \dots, m$, then the operator A given by* (1.1) *is Fredholm on the space $L^2(\mathbb{R})$ if and only if*

$$\liminf_{t^2+x^2\to\infty} |\sigma(t,x)| > 0, \tag{5.1}$$

where the function σ is defined by (1.7)*. The index of the Fredholm operator A is calculated by the formula*

$$\operatorname{ind} A = -\lim_{r\to\infty} \frac{1}{2\pi}\{\arg\, \sigma(t,x)\}_{(t,x)\in\partial\Pi_r} = \lim_{r\to\infty} \operatorname{ind} A_r, \tag{5.2}$$

where $\{\arg\, \sigma(t,x)\}_{(t,x)\in\partial\Pi_r}$ denotes the increment of $\arg\, \sigma(t,x)$ when the point (t,x) traces counter-clockwise the boundary $\partial\Pi_r$ of the square $\Pi_r = \{(t,x) : |t| < r,\ |x| < r\}$, and A_r are truncated operators for the operator A.

Proof. Since all functions a_k and b_k are in $\operatorname{alg}(SO, C(\overline{\mathbb{R}}))$, we conclude from [21, Corollary 3.7] (also see Theorem 3.1) that condition (5.1) is equivalent to the Fredholmness of the operator A on the space $L^2(\mathbb{R})$. Moreover, condition (5.1) implies that for all sufficiently large $r > 0$ the function $\sigma(t,x) = \sum_{k=1}^m a_k(t)b_k(x)$ is separated from zero on the set $\mathbb{R}^2 \setminus [-r,r]^2$. The index formula follows from [21, Corollary 3.7] (see also [18, Theorem 7.2]). □

For the remainder of this section, we assume that A satisfies condition (C1), with all $b_k \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$:

$$A = \sum_{k=1}^{m} a_k W^0(b_k), \quad a_k \in \mathrm{alg}(SO, PC), \quad b_k \in \mathrm{alg}(SO, C(\overline{\mathbb{R}})) \;, \tag{5.3}$$

for all k, and that $\tau_1 < \ldots < \tau_n$ are all possible discontinuity points on $\mathbb{R}$ for the functions a_k $(k = 1, 2, \ldots, m)$.

It follows from Lemma 3.3 and Corollary 3.2 applied to the Fredholm operator A given by (5.3) that there is a $t_0 > \max\{|\tau_1|, \ldots, |\tau_n|\}$ such that for all $t \in \mathbb{R} \setminus (-t_0, t_0)$ the functions $x \mapsto \sum_{k=1}^{m} a_k(t) b_k(x)$ are invertible in the C^*-algebra $\mathrm{alg}(SO, C(\overline{\mathbb{R}}))$. Then the operators

$$A_{t_0} := W^0\Big(\sum_{k=1}^{m} a_k(t_0) b_k\Big), \quad A_{-t_0} := W^0\Big(\sum_{k=1}^{m} a_k(-t_0) b_k\Big)$$

are invertible on the space $L^2(\mathbb{R})$.

Given $\tau \in \mathbb{R}$, let χ_τ^+ and χ_τ^- be the characteristic functions of $[\tau, +\infty)$ and $(-\infty, \tau]$, respectively. We define the operators

$$A_{t_0}^+ := \chi_{t_0}^+ A + \chi_{t_0}^- A_{t_0}, \quad A_{t_0}^- := \chi_{-t_0}^- A + \chi_{-t_0}^+ A_{-t_0}, \tag{5.4}$$

$$A_{t_0}^\diamond := \chi_{-t_0}^- A_{-t_0} + \chi_{-t_0}^+ \chi_{t_0}^- A + \chi_{t_0}^+ A_{t_0}. \tag{5.5}$$

Then

$$A_{t_0}^\pm = \sum_{k=1}^{m} a_k^\pm W^0(b_k), \quad A_{t_0}^\diamond = \sum_{k=1}^{m} a_k^\diamond W^0(b_k),$$

where the functions $a_k^\pm \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ and $a_k^\diamond \in PC$ are given by

$$a_k^+(t) = \begin{cases} a_k(t), & \text{if } t \geq t_0, \\ a_k(t_0), & \text{if } t < t_0, \end{cases} \quad a_k^-(t) = \begin{cases} a_k(t), & \text{if } t \leq -t_0, \\ a_k(-t_0), & \text{if } t > -t_0, \end{cases}$$

$$a_k^\diamond(t) = \begin{cases} a_k(t_0), & \text{if } t \geq t_0, \\ a_k(t), & \text{if } |t| < t_0, \\ a_k(-t_0), & \text{if } t \leq -t_0, \end{cases} \tag{5.6}$$

that is, the functions $a_k^\pm$ are continuous on $\mathbb{R}$ with a limit at $\mp\infty$ and the slowly oscillating behavior of a_k at $\pm\infty$, respectively, while the functions $a_k^\diamond$ have one-sided limits at $\pm\infty$ and are continuous on the set $\mathbb{R} \setminus \{\tau_1, \ldots, \tau_n\}$. In what follows, $A \simeq B$ means that the operator $A - B$ is compact.

Theorem 5.2. *If the operator A given by* (5.3) *is Fredholm on the space $L^2(\mathbb{R})$, then the operators $A_{t_0}^{\pm}$ and $A_{t_0}^{\diamond}$ given by* (5.4) *and* (5.5) *are also Fredholm on the space $L^2(\mathbb{R})$, and*

$$\operatorname{ind} A = \operatorname{ind} A_{t_0}^{+} + \operatorname{ind} A_{t_0}^{-} + \operatorname{ind} A_{t_0}^{\diamond}. \tag{5.7}$$

Proof. Along with $A_{t_0}^{+}$, we define the operator $\widehat{A}_{t_0}^{+} = \sum_{k=1}^{m} \widehat{a}_k^{+} W^0(b_k)$, where $\widehat{a}_k^{+}(t) = a_k(t)$ for $t \le t_0$ and $\widehat{a}_k^{+}(t) = a_k(t_0)$ for $t > t_0$. We start with showing that the operators $A_{t_0}^{+}$ and $\widehat{A}_{t_0}^{+}$ are Fredholm and

$$\operatorname{ind} A = \operatorname{ind} A_{t_0}^{+} + \operatorname{ind} \widehat{A}_{t_0}^{+}. \tag{5.8}$$

To this end, we consider the operator

$$B_{t_0} := A A_{t_0}^{-1} = \sum_{k=1}^{m} a_k W^0\left(b_k \left(\sum_{s=1}^{m} a_s(t_0) b_s \right)^{-1} \right). \tag{5.9}$$

It is Fredholm on the space $L^2(\mathbb{R})$, and, using the symbol map Φ defined in Theorem 3.1, we see that the Fredholm symbol $\mathcal{B}_{t_0} = \Phi(B_{t_0})$ is such that

$$\mathcal{B}_{t_0}(t_0, \eta, \mu) = I_2 \quad \text{for all} \quad \eta \in M_\infty(SO) \quad \text{and all} \quad \mu \in [0,1], \tag{5.10}$$

where I_2 is the identity 2×2 matrix. Let us show that

$$\begin{aligned} B_{t_0} &\simeq (\chi_{t_0}^{+} B_{t_0} + \chi_{t_0}^{-} I)(\chi_{t_0}^{-} B_{t_0} + \chi_{t_0}^{+} I) \\ &\simeq (\chi_{t_0}^{-} B_{t_0} + \chi_{t_0}^{+} I)(\chi_{t_0}^{+} B_{t_0} + \chi_{t_0}^{-} I). \end{aligned} \tag{5.11}$$

Indeed,

$$(\chi_{t_0}^{+} B_{t_0} + \chi_{t_0}^{-} I)(\chi_{t_0}^{-} B_{t_0} + \chi_{t_0}^{+} I) = \chi_{t_0}^{+} B_{t_0} \chi_{t_0}^{-} B_{t_0} + \chi_{t_0}^{+} B_{t_0} \chi_{t_0}^{+} I + \chi_{t_0}^{-} B_{t_0} \simeq B_{t_0}$$

if $\chi_{t_0}^{+} B_{t_0} \chi_{t_0}^{-} I \simeq 0$. To prove the compactness of $\chi_{t_0}^{+} B_{t_0} \chi_{t_0}^{-} I$, it is sufficient to check its Fredholm symbol at the points (t_0, η, μ) for $(\eta, \mu) \in M_\infty(SO) \times [0,1]$ and at the points $(\xi, x, \mu) \in \mathcal{M}_{\infty,\mathbb{R}}$ and $(\xi, \eta, \mu) \in \mathcal{M}_{\infty,\infty}$. We then obtain

$$\begin{aligned} \Phi[\chi_{t_0}^{+} B_{t_0} \chi_{t_0}^{-} I](t_0, \eta, \mu) &= \operatorname{diag}\{1,0\}\, \mathcal{B}_{t_0}(t_0, \eta, \mu) \operatorname{diag}\{0,1\} = 0_{2\times 2}, \\ \Phi[\chi_{t_0}^{+} B_{t_0} \chi_{t_0}^{-} I](\xi, x, \mu) &= \operatorname{diag}\{0,1\}\, \mathcal{B}_{t_0}(\xi, x, \mu) \operatorname{diag}\{1,0\} = 0_{2\times 2}, \\ \Phi[\chi_{t_0}^{+} B_{t_0} \chi_{t_0}^{-} I](\xi, \eta, \mu) &= \operatorname{diag}\{0,1\}\, \mathcal{B}_{t_0}(\xi, \eta, \mu) \operatorname{diag}\{1,0\} = 0_{2\times 2}, \end{aligned}$$

because (5.10) holds and

$$\begin{aligned}
\mathcal{B}_{t_0}(\xi,x,\mu) &= \operatorname{diag}\left\{\frac{\sum_{k=1}^m a_k(\xi^+)b_k(x)}{\sum_{k=1}^m a_k(t_0)b_k(x)}, \frac{\sum_{k=1}^m a_k(\xi^-)b_k(x)}{\sum_{k=1}^m a_k(t_0)b_k(x)}\right\},\\
\mathcal{B}_{t_0}(\xi,\eta,0) &= \operatorname{diag}\left\{\frac{\sum_{k=1}^m a_k(\xi^+)b_k(\eta^-)}{\sum_{k=1}^m a_k(t_0)b_k(\eta^-)}, \frac{\sum_{k=1}^m a_k(\xi^-)b_k(\eta^+)}{\sum_{k=1}^m a_k(t_0)b_k(\eta^+)}\right\},\\
\mathcal{B}_{t_0}(\xi,\eta,1) &= \operatorname{diag}\left\{\frac{\sum_{k=1}^m a_k(\xi^+)b_k(\eta^+)}{\sum_{k=1}^m a_k(t_0)b_k(\eta^+)}, \frac{\sum_{k=1}^m a_k(\xi^-)b_k(\eta^-)}{\sum_{k=1}^m a_k(t_0)b_k(\eta^-)}\right\}.
\end{aligned}$$

Since the symbol of $\chi_{t_0}^+ B_{t_0}\chi_{t_0}^- I$ is the zero matrix for all $(t,x,\mu)\in\mathcal{M}$, the operator $\chi_{t_0}^+ B_{t_0}\chi_{t_0}^- I$ is compact in view of Theorem 3.1, which gives the first relation in (5.11). The second relation in (5.11) is obtained analogously.

It follows from (5.11) and the Fredholmness of B_{t_0} (which is equivalent to the Fredholmness of A) that both the operators $\chi_{t_0}^+ B_{t_0}+\chi_{t_0}^- I$ and $\chi_{t_0}^- B_{t_0}+\chi_{t_0}^+ I$ are Fredholm. Hence the operators

$$\begin{aligned}
A_{t_0}^+ &= \chi_{t_0}^+ A+\chi_{t_0}^- A_{t_0} = (\chi_{t_0}^+ B_{t_0}+\chi_{t_0}^- I)A_{t_0},\\
\widehat{A}_{t_0}^+ &= \chi_{t_0}^- A+\chi_{t_0}^+ A_{t_0} = (\chi_{t_0}^- B_{t_0}+\chi_{t_0}^+ I)A_{t_0}
\end{aligned}\tag{5.12}$$

are Fredholm as well. Moreover, from (5.9), (5.11) and (5.12) it follows that

$$\operatorname{ind} A = \operatorname{ind} B_{t_0} = \operatorname{ind} A_{t_0}^+ + \operatorname{ind}\widehat{A}_{t_0}^+,$$

which gives (5.8).

Replacing A by $\widehat{A}_{t_0}^+$, we proceed similarly for the point $-t_0$. Let

$$B_{-t_0} := \widehat{A}_{t_0}^+ A_{-t_0}^{-1}.\tag{5.13}$$

Then $\mathcal{B}_{-t_0}(-t_0,\eta,\mu)=I_2$ for all $\eta\in M_\infty(SO)$ and all $\mu\in[0,1]$. In the same way as above, we have

$$\begin{aligned}
B_{-t_0} &\simeq (\chi_{-t_0}^+ B_{-t_0}+\chi_{-t_0}^- I)(\chi_{-t_0}^- B_{-t_0}+\chi_{-t_0}^+ I)\\
&\simeq (\chi_{-t_0}^- B_{-t_0}+\chi_{-t_0}^+ I)(\chi_{-t_0}^+ B_{-t_0}+\chi_{-t_0}^- I).
\end{aligned}\tag{5.14}$$

Since the operator B_{-t_0} is Fredholm along with $\widehat{A}_{t_0}^+$ and since $\chi_{-t_0}^- A = \chi_{-t_0}^-\widehat{A}_{t_0}^+$, we infer from (5.14) the Fredholmness of the operators

$$\begin{aligned}
A_{t_0}^- &= \chi_{-t_0}^- A+\chi_{-t_0}^+ A_{-t_0} = (\chi_{-t_0}^- B_{-t_0}+\chi_{-t_0}^+ I)A_{-t_0},\\
A_{t_0}^\diamond &= \chi_{-t_0}^+\chi_{t_0}^- A+\chi_{t_0}^+ A_{t_0}+\chi_{-t_0}^- A_{-t_0}\\
&= \chi_{-t_0}^+\widehat{A}_{t_0}^+ +\chi_{-t_0}^- A_{-t_0} = (\chi_{-t_0}^+ B_{-t_0}+\chi_{-t_0}^- I)A_{-t_0}.
\end{aligned}$$

Hence we deduce from (5.13) and (5.14) that

$$\operatorname{ind} \widehat{A}^{+}_{t_0} = \operatorname{ind} A^{-}_{t_0} + \operatorname{ind} A^{\diamond}_{t_0}. \tag{5.15}$$

Finally, (5.8) and (5.15) imply (5.7). □

Since the data functions $a^{\pm}_k, b_k$ of the operators $A^{\pm}_{t_0}$ are in $\operatorname{alg}(SO, C(\overline{\mathbb{R}}))$, the indices of the Fredholm operators $A^{\pm}_{t_0}$ are computed as in Theorem 5.1. Put

$$\sigma_{\pm}(t,x) := \sum_{k=1}^{m} a^{\pm}_k(t) b_k(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{5.16}$$

Theorem 5.3. *The indices of the Fredholm operators $A^{\pm}_{t_0}$ are calculated by*

$$\operatorname{ind} A^{\pm}_{t_0} = -\lim_{r\to\infty} \frac{1}{2\pi} \{\arg\, \sigma_{\pm}(t,x)\}_{(t,x)\in\partial\Pi_r} = \lim_{r\to\infty} \operatorname{ind}\, [A^{\pm}_{t_0}]_r, \tag{5.17}$$

where $\{\arg\, \sigma_{\pm}(t,x)\}_{(t,x)\in\partial\Pi_r}$ *denotes the increment of* $\arg\, \sigma_{\pm}(t,x)$ *when the point* (t,x) *traces counter-clockwise the boundary* $\partial\Pi_r$ *of the square* $\Pi_r = \{(t,x) : |t| < r,\ |x| < r\}$*, and* $[A^{\pm}_{t_0}]_r$ *are truncated operators for* $A^{\pm}_{t_0}$*.*

6. Separation of discontinuities

According to (5.7) and Theorem 5.3, to compute $\operatorname{ind} A$ under conditions (5.3), it remains to compute the index of the Fredholm operator $A^{\diamond}_{t_0}$ acting on the space $L^2(\mathbb{R})$ and given by

$$A^{\diamond}_{t_0} = \chi^{-}_{-t_0} A_{-t_0} + \chi^{+}_{-t_0}\chi^{-}_{t_0} A + \chi^{+}_{t_0} A_{t_0} = \sum_{k=1}^{m} a^{\diamond}_k W^0(b_k), \tag{6.1}$$

where $a^{\diamond}_k \in PC$ are given by (5.6), $b_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$, and $\tau_1 < \ldots < \tau_n$ are possible discontinuity points in $\mathbb{R}$ of the functions $a^{\diamond}_k$. Thus, $a^{\diamond}_k = a_{k,r}$ with $r = t_0$. By (3.1), with $A^{\diamond}_{t_0}$ we associate the functions

$$c^{\diamond}_{\pm}(t,x) = \sum_{k=1}^{m} a^{\diamond}_k(t \pm 0) b_k(x+0), \quad d^{\diamond}_{\pm}(t,x) = \sum_{k=1}^{m} a^{\diamond}_k(t \pm 0) b_k(x-0) \tag{6.2}$$

for all $(t,x) \in \mathfrak{M}$, where $\mathfrak{M}$ is given by (3.7).

Since the operator $A^{\diamond}_{t_0}$ is Fredholm on the space $L^2(\mathbb{R})$, it follows from Corollary 3.2 that there is an $r > 0$ such that the functions $x \mapsto c^{\diamond}_{-}(\tau_s, x)$ in

$\mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ are separated from zero for all $|x| \geq r$ and all $s = 1, 2, \ldots, n$. Then, for every $k = 1, 2, \ldots, m$ and every $s = 1, 2, \ldots, n$ there exist functions $\widehat{b}_{k,s} \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ such that $\widehat{b}_{k,s}(x) = b_k(x)$ for all $|x| \geq r$ and the functions

$$\widehat{c}_s := \sum_{k=1}^{m} a_k(\tau_s - 0)\widehat{b}_{k,s} \quad (s = 1, 2, \ldots, n)$$

are invertible in $\mathrm{alg}(SO, C(\overline{\mathbb{R}}))$. Indeed, fix s and $a_k(\tau_s - 0) \neq 0$, and for $x \in [-r, r]$ put

$$\widehat{c}_s(x) = \left(\sum_{k=1}^{m} a_k(\tau_s - 0)b_k(-r)\right)^{(r-x)/(2r)} \left(\sum_{k=1}^{m} a_k(\tau_s - 0)b_k(r)\right)^{(r+x)/(2r)},$$

while $\widehat{c}_s(x) := \sum_{k=1}^{m} a_k(\tau_s - 0)b_k(x)$ for all other $x \in \mathbb{R}$. Then $\widehat{c}_s$ is invertible in the C^*-algebra $\mathrm{alg}(SO, C(\overline{\mathbb{R}}))$. Choosing now arbitrary restrictions on $[-r, r]$ of functions $\widehat{b}_{j,s} \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ for $j \in \{1, 2, \ldots, m\} \setminus \{k\}$, it is sufficient to take $\widehat{b}_{k,s} = \big(\widehat{c}_s - \sum_{j \neq k} a_j(\tau_s - 0)\widehat{b}_{j,s}\big)/a_k(\tau_s - 0)$.

Hence the operators

$$\widetilde{A}_{\tau_s}^{-} := \sum_{k=1}^{m} a_k(\tau_s - 0)W^0(\widehat{b}_{k,s}) = W^0(\widehat{c}_s) \quad (s = 1, 2, \ldots, n) \tag{6.3}$$

are invertible on the space $L^2(\mathbb{R})$, and the operator

$$D_{\tau_n} = A_{t_0}^{\diamond}(\widetilde{A}_{\tau_n}^{-})^{-1} \tag{6.4}$$

is Fredholm on this space.

Lemma 6.1. *The operator $\chi_{\tau_n}^{-} D_{\tau_n} \chi_{\tau_n}^{+} I$ is compact on the space $L^2(\mathbb{R})$, and*

$$D_{\tau_n} \simeq (\chi_{\tau_n}^{-} D_{\tau_n} + \chi_{\tau_n}^{+} I)(\chi_{\tau_n}^{+} D_{\tau_n} + \chi_{\tau_n}^{-} I). \tag{6.5}$$

Proof. Applying (6.2), we deduce from (3.4) that

$$\mathcal{A}_{t_0}^{\diamond}(t, x, \mu) = \begin{bmatrix} c_{+}^{\diamond}(t, x)\mu + d_{+}^{\diamond}(t, x)(1 - \mu) & (c_{+}^{\diamond}(t, x) - d_{+}^{\diamond}(t, x))\nu(\mu) \\ (c_{-}^{\diamond}(t, x) - d_{-}^{\diamond}(t, x))\nu(\mu) & c_{-}^{\diamond}(t, x)(1 - \mu) + d_{-}^{\diamond}(t, x)\mu \end{bmatrix} \tag{6.6}$$

for all $(t,x,\mu) \in \mathcal{M}$. On the other hand, since $\widehat{c}_n(x+0) = c_-^\diamond(\tau_n,x)$ and $\widehat{c}_n(x-0) = d_-^\diamond(\tau_n,x)$ for all $x \in M_\infty(SO)$, we infer for $(t,x,\mu) \in \mathcal{M}_{\mathbb{R},\infty}$ that

$$\begin{aligned} &\widetilde{\mathcal{A}}_{\tau_n}^-(t,x,\mu) \\ &\quad = \begin{bmatrix} c_-^\diamond(\tau_n,x)\mu + d_-^\diamond(\tau_n,x)(1-\mu) & (c_-^\diamond(\tau_n,x) - d_-^\diamond(\tau_n,x))\nu(\mu) \\ (c_-^\diamond(\tau_n,x) - d_-^\diamond(\tau_n,x))\nu(\mu) & c_-^\diamond(\tau_n,x)(1-\mu) + d_-^\diamond(\tau_n,x)\mu \end{bmatrix}, \\ &\big[\widetilde{\mathcal{A}}_{\tau_n}^-(t,x,\mu)\big]^{-1} = \big[\det \widetilde{\mathcal{A}}_{\tau_n}^-(t,x,\mu)\big]^{-1} \\ &\quad \times \begin{bmatrix} c_-^\diamond(\tau_n,x)(1-\mu) + d_-^\diamond(\tau_n,x)\mu & -(c_-^\diamond(\tau_n,x) - d_-^\diamond(\tau_n,x))\nu(\mu) \\ -(c_-^\diamond(\tau_n,x) - d_-^\diamond(\tau_n,x))\nu(\mu) & c_-^\diamond(\tau_n,x)\mu + d_-^\diamond(\tau_n,x)(1-\mu) \end{bmatrix}. \end{aligned} \tag{6.7}$$

To prove the compactness of the operator $\chi_{\tau_n}^- D_{\tau_n} \chi_{\tau_n}^+ I$, where D_{τ_n} is given by (6.4), it suffices to check that its Fredholm symbol vanishes at the points (τ_n, x, μ) for $(x,\mu) \in M_\infty(SO) \times [0,1]$ and at the points $(t,x,\mu) \in \mathcal{M}_{\infty,\mathbb{R}} \cup \mathcal{M}_{\infty,\infty}$. Writing $\mathcal{D}_{\tau_n} := \Phi(D_{\tau_n})$, we get

$$\Phi\big[\chi_{\tau_n}^- D_{\tau_n} \chi_{\tau_n}^+ I\big](\tau_n,x,\mu) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathcal{D}_{\tau_n}(\tau_n,x,\mu) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ d_{\tau_n}(\tau_n,x,\mu) & 0 \end{bmatrix},$$

where, by (6.6) and (6.7), $d_{\tau_n}(\tau_n,x,\mu) = 0$. Thus,

$$\Phi\big[\chi_{\tau_n}^- D_{\tau_n} \chi_{\tau_n}^+ I\big](t,x,\mu) = 0_{2\times 2} \ \text{ for all } \ (t,x,\mu) \in \mathcal{M}_{\mathbb{R},\infty}.$$

On the other hand, since the matrices $\mathcal{A}_{t_0}^\diamond(t,x,\mu)$ and $\widetilde{\mathcal{A}}_{\tau_n}^-(t,x,\mu)$ are diagonal for all $(t,x,\mu) \in \mathcal{M}_{\infty,\mathbb{R}} \cup \mathcal{M}_{\infty,\infty}$, we again conclude that

$$\Phi\big[\chi_{\tau_n}^- D_{\tau_n} \chi_{\tau_n}^+ I\big](t,x,\mu) = 0_{2\times 2} \ \text{ for all } \ (t,x,\mu) \in \mathcal{M}_{\infty,\mathbb{R}} \cup \mathcal{M}_{\infty,\infty}.$$

Hence, the Fredholm symbol of the operator $\chi_{\tau_n}^- D_{\tau_n} \chi_{\tau_n}^+ I$ is the zero matrix function on $\mathcal{M}$, and therefore, by Theorem 3.1, this operator is compact on the space $L^2(\mathbb{R})$, which immediately implies (6.5). $\square$

Theorem 6.2. *If the operator $A_{t_0}^\diamond$ is Fredholm on the space $L^2(\mathbb{R})$, then the operators*

$$\widehat{A}_{\tau_n}^- := \chi_{\tau_n}^- A_{t_0}^\diamond + \chi_{\tau_n}^+ \widetilde{A}_{\tau_n}^-, \quad \widetilde{A}_{\tau_n}^+ := \chi_{\tau_n}^+ A_{t_0}^\diamond + \chi_{\tau_n}^- \widetilde{A}_{\tau_n}^-, \tag{6.8}$$

where $\widetilde{A}_{\tau_n}^-$ is given by (6.3), also are Fredholm on this space, and

$$\operatorname{ind} A_{t_0}^\diamond = \operatorname{ind} \widehat{A}_{\tau_n}^- + \operatorname{ind} \widetilde{A}_{\tau_n}^+. \tag{6.9}$$

Proof. We infer from (6.2), (3.4) and (3.9) that

$$\widetilde{\mathcal{A}}^{+}_{\tau_n}(t,x,\mu) = \mathcal{A}^{\diamond}_{t_0}(t,x,\mu) \quad\text{if}\quad (t,x,\mu)\in[\tau_n,+\infty)\times M_\infty(SO)\times[0,1],$$
$$\det\widetilde{\mathcal{A}}^{+}_{\tau_n}(t,x,\mu) = c^{\diamond}_{-}(\tau_n,x)d^{\diamond}_{-}(\tau_n,x) \quad\text{if}\quad (t,x,\mu)\in(-\infty,\tau_n)\times M_\infty(SO)\times[0,1],$$
$$\det\widetilde{\mathcal{A}}^{+}_{\tau_n}(t,x,\mu) = \widehat{c}_n(x)d^{\diamond}_{-}(t_0,x) \quad\text{if}\quad (t,x,\mu)\in M_\infty(SO)\times\mathbb{R}\times[0,1],$$

and

$$\det\widetilde{\mathcal{A}}^{+}_{\tau_n}(t,x,0) = c^{\diamond}_{-}(t_0,x)d^{\diamond}_{-}(\tau_n,x), \quad \det\widetilde{\mathcal{A}}^{+}_{\tau_n}(t,x,1) = c^{\diamond}_{-}(\tau_n,x)d^{\diamond}_{-}(t_0,x)$$

for $(t,x)\in M_\infty(SO)\times M_\infty(SO)$. Then from Corollary 3.2 and Theorem 3.1 it follows that the operator $\widetilde{A}^{+}_{\tau_n}$ is Fredholm on the space $L^2(\mathbb{R})$ along with $A^{\diamond}_{t_0}$, which implies the Fredholmness of the operator $\chi^{+}_{\tau_n}D_{\tau_n}+\chi^{-}_{\tau_n}I$ in view of the equality

$$\widetilde{A}^{+}_{\tau_n} = \big(\chi^{+}_{\tau_n}D_{\tau_n}+\chi^{-}_{\tau_n}I\big)\widetilde{A}^{-}_{\tau_n} \tag{6.10}$$

(see (6.4) and (6.8)) and the invertibility of the operator $\widetilde{A}^{-}_{\tau_n}$. Since the operators D_{τ_n} and $\chi^{+}_{\tau_n}D_{\tau_n}+\chi^{-}_{\tau_n}I$ are Fredholm on $L^2(\mathbb{R})$, so is the operator $\chi^{-}_{\tau_n}D_{\tau_n}+\chi^{+}_{\tau_n}I$ due to (6.5), which according to the equality

$$\widehat{A}^{-}_{\tau_n} = (\chi^{-}_{\tau_n}D_{\tau_n}+\chi^{+}_{\tau_n}I)\widetilde{A}^{-}_{\tau_n} \tag{6.11}$$

(see (6.4) and (6.8)) and the invertibility of the operator $\widetilde{A}^{-}_{\tau_n}$ implies the Fredholmness of the operator $\widehat{A}^{-}_{\tau_n}$ on the space $L^2(\mathbb{R})$.

Finally, from (6.4) and (6.5) it follows that

$$\operatorname{ind}A^{\diamond}_{t_0} = \operatorname{ind}D_{\tau_n} = \operatorname{ind}(\chi^{-}_{\tau_n}D_{\tau_n}+\chi^{+}_{\tau_n}I)+\operatorname{ind}(\chi^{+}_{\tau_n}D_{\tau_n}+\chi^{-}_{\tau_n}I),$$

which gives (6.9) because

$$\operatorname{ind}(\chi^{-}_{\tau_n}D_{\tau_n}+\chi^{+}_{\tau_n}I) = \operatorname{ind}\widehat{A}^{-}_{\tau_n}, \quad \operatorname{ind}(\chi^{+}_{\tau_n}D_{\tau_n}+\chi^{-}_{\tau_n}I) = \operatorname{ind}\widetilde{A}^{+}_{\tau_n}$$

according to (6.10) and (6.11). □

Taking $s=n-1,n-2,\dots,1$ for $n\ge 2$, we recursively define the operators

$$\widehat{A}^{-}_{\tau_s} := \chi^{-}_{\tau_s}\widehat{A}^{-}_{\tau_{s+1}}+\chi^{+}_{\tau_s}\widetilde{A}^{-}_{\tau_s}, \quad \widehat{A}^{+}_{\tau_s} := \chi^{+}_{\tau_s}\widehat{A}^{-}_{\tau_{s+1}}+\chi^{-}_{\tau_s}\widetilde{A}^{-}_{\tau_s}. \tag{6.12}$$

It is easily seen that

$$\widehat{A}^{-}_{\tau_s} = \chi^{-}_{\tau_s}A^{\diamond}_{t_0}+\chi^{+}_{\tau_s}\widetilde{A}^{-}_{\tau_s} \quad\text{for all}\quad s=1,2,\dots,n. \tag{6.13}$$

Corollary 6.3. *If the operator $A^{\diamond}_{t_0}$ given by* (6.1) *is Fredholm on the space $L^2(\mathbb{R})$ and $n \ge 1$, then the operators* (6.12) *for all $s = 1, 2, \dots, n-1$ and the operators* (6.8) *also are Fredholm on the space $L^2(\mathbb{R})$, and*

$$\operatorname{ind} A^{\diamond}_{t_0} = \operatorname{ind} \widehat{A}^{-}_{\tau_1} + \sum_{s=1}^{n} \operatorname{ind} \widetilde{A}^{+}_{\tau_s}. \tag{6.14}$$

Proof. For $n = 1$, the corollary follows from Theorem 6.2. Let now $n \ge 2$. By analogy with Theorem 6.2, one can prove that for every $s = n-1, n-2, \dots, 1$ the Fredholmness of the operator $\widehat{A}^{-}_{\tau_{s+1}}$ given by (6.13) and the invertibility of the operator $\widetilde{A}^{-}_{\tau_s}$ on the space $L^2(\mathbb{R})$ implies the Fredholmness of both the operators $\widetilde{A}^{+}_{\tau_s}$ and $\widehat{A}^{-}_{\tau_s}$ on this space, and also the fulfilment of the recursive relations

$$\operatorname{ind} \widehat{A}^{-}_{\tau_{s+1}} = \operatorname{ind} \widehat{A}^{-}_{\tau_s} + \operatorname{ind} \widetilde{A}^{+}_{\tau_s}. \tag{6.15}$$

Finally, applying (6.9) and then (6.15) for all $s = n-1, n-2, \dots, 1$, we arrive at the equality (6.14). $\square$

It is easily seen that

$$\widetilde{A}^{+}_{\tau_n} = \sum_{k=1}^{m} \big[\chi^{-}_{\tau_n} a^{\diamond}_k(\tau_n - 0) W^0(\widehat{b}_{k,n}) + \chi^{+}_{\tau_n} a^{\diamond}_k W^0(b_k)\big], \tag{6.16}$$

where, for all $k = 1, 2, \dots, m$, the functions $\chi^{-}_{\tau_n} a^{\diamond}_k(\tau_n - 0) + \chi^{+}_{\tau_n} a^{\diamond}_k$ can have a discontinuity on $\mathbb{R}$ only at the point τ_n, while, for every $s = n-1, n-2, \dots, 1$,

$$\begin{aligned} \widetilde{A}^{+}_{\tau_s} = \sum_{k=1}^{m} \Big(& \chi^{-}_{\tau_s} a^{\diamond}_k(\tau_s - 0) W^0(\widehat{b}_{k,s}) \\ & + \chi^{+}_{\tau_s} \big[\chi^{-}_{\tau_{s+1}} a^{\diamond}_k W^0(b_k) + \chi^{+}_{\tau_{s+1}} a^{\diamond}_k(\tau_{s+1} - 0) W^0(\widehat{b}_{k,s+1})\big]\Big). \end{aligned} \tag{6.17}$$

It follows from [9, Lemma 7.1] that the operators $\chi^{+}_{\tau_s}\chi^{-}_{\tau_{s+1}} a^{\diamond}_k W^0(b_k - \widehat{b}_{k,s+1})$, for $s = 1, 2, \dots, n-1$ and $k = 1, 2, \dots, m$, are compact on the space $L^2(\mathbb{R})$ because the functions $\chi^{+}_{\tau_s}\chi^{-}_{\tau_{s+1}} a^{\diamond}_k$ and $b_k - \widehat{b}_{k,s+1}$ equal zero at neighborhoods of ∞. Hence, we infer from (6.17) that for every $s = 1, 2, \dots, n-1$,

$$\widetilde{A}^{+}_{\tau_s} \simeq \sum_{k=1}^{m} \Big(\chi^{-}_{\tau_s} a^{\diamond}_k(\tau_s - 0) W^0(\widehat{b}_{k,s}) + \big[\chi^{+}_{\tau_s}\chi^{-}_{\tau_{s+1}} a^{\diamond}_k + \chi^{+}_{\tau_{s+1}} a^{\diamond}_k(\tau_{s+1} - 0)\big] W^0(\widehat{b}_{k,s+1})\Big), \tag{6.18}$$

where, for all $k = 1, 2, \dots, m$ and every $s = 1, 2, \dots, n-1$, the functions

$$\chi^-_{\tau_s} a^\diamond_k(\tau_s - 0) + \chi^+_{\tau_s}\chi^-_{\tau_{s+1}} a^\diamond_k + \chi^+_{\tau_{s+1}} a^\diamond_k(\tau_{s+1} - 0)$$

can have a discontinuity on $\mathbb{R}$ only at the point τ_s.

On the other hand, we see that

$$\widehat{A}^-_{\tau_1} = \chi^-_{\tau_1} A^\diamond_{t_0} + \chi^+_{\tau_1} \widetilde{A}^-_{\tau_1} = \sum_{k=1}^{m} \big(\chi^-_{\tau_1} a^\diamond_k W^0(b_k) + \chi^+_{\tau_1} a^\diamond_k(\tau_1 - 0) W^0(\widehat{b}_{k,1})\big), \tag{6.19}$$

where $\chi^-_{\tau_1} a^\diamond_k + \chi^+_{\tau_1} a^\diamond_k(\tau_1 - 0) \in C(\overline{\mathbb{R}})$ for all $k = 1, 2, \dots, m$.

7. Index of the operator $A^\diamond_{t_0}$

In this section we compute the index of the Fredholm operator

$$A^\diamond_{t_0} = \sum_{k=1}^{m} a^\diamond_k W^0(b_k), \tag{7.1}$$

where $a^\diamond_k \in PC$, $b_k \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$, $a^\diamond_k(t) = a_k(t)$ for all $t \in [-t_0, t_0]$, $a^\diamond_k(\pm t) = a_k(\pm t_0)$ for all $t > t_0$, and $\tau_1 < \ldots < \tau_n$ are all possible discontinuity points of the functions a_k on $(-t_0, t_0)$.

Theorem 7.1. *The index of the Fredholm operator* (7.1) *on the space* $L^2(\mathbb{R})$ *is given by*

$$\begin{aligned}
\mathrm{ind}\, A^\diamond_{t_0} &= \lim_{r\to\infty} \mathrm{ind}[A^\diamond_{t_0}]_r = -\frac{1}{2\pi} \lim_{r\to\infty} \Bigg(\left\{ \arg\left(\frac{\sum_{k=1}^m a_k(t_0) b_k(x)}{\sum_{k=1}^m a_k(-t_0) b_k(x)} \right) \right\}_{x\in[-r,r]} \\
&+ \sum_{s=1}^{n} \left\{ \arg\left(\frac{\sum_{k=1}^m a_k(\tau_s + 0) b_k(-r)}{\sum_{k=1}^m a_k(\tau_s + 0) b_k(r)} \mu + \frac{\sum_{k=1}^m a_k(\tau_s - 0) b_k(-r)}{\sum_{k=1}^m a_k(\tau_s - 0) b_k(r)} (1-\mu) \right) \right\}_{\mu\in[0,1]} \\
&+ \sum_{s=1}^{n+1} \left\{ \arg\left(\frac{\sum_{k=1}^m a_k(t) b_k(-r)}{\sum_{k=1}^m a_k(t) b_k(r)} \right) \right\}_{t\in[\tau_{s-1}+0, \tau_s - 0]} \Bigg),
\end{aligned} \tag{7.2}$$

where $\tau_0 = -t_0$ *and* $\tau_{n+1} = t_0$.

Proof. To calculate $\mathrm{ind}\, A^\diamond_{t_0}$, we approximate the piecewise continuous functions $a^\diamond_k$ by piecewise constant (step) functions h_k on the compact set $[-t_0, t_0]$ such that $h_k(\pm t_0 \mp 0) = a_k(\pm t_0)$ and $h_k(\tau_s \pm 0) = a_k(\tau_s \pm 0)$, respectively, and $t_1 < t_2 < \ldots < t_p$ are all possible discontinuities of the functions h_k $(k = 1, 2, \ldots, m)$ in $(-t_0, t_0)$, where the set $\{t_1, \ldots, t_p\}$ contains the set $\{\tau_1, \ldots, \tau_n\}$. Put $h_k(\pm t) = a_k(\pm t_0)$ for all $t \geq t_0$. Then the functions h_k are continuous at the points $\pm t_0$.

In view of the stability of operator indices there is an $\varepsilon > 0$ such that the operator $Q := \sum_{k=1}^{m} h_k W^0(b_k)$ is Fredholm on the space $L^2(\mathbb{R})$ if

$$\|a_k - h_k\|_{L^\infty[-t_0,t_0]} < \varepsilon \quad \text{for all} \quad k = 1, 2, \ldots, m,$$

and then $\operatorname{ind} A^\diamond_{t_0} = \operatorname{ind} Q$.

Further, to the operator Q with piecewise constant coefficients h_k and functions $b_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$ we apply the results of Section 7. By analogy with (6.16), (6.18) and (6.19), we introduce the operators

$$\begin{aligned}
\widehat{Q}^-_{t_1} &= \sum_{k=1}^{m} \big[\chi^-_{t_1} a_k(-t_0) W^0(b_k) + \chi^+_{t_1} h_k(t_1 - 0) W^0(\widehat{b}_{k,1})\big],\\
\widetilde{Q}^+_{t_j} &= \sum_{k=1}^{m} \big[\chi^-_{t_j} h_k(t_j - 0) W^0(\widehat{b}_{k,j}) + \chi^+_{t_j} h_k(t_{j+1} - 0) W^0(\widehat{b}_{k,j+1})\big]\\
&\quad \text{for} \quad j = 1, 2, \ldots, p-1,\\
\widetilde{Q}^+_{t_p} &= \sum_{k=1}^{m} \big[\chi^-_{t_p} h_k(t_p - 0) W^0(\widehat{b}_{k,p}) + \chi^+_{t_p} a_k(t_0) W^0(b_k)\big],
\end{aligned} \tag{7.3}$$

where for all $j = 1, 2, \ldots, p$ the functions

$$g_j(x) := \sum_{k=1}^{m} h_k(t_j - 0)\widehat{b}_{k,j}(x), \quad x \in \mathbb{R}, \tag{7.4}$$

are invertible in the C^*-algebra $\operatorname{alg}(SO, C(\overline{\mathbb{R}}))$, and therefore the operators $W^0(g_j)$ are invertible on the space $L^2(\mathbb{R})$. To this end we take $\widehat{b}_{k,j}(x) = b_k(x)$ for all $|x| \geq r$, where $r > 0$ is sufficiently large, which implies that the functions $\sum_{k=1}^{m} h_k(t_j - 0)\widehat{b}_{k,j}$ are separated from zero on $\mathbb{R} \setminus [-r, r]$ for all $j = 1, 2, \ldots, p$, and then extend these functions to $[-r, r]$ to get functions (7.4) invertible in $\operatorname{alg}(SO, C(\overline{\mathbb{R}}))$.

By Corollary 6.3, the operators $\widehat{Q}^-_{t_1}$ and $\widetilde{Q}^+_{t_j}$ for all $j = 1, 2, \ldots, p$ are Fredholm on the space $L^2(\mathbb{R})$ and

$$\operatorname{ind} A^\diamond_{t_0} = \operatorname{ind} Q = \operatorname{ind} \widehat{Q}^-_{t_1} + \sum_{j=1}^{p} \operatorname{ind} \widetilde{Q}^+_{t_j}. \tag{7.5}$$

Since the functions (7.4) are invertible in $\mathrm{alg}(SO, C(\overline{\mathbb{R}}))$, the operators (7.3) are Fredholm simultaneously with the operators

$$\begin{aligned} &\widehat{B}_{t_1}^- := \chi_{t_1}^+ W^0(g_1/\sigma(-t_0,\cdot)) + \chi_{t_1}^- I, \\ &\widetilde{B}_{t_j}^+ := \chi_{t_j}^+ W^0(g_{j+1}/g_j) + \chi_{t_j}^- I \quad (j = 1, 2, \dots, p-1), \\ &\widetilde{B}_{t_p}^+ := \chi_{t_p}^+ W^0(\sigma(t_0,\cdot)/g_p) + \chi_{t_p}^- I, \end{aligned} \tag{7.6}$$

where $\sigma(t,x) = \sum_{k=1}^m a_k(t)b_k(x)$ for $(t,x) \in \mathbb{R} \times \mathbb{R}$, and

$$\mathrm{ind}\,\widehat{Q}_{t_1}^- = \mathrm{ind}\,\widehat{B}_{t_1}^-, \quad \mathrm{ind}\,\widetilde{Q}_{t_j}^+ = \mathrm{ind}\,\widetilde{B}_{t_j}^+ \quad (j = 1, 2, \dots, p). \tag{7.7}$$

Hence from (7.5) and (7.7) it follows that

$$\mathrm{ind}\,A_{t_0}^\diamond = \mathrm{ind}\,\widehat{B}_{t_1}^- + \sum_{j=1}^p \mathrm{ind}\,\widetilde{B}_{t_j}^+. \tag{7.8}$$

Applying now Theorem 4.2 to the operators (7.6), we conclude that for every sufficiently large $r > 0$ the truncated operators

$$\begin{aligned} &[\widehat{B}_{t_1}^-]_r := \chi_{t_1}^+ W^0([g_1/\sigma(-t_0,\cdot)]_r) + \chi_{t_1}^- I, \\ &[\widetilde{B}_{t_j}^+]_r := \chi_{t_j}^+ W^0([g_{j+1}/g_j]_r) + \chi_{t_j}^- I \quad (j = 1, 2, \dots, p-1), \\ &[\widetilde{B}_{t_p}^+]_r := \chi_{t_p}^+ W^0([\sigma(t_0,\cdot)/g_p]_r) + \chi_{t_p}^- I \end{aligned}$$

are Fredholm on the space $L^2(\mathbb{R})$, and

$$\begin{aligned} \mathrm{ind}\,\widehat{B}_{t_1}^- &= \lim_{r\to\infty} \mathrm{ind}\,[\widehat{B}_{t_1}^-]_r = -\frac{1}{2\pi} \lim_{r\to\infty} \Big(\big\{ \arg[g_1(x)/\sigma(-t_0,x)] \big\}_{x\in[-r,r]} \\ &\quad + \big\{ \arg\big([g_1(-r)/\sigma(-t_0,-r)]\mu + [g_1(r)/\sigma(-t_0,r)](1-\mu)\big) \big\}_{\mu\in[0,1]} \Big), \\ \mathrm{ind}\,\widetilde{B}_{t_j}^+ &= \lim_{r\to\infty} \mathrm{ind}\,[\widetilde{B}_{t_j}^+]_r = -\frac{1}{2\pi} \lim_{r\to\infty} \Big(\big\{ \arg[g_{j+1}(x)/g_j(x)] \big\}_{x\in[-r,r]} \\ &\quad + \big\{ \arg\big([g_{j+1}(-r)/g_j(-r)]\mu + [g_{j+1}(r)/g_j(r)](1-\mu)\big) \big\}_{\mu\in[0,1]} \Big), \\ &\quad\ \text{for all } \ j = 1, 2, \dots, p-1, \\ \mathrm{ind}\,\widetilde{B}_{t_p}^+ &= \lim_{r\to\infty} \mathrm{ind}\,[\widetilde{B}_{t_p}^+]_r = -\frac{1}{2\pi} \lim_{r\to\infty} \Big(\big\{ \arg[\sigma(t_0,x)/g_p(x)] \big\}_{x\in[-r,r]} \\ &\quad + \big\{ \arg\big([\sigma(t_0,-r)/g_p(-r)]\mu + [\sigma(t_0,r)/g_p(r)](1-\mu)\big) \big\}_{\mu\in[0,1]} \Big). \end{aligned}$$

Substituting the latter formulas into (7.8) and noting that

$$\{\arg f(x)\}_{x\in[\nu_1,\nu_2]} + \{\arg g(x)\}_{x\in[\nu_1,\nu_2]} = \{\arg[f(x)g(x)]\}_{x\in[\nu_1,\nu_2]},$$

we obtain

$$\begin{aligned}\operatorname{ind} A^{\diamond}_{t_0} = -\frac{1}{2\pi}\lim_{r\to\infty}\Big(&\{\arg[\sigma(t_0,x)/\sigma(-t_0,x)]\}_{x\in[-r,r]}\\ &+\Big\{\arg\big([g_1(-r)/\sigma(-t_0,-r)]\mu + [g_1(r)/\sigma(-t_0,r)](1-\mu)\big)\Big\}_{\mu\in[0,1]}\\ &+\sum_{j=1}^{p-1}\Big\{\arg\big([g_{j+1}(-r)/g_j(-r)]\mu + [g_{j+1}(r)/g_j(r)](1-\mu)\big)\Big\}_{\mu\in[0,1]}\\ &+\Big\{\arg\big([\sigma(t_0,-r)/g_p(-r)]\mu + [\sigma(t_0,r)/g_p(r)](1-\mu)\big)\Big\}_{\mu\in[0,1]}\Big).\end{aligned} \tag{7.9}$$

Since $h_k(t_1-0) = h_k(-t_0) = a_k(-t_0)$, we conclude that

$$g_1(\pm r)/\sigma(-t_0,\pm r) = \frac{\sum_{k=1}^m h_k(t_1-0)b_k(\pm r)}{\sum_{k=1}^m a_k(-t_0)b_k(\pm r)} = 1,$$

and therefore

$$\Big\{\arg\big([g_1(-r)/\sigma(-t_0,-r)]\mu + [g_1(r)/\sigma(-t_0,r)](1-\mu)\big)\Big\}_{\mu\in[0,1]} = 0. \tag{7.10}$$

Because the functions h_k are piecewise constant and hence $h_k(t_{j+1}-0) = h_k(t_j+0)$ and $a_k(t_0) = h_k(t_p+0)$, we see that, for all sufficiently large $r>0$,

$$\begin{aligned} g_{j+1}(\pm r)/g_j(\pm r) &= \frac{\sum_{k=1}^m h_k(t_j+0)b_k(\pm r)}{\sum_{k=1}^m h_k(t_j-0)b_k(\pm r)} \quad (j=1,2,\dots,p-1),\\ \sigma(t_0,\pm r)/g_p(\pm r) &= \frac{\sum_{k=1}^m h_k(t_p+0)b_k(\pm r)}{\sum_{k=1}^m h_k(t_p-0)b_k(\pm r)}. \end{aligned} \tag{7.11}$$

Consequently, we deduce from (7.11) that

$$\begin{aligned}&\Big\{\arg\big([g_{j+1}(-r)/g_j(-r)]\mu + [g_{j+1}(r)/g_j(r)](1-\mu)\big)\Big\}_{\mu\in[0,1]}\\ &=\left\{\arg\left(\frac{\sum_{k=1}^m h_k(t_j+0)b_k(-r)}{\sum_{k=1}^m h_k(t_j-0)b_k(-r)}\mu + \frac{\sum_{k=1}^m h_k(t_j+0)b_k(r)}{\sum_{k=1}^m h_k(t_j-0)b_k(r)}(1-\mu)\right)\right\}_{\mu\in[0,1]}\\ &=\left\{\arg\left(\frac{\sum_{k=1}^m h_k(t_j+0)b_k(-r)}{\sum_{k=1}^m h_k(t_j+0)b_k(r)}\mu + \frac{\sum_{k=1}^m h_k(t_j-0)b_k(-r)}{\sum_{k=1}^m h_k(t_j-0)b_k(r)}(1-\mu)\right)\right\}_{\mu\in[0,1]}\end{aligned} \tag{7.12}$$

for all $j = 1, 2, \dots, p-1$ and, analogously,

$$\begin{aligned}
&\big\{ \arg\big([\sigma(t_0,-r)/g_p(-r)]\mu + [\sigma(t_0,r)/g_p(r)](1-\mu)\big)\big\}_{\mu\in[0,1]} \\
&= \left\{\arg\left(\frac{\sum_{k=1}^m h_k(t_p+0)b_k(-r)}{\sum_{k=1}^m h_k(t_p+0)b_k(r)}\mu + \frac{\sum_{k=1}^m h_k(t_p-0)b_k(-r)}{\sum_{k=1}^m h_k(t_p-0)b_k(r)}(1-\mu)\right)\right\}_{\mu\in[0,1]}.
\end{aligned} \tag{7.13}$$

It remains to observe that if all the coefficients a_k are continuous at the points $t_{q_{s-1}+1}, \dots, t_{q_s-1}$ for all $s = 1, 2, \dots, n+1$, where $q_0 = 0$ and $q_{n+1} = p+1$, and the function $t \mapsto \big[\sum_{k=1}^m a_k(t)b_k(-r)\big]/\big[\sum_{k=1}^m a_k(t)b_k(r)\big]$ is discontinuous at the points $t_{q_s} = \tau_s$ for all $s = 1, 2, \dots, n$, then for all sufficiently large $r > 0$ and all $s = 1, 2, \dots, n+1$,

$$\begin{aligned}
&\sum_{j=q_{s-1}+1}^{q_s-1}\left\{\arg\left(\frac{\sum_{k=1}^m h_k(t_j+0)b_k(-r)}{\sum_{k=1}^m h_k(t_j+0)b_k(r)}\mu + \frac{\sum_{k=1}^m h_k(t_j-0)b_k(-r)}{\sum_{k=1}^m h_k(t_j-0)b_k(r)}(1-\mu)\right)\right\}_{\mu\in[0,1]} \\
&\quad = \left\{\arg\left(\frac{\sum_{k=1}^m a_k(t)b_k(-r)}{\sum_{k=1}^m a_k(t)b_k(r)}\right)\right\}_{t\in[\tau_{s-1}+0,\tau_s-0]},
\end{aligned} \tag{7.14}$$

where $\tau_0 = -t_0$, $\tau_{n+1} = t_0$ and, for a continuous branch of $\arg f$ on $[\tau_{s-1}, \tau_s]$,

$$\{\arg f(t)\}_{t\in[\tau_{s-1}+0,\tau_s-0]} = \arg f(\tau_s - 0) - \arg f(\tau_{s-1} + 0).$$

Consequently, taking into account the equalities $h_k(\tau_s \pm 0) = a_k(\tau_s \pm 0)$ $(s = 1, 2, \dots, n)$, we infer from (7.9), (7.10) and (7.12)–(7.14) that

$$\begin{aligned}
\operatorname{ind} A_{t_0}^{\diamond} &= -\frac{1}{2\pi}\lim_{r\to\infty}\Bigg(\left\{\arg\left(\frac{\sum_{k=1}^m a_k(t_0)b_k(x)}{\sum_{k=1}^m a_k(-t_0)b_k(x)}\right)\right\}_{x\in[-r,r]} \\
&+\sum_{s=1}^{n}\left\{\arg\left(\frac{\sum_{k=1}^m a_k(\tau_s+0)b_k(-r)}{\sum_{k=1}^m a_k(\tau_s+0)b_k(r)}\mu + \frac{\sum_{k=1}^m a_k(\tau_s-0)b_k(-r)}{\sum_{k=1}^m a_k(\tau_s-0)b_k(r)}(1-\mu)\right)\right\}_{\mu\in[0,1]} \\
&+\sum_{s=1}^{n+1}\left\{\arg\left(\frac{\sum_{k=1}^m a_k(t)b_k(-r)}{\sum_{k=1}^m a_k(t)b_k(r)}\right)\right\}_{t\in[\tau_{s-1}+0,\tau_s-0]}\Bigg).
\end{aligned} \tag{7.15}$$

Moreover, it follows straightforwardly from (7.15) and the index formula in Theorem 3.4 for the truncated operators $[A_{t_0}^{\diamond}]_r$ that $\operatorname{ind} A_{t_0}^{\diamond} = \lim\limits_{r\to\infty} \operatorname{ind}[A_{t_0}^{\diamond}]_r$, which completes the proof. $\square$

8. Index formula for the operator A in the case (C1)

Applying Theorems 5.1–5.3 and 7.1, we compute $\operatorname{ind} A$ in the case (C1).

Theorem 8.1. *If the operator $A = \sum_{k=1}^{m} a_k W^0(b_k)$, with functions $a_k, b_k \in \operatorname{alg}(SO, PC)$ having finite sets of discontinuities, is Fredholm on the space $L^2(\mathbb{R})$ and either all a_k or all b_k are in $\operatorname{alg}(SO, C(\overline{\mathbb{R}}))$, then*

$$\operatorname{ind} A = \lim_{r\to\infty} \operatorname{ind} A_r, \tag{8.1}$$

where $\operatorname{ind} A_r$ *is calculated by* (3.10).

Proof. (i) If $a_k, b_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$ for all $k = 1, 2, \dots, m$, then (8.1) is proved in Theorem 5.1.

(ii) Let now all $a_k \in \operatorname{alg}(SO, PC)$ and all $b_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$. Then, by (5.16) and (5.17), where $a_k^{\pm}(t) = a_k(t)$ if $\pm t \geq t_0$ and $a_k^{\pm}(t) = a_k(\pm t_0)$ if $\pm t < t_0$, we obtain

$$\begin{aligned}
&-2\pi \operatorname{ind} A_{t_0}^{+} = \lim_{r\to\infty} \Big\{ \arg \sum_{k=1}^{m} a_k^{+}(t) b_k(x) \Big\}_{(t,x)\in\partial\Pi_r} \\
&= \lim_{r\to\infty} \left(\left\{ \arg \frac{\sum_{k=1}^{m} a_k(t) b_k(-r)}{\sum_{k=1}^{m} a_k(t) b_k(r)} \right\}_{t\in[t_0,r]} + \left\{ \arg \frac{\sum_{k=1}^{m} a_k(r) b_k(x)}{\sum_{k=1}^{m} a_k(t_0) b_k(x)} \right\}_{x\in[-r,r]} \right), \\
&-2\pi \operatorname{ind} A_{t_0}^{-} = \lim_{r\to\infty} \Big\{ \arg \sum_{k=1}^{m} a_k^{-}(t) b_k(x) \Big\}_{(t,x)\in\partial\Pi_r} \\
&= \lim_{r\to\infty} \left(\left\{ \arg \frac{\sum_{k=1}^{m} a_k(t) b_k(-r)}{\sum_{k=1}^{m} a_k(t) b_k(r)} \right\}_{t\in[-r,-t_0]} + \left\{ \arg \frac{\sum_{k=1}^{m} a_k(-t_0) b_k(x)}{\sum_{k=1}^{m} a_k(-r) b_k(x)} \right\}_{x\in[-r,r]} \right).
\end{aligned}$$

Applying the latter formulas, we infer from (5.7) and (7.2) that

$$\begin{aligned}
&\operatorname{ind} A = \operatorname{ind} A_{t_0}^{+} + \operatorname{ind} A_{t_0}^{-} + \operatorname{ind} A_{t_0}^{\diamond} \\
&= -\frac{1}{2\pi} \lim_{r\to\infty} \left(\left\{ \arg \left(\frac{\sum_{k=1}^{m} a_k(r) b_k(x)}{\sum_{k=1}^{m} a_k(-r) b_k(x)} \right) \right\}_{x\in[-r,r]} \right. \\
&+ \sum_{s=1}^{n} \left\{ \arg \left(\frac{\sum_{k=1}^{m} a_k(\tau_s+0) b_k(-r)}{\sum_{k=1}^{m} a_k(\tau_s+0) b_k(r)} \mu + \frac{\sum_{k=1}^{m} a_k(\tau_s-0) b_k(-r)}{\sum_{k=1}^{m} a_k(\tau_s-0) b_k(r)} (1-\mu) \right) \right\}_{\mu\in[0,1]} \\
&\left. + \sum_{s=1}^{n+1} \left\{ \arg \left(\frac{\sum_{k=1}^{m} a_k(t) b_k(-r)}{\sum_{k=1}^{m} a_k(t) b_k(r)} \right) \right\}_{t\in[\tau_{s-1}+0,\tau_s-0]} \right),
\end{aligned} \tag{8.2}$$

where now $\tau_0 = -r$ and $\tau_{n+1} = r$. Moreover, we see from (8.2) and Theorem 3.4 that again (8.1) holds.

(iii) Finally, let all $a_k \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ and all $b_k \in \mathrm{alg}(SO, PC)$. Using the Fourier transform and setting $\widehat{a}_k(x) = a_k(-x)$, we obtain

$$(\mathcal{F}A\mathcal{F}^{-1})^* = \sum_{k=1}^{m}(W^0(\widehat{a}_k)b_kI)^* = \sum_{k=1}^{m}\overline{b}_kW^0(\overline{\widehat{a}}_k).$$

We then deduce from part (ii) with $a_k = \overline{b}_k$ and $b_k = \overline{\widehat{a}}_k$ that

$$\begin{aligned}\mathrm{ind}\, A &= -\,\mathrm{ind}\,(\mathcal{F}A\mathcal{F}^{-1})^* \\ &= -\,\mathrm{ind}\left[\sum_{k=1}^{m}\overline{b}_kW^0(\overline{\widehat{a}}_k)\right] = -\lim_{r\to\infty}\mathrm{ind}\left[\sum_{k=1}^{m}\overline{b}_kW^0(\overline{\widehat{a}}_k)\right]_r \\ &= -\lim_{r\to\infty}\mathrm{ind}\,(\mathcal{F}A_r\mathcal{F}^{-1})^* = \lim_{r\to\infty}\mathrm{ind}\, A_r, \end{aligned} \tag{8.3}$$

which completes the proof. $\square$

9. Index formula for the operator A in the case (C2)

In this section, we return to the general case of the Fredholm operator (1.1) with $a_k, b_k \in \mathrm{alg}(SO, PC)$. We show that under conditions (1.2)–(1.3) the computation of the index of A can be reduced to the case where at least all a_k or all b_k are in $\mathrm{alg}(SO, C(\overline{\mathbb{R}}))$.

Given functions $a_k, b_k \in \mathrm{alg}(SO, PC)$ with finite sets of discontinuities, we can always take functions $\widetilde{a}_k, \widetilde{b}_k \in \mathrm{alg}(SO, C(\overline{\mathbb{R}}))$ in (1.2) such that, for some $r > 0$,

$$\widetilde{a}_k(t) - a_k(t) = \widetilde{b}_k(x) - b_k(x) = 0 \quad \text{for} \quad |t|, |x| > r \quad \text{and} \quad k = 1, 2, \ldots, m. \tag{9.1}$$

Let

$$\widetilde{\sigma}(t, x) := \sum_{k=1}^{m}\widetilde{a}_k(t)\widetilde{b}_k(x) \quad \text{for all} \quad (t, x) \in \mathbb{R}\times\mathbb{R} \tag{9.2}$$

similarly to the function σ given by (1.7), and consider the functions

$$\begin{aligned} c_\pm(t, x) &= \sigma(t \pm 0, x + 0), \quad d_\pm(t, x) = \sigma(t \pm 0, x - 0), \\ \widetilde{c}_\pm(t, x) &= \widetilde{\sigma}(t \pm 0, x + 0), \quad \widetilde{d}_\pm(t, x) = \widetilde{\sigma}(t \pm 0, x - 0) \end{aligned}$$

defined on the set $M(SO) \times M(SO)$ according to (3.1).

By Corollary 3.2, conditions (1.2)–(1.3) are equivalent to the following:

There exist functions $\widetilde{a}_k, \widetilde{b}_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$ *satisfying* (9.1) *such that for some* $r > 0$ *the function* (9.2) *is separated from zero on the set* $\mathbb{R}^2 \setminus [-r, r]^2$ *or, equivalently,*

$$\widetilde{c}_\pm(t, x) \neq 0, \quad \widetilde{d}_\pm(t, x) \neq 0 \tag{9.3}$$

for all $(t, x) \in (\mathbb{R} \times M_\infty(SO)) \cup (M_\infty(SO) \times \mathbb{R}) \cup (M_\infty(SO) \times M_\infty(SO))$.

Note that if the operator A is Fredholm, then from (9.1) and Corollary 3.2 it follows that for some $r > 0$ condition (9.3) is fulfilled on the set

$$\big((M(SO) \setminus [-r, r]) \times M_\infty(SO)\big) \cup \big(M_\infty(SO) \times (M(SO) \setminus [-r, r])\big).$$

Hence, we only need to define the functions $\widetilde{a}_k, \widetilde{b}_k$ on the segment $[-r, r]$ to ensure the invertibility of the functions $\widetilde{c}_\pm$ and $\widetilde{d}_\pm$ on the set $([-r, r] \times M_\infty(SO)) \cup (M_\infty(SO) \times [-r, r])$.

We now present a sufficient condition for the fulfilment of (9.1) and (9.3).

Lemma 9.1. *If the Fredholm operator* A *is given by* (1.1) *with functions* $a_k, b_k \in \operatorname{alg}(SO, PC)$ *that have finite sets of discontinuities and satisfy the conditions*

$$\begin{cases} c_+(t, x)\mu + c_-(t, x)(1 - \mu) \neq 0, & (t, x, \mu) \in \mathbb{R} \times M_\infty(SO) \times [0, 1], \\ d_+(t, x)\mu + d_-(t, x)(1 - \mu) \neq 0, & (t, x, \mu) \in \mathbb{R} \times M_\infty(SO) \times [0, 1], \end{cases} \tag{9.4}$$

and

$$\begin{cases} c_+(t, x)\mu + d_+(t, x)(1 - \mu) \neq 0, & (t, x, \mu) \in M_\infty(SO) \times \mathbb{R} \times [0, 1], \\ c_-(t, x)\mu + d_-(t, x)(1 - \mu) \neq 0, & (t, x, \mu) \in M_\infty(SO) \times \mathbb{R} \times [0, 1], \end{cases} \tag{9.5}$$

then there exist functions $\widetilde{a}_k, \widetilde{b}_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$ *such that conditions* (9.1) *and* (9.3) *are fulfilled.*

Proof. By Corollary 3.2, the Fredholmness of the operator A implies that

$$\sigma(t \pm 0, x + 0) = c_\pm(t, x) \neq 0, \quad \sigma(t \pm 0, x - 0) = d_\pm(t, x) \neq 0$$

for all $(t, x) \in (\mathbb{R} \times M_\infty(SO)) \cup (M_\infty(SO) \times \mathbb{R}) \cup (M_\infty(SO) \times M_\infty(SO))$. In particular, we may choose $r > 0$ such that all discontinuities on $\mathbb{R}$ of the functions a_k, b_k for all $k = 1, 2, \ldots, m$ lie in the segment $[-r, r]$ and the function σ given by (1.7) is separated from zero on the set $\mathbb{R}^2 \setminus [-r, r]^2$.

We first define functions $\widetilde{a}_k, \widetilde{b}_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$ $(k = 1, 2, \ldots, m)$ on the set $\mathbb{R} \setminus [-r, r]$ by (9.1). Then the function $\widetilde{\sigma}$ given by (9.2) is separated from zero on the set $(\mathbb{R} \setminus [-r, r])^2$. In particular, $\widetilde{c}_\pm(t, x) \neq 0$ and $\widetilde{d}_\pm(t, x) \neq 0$ for all $(t, x) \in M_\infty(SO) \times M_\infty(SO)$.

Now we define $\widetilde{a}_k(t)$ for $t \in [-r, r]$. According to (9.4), we can also assume that $r > 0$ is chosen such that for all $t \in [-r, r]$ and all $x \in \mathbb{R} \setminus [-r, r]$, the function

$$c_+(t, x)\mu + c_-(t, x)(1 - \mu) = \sum_{k=1}^{m} \big(a_k(t + 0)\mu + a_k(t - 0)(1 - \mu)\big) b_k(x) \quad (9.6)$$

is separated from zero for any $\mu \in [0, 1]$. Around each discontinuity point τ_s of the functions a_k, we choose points $t_{0,s}$, $t_{1,s}$ such that $\tau_s \in (t_{0,s}, t_{1,s})$ and

$$|a_k(\tau_s + 0) - a_k(t_{1,s})| < \varepsilon/2, \quad |a_k(\tau_s - 0) - a_k(t_{0,s})| < \varepsilon/2.$$

Then from (9.6) it follows that for sufficiently small $\varepsilon > 0$ the function

$$\sum_{k=1}^{m} (a_k(t_{1,s})\mu + a_k(t_{0,s})(1 - \mu)) b_k(x) \quad (9.7)$$

is separated from zero for any $\mu \in [0, 1]$ and all $|x| > r$. Assuming that $(t_{0,s}, t_{1,s}) \cap (t_{0,j}, t_{1,j}) = \emptyset$ for $s \neq j$, we define the functions

$$\widetilde{a}_k(t) = \begin{cases} a_k(t_{0,s}) + \dfrac{a_k(t_{1,s}) - a_k(t_{0,s})}{t_{1,s} - t_{0,s}}(t - t_{0,s}) & \text{if } \ t \in (t_{0,s}, t_{1,s}), \\ a_k(t) & \text{if } \ t \notin \bigcup_{s=1}^{n} (t_{0,s}, t_{1,s}). \end{cases}$$

The functions $\widetilde{a}_k$ are continuous on $[-r, r]$, and hence $\widetilde{a}_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}}))$. Moreover, since $\widetilde{a}_k(t) = a_k(t_{1,s})\mu + a_k(t_{0,s})(1 - \mu)$ for $t \in [t_{0,s}, t_{1,s}]$ with $\mu = \frac{t - t_{0,s}}{t_{1,s} - t_{0,s}} \in [0, 1]$, we infer from (9.6) and (9.7) that the function

$$\widetilde{\sigma}(t, x) = \sum_{k=1}^{m} \widetilde{a}_k(t) \widetilde{b}_k(x) = \sum_{k=1}^{m} \widetilde{a}_k(t) b_k(x)$$

is separated from zero for all $t \in \mathbb{R}$ and all $x \in \mathbb{R} \setminus [-r, r]$.

Applying (9.5) and noting that for $t \in \mathbb{R} \setminus [-r, r]$ and $x \in \mathbb{R}$ the function

$$c_+(t, x)\mu + d_+(t, x)(1 - \mu) = \sum_{k=1}^{m} a_k(t)(b_k(x + 0)\mu + b_k(x - 0)(1 - \mu)) \quad (9.8)$$

is separated from zero for any $\mu \in [0, 1]$, we can proceed in the same way as above by defining $\widetilde{b}_k$ on $[-r, r]$ and showing that the function $\widetilde{\sigma}(t, x)$ is also separated from zero for all $t \in \mathbb{R} \setminus [-r, r]$ and all $x \in \mathbb{R}$.

Thus, the function $\widetilde{\sigma}(t, x)$ is separated from zero on the set $\mathbb{R}^2 \setminus [-r, r]^2$, which is equivalent to (9.3). $\square$

Note that Lemma 9.1 remains valid with μ replaced by any continuous complex function f defined for $\mu \in [0, 1]$ and such that $f(0) = 0$, $f(1) = 1$.

In the remainder of this section, we let A be a Fredholm operator given by (1.1) and such that condition (9.3) is satisfied. With such an A, we associate the operator

$$A_{\infty,\infty} := \sum_{k=1}^{m} \widetilde{a}_k W^0(\widetilde{b}_k), \quad \widetilde{a}_k, \widetilde{b}_k \in \operatorname{alg}(SO, C(\overline{\mathbb{R}})), \quad (9.9)$$

which is Fredholm on the space $L^2(\mathbb{R})$ according to Theorem 5.1 because

$$\liminf_{t^2+x^2 \to \infty} |\widetilde{\sigma}(t, x)| > 0$$

in view of conditions (1.2)–(1.3) or, equivalently, condition (9.3).

By analogy with [9], we introduce the following operators associated with the operator A:

$$\begin{aligned} A_{\mathbb{R},\infty} &:= \sum_{k=1}^{m} a_k W^0(\widetilde{b}_k), \quad & A_{0,\infty} &:= \sum_{k=1}^{m} (a_k - \widetilde{a}_k) W^0(\widetilde{b}_k), \\ A_{\infty,\mathbb{R}} &:= \sum_{k=1}^{m} \widetilde{a}_k W^0(b_k), \quad & A_{\infty,0} &:= \sum_{k=1}^{m} \widetilde{a}_k W^0(b_k - \widetilde{b}_k). \end{aligned} \quad (9.10)$$

Since by (9.1) and [9, Lemma 7.1] the operator

$$A_{0,0} := \sum_{k=1}^{m} (a_k - \widetilde{a}_k) W^0(b_k - \widetilde{b}_k)$$

is compact on every Lebesgue space $L^p(\mathbb{R})$, $1 < p < \infty$, we infer that

$$A = A_{0,\infty} + A_{\infty,\infty} + A_{\infty,0} + A_{0,0} \simeq A_{0,\infty} + A_{\infty,\infty} + A_{\infty,0}. \tag{9.11}$$

Denoting by $A^{(-1)}_{\infty,\infty}$ a regularizer of the Fredholm operator $A_{\infty,\infty}$ in the C^*-algebra $\mathcal{A}_{[SO,C(\overline{\mathbb{R}})]}$, we conclude that the operator

$$B := A_{0,\infty} A^{(-1)}_{\infty,\infty} A_{\infty,0} = \sum_{k=1}^{m} \sum_{j=1}^{m} (a_k - \widetilde{a}_k) W^0(\widetilde{b}_k) A^{(-1)}_{\infty,\infty} \widetilde{a}_j W^0(b_j - \widetilde{b}_j) \tag{9.12}$$

is compact on every Lebesgue space $L^p(\mathbb{R})$, $1 < p < \infty$. Indeed, representing the functions $a_k - \widetilde{a}_k$ in the form $a_k - \widetilde{a}_k = \widehat{a}_k c_k$, where $\widehat{a}_k \in \operatorname{alg}(PC, SO)$ and $c_k \in C(\dot{\mathbb{R}})$ with $c_k(\infty) = 0$, and using the compactness of commutators of $c_k I$ and operators in $\mathcal{A}_{[SO,C(\overline{\mathbb{R}})]}$ (see, e.g., [1, Theorem 4.2]), we get

$$B \simeq \sum_{k=1}^{m} \sum_{j=1}^{m} \widehat{a}_k W^0(\widetilde{b}_k) A^{(-1)}_{\infty,\infty} \widetilde{a}_j c_k W^0(b_j - \widetilde{b}_j),$$

where the operators $c_k W^0(b_j - \widetilde{b}_j)$ are compact according to [9, Lemma 7.1]. Then, similarly to the proof of [9, Theorem 7.7], we deduce from (9.10), (9.11) and the compactness of operator (9.12) that

$$A \simeq A_{0,\infty} + A_{\infty,\infty} + A_{\infty,0} + B \simeq A_{\mathbb{R},\infty} A^{(-1)}_{\infty,\infty} A_{\infty,\mathbb{R}} \simeq A_{\infty,\mathbb{R}} A^{(-1)}_{\infty,\infty} A_{\mathbb{R},\infty}. \tag{9.13}$$

Since the operator A is Fredholm, we conclude from the last two relations in (9.13) that both the operators $A_{\mathbb{R},\infty}$ and $A_{\infty,\mathbb{R}}$ are also Fredholm. Indeed, let $A^{(-1)}$ be a regularizer of the operator A. Then the operators

$$A^{(-1)}_{\infty,\infty} A_{\infty,\mathbb{R}} A^{(-1)} \quad \text{and} \quad A^{(-1)} A_{\infty,\mathbb{R}} A^{(-1)}_{\infty,\infty}$$

are, respectively, the right and the left regularizers of the operator $A_{\mathbb{R},\infty}$, while the operators

$$A^{(-1)}_{\infty,\infty} A_{\mathbb{R},\infty} A^{(-1)} \quad \text{and} \quad A^{(-1)} A_{\mathbb{R},\infty} A^{(-1)}_{\infty,\infty}$$

are, respectively, the right and the left regularizers of the operator $A_{\infty,\mathbb{R}}$. Hence, in the case (C2) we obtain another decomposition of the operator A, which has the form

$$A \simeq A_{\mathbb{R},\infty} A^{(-1)}_{\infty,\infty} A_{\infty,\mathbb{R}}, \tag{9.14}$$

where all operators on the right are Fredholm, and gives the following.

Lemma 9.2. *If the operator A given by* (1.1) *is Fredholm on the space $L^2(\mathbb{R})$, and satisfies conditions* (1.2)–(1.3) *(or condition* (9.3)*), then so are the operators $A_{\infty,\infty}$, $A_{\mathbb{R},\infty}$ and $A_{\infty,\mathbb{R}}$ in* (9.14)*, and therefore*

$$\operatorname{ind} A = \operatorname{ind} A_{\mathbb{R},\infty} + \operatorname{ind} A_{\infty,\mathbb{R}} - \operatorname{ind} A_{\infty,\infty}. \tag{9.15}$$

Making use of Lemma 9.2, Theorem 8.1 and Lemma 3.3, we now obtain the main result of this paper, stated already in Theorem 1.1.

Theorem 9.3. *If the operator $A = \sum_{k=1}^{m} a_k W^0(b_k)$, with data functions $a_k, b_k \in \operatorname{alg}(SO, PC)$ admitting finite sets of discontinuities and satisfying conditions* (1.2)–(1.3)*, is Fredholm on the space $L^2(\mathbb{R})$, then its index is given by*

$$\operatorname{ind} A = \lim_{r\to\infty} \operatorname{ind} A_r, \tag{9.16}$$

where $\operatorname{ind} A_r$ *is calculated in Theorem* 3.4.

Proof. Since the data functions of the operators $A_{\mathbb{R},\infty}$, $A_{\infty,\mathbb{R}}$ and $A_{\infty,\infty}$ correspond to the case (C1) and these operators are Fredholm on the space $L^2(\mathbb{R})$ by Lemma 9.2, we infer from Theorem 8.1 and equality (9.15) that

$$\begin{aligned}\operatorname{ind} A &= \operatorname{ind} A_{\mathbb{R},\infty} + \operatorname{ind} A_{\infty,\mathbb{R}} - \operatorname{ind} A_{\infty,\infty} \\ &= \lim_{r\to\infty} \big(\operatorname{ind}\,[A_{\mathbb{R},\infty}]_r + \operatorname{ind}\,[A_{\infty,\mathbb{R}}]_r - \operatorname{ind}\,[A_{\infty,\infty}]_r \big).\end{aligned} \tag{9.17}$$

Since decomposition (9.14) holds, the corresponding truncated operators are Fredholm on the space $L^2(\mathbb{R})$ as well (see Lemma 3.3), and

$$A_r \simeq [A_{\mathbb{R},\infty}]_r ([A_{\infty,\infty}]_r)^{(-1)} [A_{\infty,\mathbb{R}}]_r,$$

which implies the quality

$$\operatorname{ind}\,[A_{\mathbb{R},\infty}]_r + \operatorname{ind}\,[A_{\infty,\mathbb{R}}]_r - \operatorname{ind}\,[A_{\infty,\infty}]_r = \operatorname{ind} A_r. \tag{9.18}$$

Finally, by (9.17) and (9.18), we obtain (9.16). □

Applying now Theorems 8.1 and 9.3, we immediately infer Theorem 1.1, with index formula (1.6), from Theorem 3.4.

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