Notes on the Atiyah-Singer index theorem

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Introduction

In these notes, we make a review of the classic index theorem from Atiyah and Singer, which computes the Fredholm index of an elliptic pseudodifferential operator on a compact smooth manifold without boundary depending only on topological invariants associated to its symbol.

The approach followed here is the one given in [3], the so-called 'embedding' proof, which is K-theoretic in nature and makes use of push-forward maps and their functoriality properties. Atiyah and Singer showed that each elliptic pseudodifferential operator on a closed manifold M defines, through its symbol, a class in the K-theory of TM, the tangent space of M, such that $K^0(TM)$ is exhausted by these classes (in fact, this was one of the main motivations for the development of topological K-theory by Atiyah and Hirzebruch). The Fredholm index depends only on this symbol class, so that it can be regarded as a K-theory map. The main point in the line of proof given in [3] is to take an embedding $i : M \to \mathbb{R}^n$ and to consider an associated map in K-theory

$$i_! : K^0(TM) \to K^0(T\mathbb{R}^n) = K^0(\mathbb{R}^{2n}),$$

the *push-forward* map, which is shown to behave functorially with respect to the Fredholm index. This fact enables one to reduce the proof of the index formula to $\mathbb{R}^n$, where the index coincides with the Bott Periodicity map in K-theory.

It should be noted that there are many different approaches to prove the classic index formula. The proof presented here has the advantage of applying to operators equivariant with respect to an action of a compact Lie group, and it generalises easily to the case of families [5]. Atiyah and Singer's first proof [2] relied on the cobordism invariance of the index, following Hirzebruch's proof of the signature theorem (see also Palais' book [15] for an extensive review). On the analytic side, Atiyah, Bott and Patodi gave yet another proof of the index formula using an asymptotic expansion of the kernel of the heat operator [6] (this had an immediate generalization to an index formula on compact manifolds with boundary [7], see also [14]). On a very different line, Connes gave a proof of the Atiyah-Singer index theorem totally within the framework of noncommutative geometry in[9], using the tangent groupoid of a manifold and deformations of C^*-algebras. There is also a remarkable generalization of the index formula for longitudinally elliptic operators on foliations [10], using an analogue of the push-forward map defined above.

We now review the contents of this paper. We start with recalling the main results on Fredholm operators in Section 1. In Section 2, we develop the theory of pseudodifferential operators on manifolds and their symbols, which are functions on the cotangent space. Section 3 is devoted to examples: we consider the de Rham operator, the Hodge-* operator and the Doulbeault operator, for which the Fredholm index coincides with well-known topological invariants. In Section 4, we make a review of topological K-theory and the K-theory symbol class is defined. The Fredholm index is then regarded as a homomorphism of K-groups. The proof of Atiyah and Singer's index formula in K-theory is sketched in Section 5, following an axiomatic approach. The better known index formula in cohomology is then deduced in Section 6. Finally, in Section 7 we define the K-groups associated to a C^*-algebra and place the Fredholm index in the context of K-theory for C^*-algebras, showing that it coincides with the connecting map of a suitable long exact sequence in K-theory. This is the starting point of noncommutative index theory.

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1 The Fredholm index

For separable, complex Hilbert spaces H_1 and H_2, we call an operator $P : H_1 \to H_2$ a *Fredholm operator* if its kernel and cokernel are finite dimensional (where $\ker P = \{u \in H_1 \ : \ Pu = 0\}$ and $\operatorname{coker} P = H_2/\operatorname{Im} P$.) In this case, the *Fredholm index* of P is defined by

$$\operatorname{ind} P = \dim\ker P - \dim\operatorname{coker} P.$$

Of course, if H_1 and H_2 are finite dimensional, then every operator is Freholm with index 0. Less trivial examples of Fredholm operators are given by operators of the form $I + K$, where $I : H_1 \to H_1$ is the identity and K is a compact operator. Typically, we will be dealing with bounded operators on Hilbert spaces of sections of vector bundles.

We let $\mathcal{B}(H_1, H_2)$ be the class of bounded operators and $\mathcal{F}(H_1, H_2)$ the class of bounded Fredholm operators from H_1 to H_2 (if $H_1 = H_2$, we write $\mathcal{F}(H_1, H_2) = \mathcal{F}(H_1)$). If $P \in \mathcal{F}(H_1, H_2)$ then P has closed range and, if $P^* : H_2 \to H_1$ is the Hilbert space adjoint, then $\ker P \cong \operatorname{coker} P^*$, $\operatorname{coker} P \cong \ker P^*$. Hence P^* is also Fredholm, with

$$\operatorname{ind}(P^*) = -\operatorname{ind}(P).$$

By Atkinson's theorem, if P is bounded then P is Fredholm if and only if there exist operators $Q : H_2 \to H_1$ and $Q' : H_1 \to H_2$ such that $QP - I$ and $PQ' - I$ are compact operators on H_1 and H_2, respectively.

Using this characterization, one can see that if H_3 is also an Hilbert space and $P \in \mathcal{F}(H_1, H_2)$, $Q \in \mathcal{F}(H_2, H_3)$ then $QP \in \mathcal{F}(H_1, H_3)$ and

$$\mathrm{ind}(QP) = \mathrm{ind}(P) + \mathrm{ind}(Q).$$

On the other hand, we can also see that $\mathcal{F}(H_1, H_2)$ is open, and more over, that the Fredholm index remains invariant under homotopy, that is, it is constant on each connected component. The index induces then a map

$$[\mathcal{F}(H_1, H_2)] \to \mathbb{Z}. \tag{1}$$

In particular, if $P \in \mathcal{F}(H_1, H_2)$ and K is a compact operator, then $P+K$ is also Fredholm and we have

$$\mathrm{ind}(P + K) = \mathrm{ind}(P)$$

(since the ideal of compact operators is connected).

If $[\mathcal{F}]$ denotes the set of homotopy classes of bounded Fredholm operators $H \to H$, where H is an infinite dimensional Hilbert space, we have now that $[\mathcal{F}]$ is a group and that the Fredholm index is a group homomorphism. This map is clearly surjective and one can check also that if two operators have the same index then they are homotopic, so that we have in fact an isomorphism

$$\mathrm{ind} : [\mathcal{F}] \to \mathbb{Z}.$$

Note that, in this case, $\mathcal{F}$ coincides with the group of units in the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$.

There are many well-known invariants given by Fredholm indices of operators: the Euler characteristic and the signature of a manifold, for instance, coincide with indices of Dirac type operators . Also, the winding number of a curve coincides with the index of the induced Toepliz operator.

The homotopy invariance of the index famously led Gelfand to suggest that there might be a way of computing it from invariants of the space. As Atiyah and Singer showed, the algebraic topology best suited to tackle this problem is K-theory and the class of operators turns out to be the class of pseudodifferential operators.

2 Pseudodifferential operators

We review here a few facts of the theory of pseudodifferential operators on manifolds (see for instance [11, 16, 18]). For $U \subset \mathbb{R}^n$ open, $m \in \mathbb{Z}$, we define a class $S^m(U \times \mathbb{R}^n)$ of $a \in C^\infty(U \times \mathbb{R}^n)$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{K\alpha\beta} (1 + |\xi|)^{m-|\beta|},$$

for multi-indices α, β and $x \in K \subset U$ compact, $\xi \in \mathbb{R}^n$. We assume that a is *classical*, that is, it has an infinite expansion as $\sum a_{m-k}$,[1] with a_{m-k} positively homogeneous of degree $m-k$ in ξ. We call $S^m(U \times \mathbb{R}^n)$ the class of *symbols of order* m

An element $a \in S^m(U \times \mathbb{R}^n)$ defines an operator $A : C_c^\infty(U) \to C^\infty(U)$ given by

$$Au(x) = a(\cdot, D)u(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} a(x,\xi) e^{ix\cdot\xi} \widehat{u}(\xi) d\xi. \tag{2}$$

An operator of the form $A = a(\cdot, D)$, with $a \in S^m(U \times \mathbb{R}^n)$ is said to be a *pseudodifferential operator of order* m on U.

The class of pseudodifferential operators of order m is denoted by $\Psi^m(U)$. It is clear that $\Psi^j(U) \subset \Psi^m(U)$ for $j < m$; if $P \in \Psi^m(U)$ for all $m \in \mathbb{Z}$, then P is said to be of order $-\infty$, $P \in \Psi^{-\infty}(U)$.

Of course, any differential operator is also pseudodifferential, and its symbol is given by the characteristic polynomial. Another example is given by *regularizing* operators, that is, operators given by a smooth kernel, which are pseudodifferential of order $-\infty$. In fact, parametrices of elliptic differential operators, that is, inverses modulo regularizing operators, are pseudodifferential and this was one of the motivations to consider such classes.

The *principal symbol* $\sigma_m(A)$ of a pseudodifferential operator A of order m is defined as the class of a in $S^m(U \times \mathbb{R}^n)/S^{m-1}(U \times \mathbb{R}^n)$, that is, the leading term in the expansion of a as a classical symbol. It is positively homogeneous of order m on ξ and smooth for $\xi \neq 0$. For any m, the class of such symbols can be identified with $C^\infty(U \times S^{n-1})$, where $S^{n-1} = \{\xi \in \mathbb{R}^n \ : \ \|\xi\| = 1\}$.

Now we define pseudodifferential operators on sections of vector bundles. Locally, these are just vector valued functions on open sets of $\mathbb{R}^n$. Let M be a smooth Riemannian manifold, with a positive smooth measure, and let E, F be complex vector bundles over M, endowed with hermitian structures. We define a pseudodifferential operator operator of order m $P : C_c^\infty(M; E) \to C^\infty(M; F)$ requiring that on every coordinate chart of M trivializing E and F, such P yields a matrix of pseudodifferential operators of order m on U. Conversely, given order m pseudodifferential operators on a covering of M by coordinate charts, we can use a partition of unity to define a pseudodifferential operator of order m on M. (Note that the difference between two local representations yields a regularizing operator).

Remark 2.1 *Since pseudodifferential operators are not local, we need to check the pseudodifferential operator property on each chart, which is not*

[1] This means that $a - \sum_{k=0}^N a_{m-k} \in S^{m-1-N}(U \times \mathbb{R}^n)$.

very practical. If we have a priori that P is pseudolocal, which reduces to the kernel of P being smooth outside the diagonal, then it suffices to consider an atlas for M trivializing E, F.

The local principal symbols defined on $U \times \mathbb{R}^n$ transform to define globally defined functions on the cotangent bundle T^*M. We obtain a function on T^*M with values in $\mathrm{Hom}(E,F)$, that is a section $\sigma_m(P) : T^*M \to \mathrm{Hom}(\pi^*E, \pi^*F)$, with $\pi : T^*M \to M$ the projection, such that $\sigma_m(x,\xi) \in \mathrm{Hom}(E_x \to F_x)$. To our purposes, $\sigma_m(P)$ is best regarded as a bundle map

$$\sigma_m(P) : \pi^*E \to \pi^*F. \tag{3}$$

The globally defined symbol is smooth outside the zero-section and positively homogeneous of degree m on the fibers. We denote the class of such sections by $S^m(T^*M; E, F)/S^{m-1}(T^*M; E, F)$, and this space can be identified, for any m, with the space of smooth sections on $S^*M = \{(x,\xi) \in T^*M \ : \ \|\xi\| = 1\|\}$, the sphere bundle of T^M. Note that, again, a local representation only defines σ_m modulo regularizing operators. Using partitions of unity, it is easy to see that the *symbol map* defined by

$$\sigma_m : \Psi^m(M; E, F) \to S^m(T^*M; E, F)/S^{m-1}(T^*M; E, F), \quad P \mapsto \sigma_m(P), \tag{4}$$

is surjective and $\ker \sigma_m = \Psi^{m-1}(M; E, F)$.

¿From now on, we assume that M is a compact manifold, without boundary. Using the Hermitian structure in E, we define an inner product in $C^\infty(M; E)$

$$(u, v) := \int_M < u(x), v(x) >_E dx.$$

Let $L^2(M; E)$ be the completion of $C^\infty(M; E)$ with respect to the norm induced by the inner-product. We can also define Sobolev spaces $H^s(M; E)$, for $s \in \mathbb{R}$, using partitions of unit and the Sobolev norms in $\mathbb{R}^n$.

Theorem 2.2 *Let $P \in \Psi^m(M; E, F)$. Then P can be extended as a bounded operator $H^s(M; E, F) \to H^{s-m}(M; E, F)$, for any $s \in \mathbb{R}$. Moreover, if $m < 0$, then P is compact.*

(The compactness part of the result above is a consequence of the *Rellich lemma*.) In particular, an operator of order 0 is bounded $L^2(M; E, F) \to L^2(M; E, F)$. Also, if $P \in \Psi^{-\infty}(M; E, F)$, then P is compact and regularizing, so that in fact $\Psi^{-\infty}(M; E, F)$ coincides with the class of operators given by smooth kernels. One can prove also that the kernel of any pseudodifferential operator is smooth outside the diagonal.

We will now be concerned with properties of the principal symbol map (4). Recall that given a bounded operator $P : C^\infty(M; E) \to C^\infty(M; F)$, its adjoint is the operator $P^* : C^\infty(M; F) \to C^\infty(M; E)$ such that $(P^*u, v) = (u, Pv)$ for $u \in C^\infty(M; F)$, $v \in C^\infty(M; E)$. If $P \in \Psi^m(M; E, F)$ then $P^* \in \Psi^m(M; F, E)$ and

$$\sigma_m(P^*) = \sigma_m^*(P),$$

where σ_m denotes the principal symbol map. Also, if $P \in \Psi^m(M; E, F)$ and $Q \in \Psi^{m'}(M; F, G)$ then $QP \in \Psi^{m+m'}(M; E, G)$ and

$$\sigma_{m+m'}(QP) = \sigma_{m'}(Q)\sigma_m(P).$$

Let $P \in \Psi^m(M; E, F)$. Then P is *elliptic* if its symbol is invertible outside the zero-section, that is if $\sigma_m(P)(x, \xi)$ is an isomorphism for $\xi \neq 0$ (in particular, E and F have the same fiber-dimension).

We have the following:

Theorem 2.3 *If $P \in \Psi^m(M; E, F)$ is elliptic, then P is a Fredholm operator. Moreover, its index only depends on the homotopy class of the principal symbol $\sigma_m(P)$.*

Proof. Let P be elliptic, with principal symbol $p \in S^m(T^*M; E, F)$; then there is $q \in S^{-m}(T^*M; F, E)$ with $pq = I$. From the symbolic calculus, there is then a pseudodifferential operator Q of order $-m$ such that $\sigma_0(PQ-I) = 0$, that is, $PQ - I \in \Psi^{-1}(M; E)$ and hence $PQ - I$ is a compact operator.

For the second part, if $\sigma_m(P) = \sigma_m(Q)$, then also $\sigma_m(P_t) = \sigma_m(P)$, with $P_t = tP + (1-t)Q$, $0 \leq t \leq 1$, so that each P_t is elliptic, hence Fredholm. From the homotopy invariance of the index, we have then $\mathrm{ind}(P) = \mathrm{ind}(Q)$. □

We claim now that in what concerns the index of elliptic pseudodifferential operators , it suffices to consider operators of order 0. In fact, given an arbitrary pseudodifferential operator P of order m, we can always find an elliptic pseudodifferential operator Q of order 0, namely, an operator with principal symbol given by

$$\sigma_0(Q)(x, \xi) := \frac{\sigma_m(P)(x, \xi)}{\|\xi\|^m} = \sigma_m(P)\left(x, \frac{\xi}{\|\xi\|}\right),$$

such that $\mathrm{ind}(Q) = \mathrm{ind}(P)$. We will denote from now on by $\Psi(M; E, F)$ the class of pseudodifferential operators of order 0. Note that $\Psi(M; E, F) \subset \mathcal{B}(L^2(M; E), L^2(M; F)$. Endowing $\Psi(M; E, F)$ with the induced topology, and the space of symbols $C^\infty(S^*M; E, F)$ with the sup-norm topology, one proves that the symbol map is bounded. If $E = F$, then $\Psi(M;E)$ and $S^0(M;E)$ are *-algebras, and the symbol map leads to the following exact sequence of C^*-algebras:

$$0 \longrightarrow \mathcal{K}(M;E) \longrightarrow \overline{\Psi}(M,E) \xrightarrow{\sigma} C(S^*M;E) \longrightarrow 0 \quad (5)$$

where σ denotes the extended symbol map. As we shall see in §.7 in the framework of K-theory for C^*-algebras, the Fredholm index is closely related to the above sequence.

3 Examples

We will now see some examples of elliptic operators and their indices (for a more detailed account, check [13, 15]). Throughout this section, M will be a compact oriented manifold.

Example 3.1 *Let $\wedge^*(T^*M)$ denote the space of smooth complex forms on M regarded as an Hermitian vector bundle over M and denote by $C^\infty(\wedge^*(T^*M))$ the space of smooth sections. Consider the operator*

$$d : C^\infty(\wedge^*(T^*M)) \to C^\infty(\wedge^*(T^*M))$$

given by exterior derivative on forms, and let d^* be its formal adjoint; these are differential operators of order 1 *and* -1*, respectively. We let $D := d + d^*$ and $D^2 = dd^* + d^*d$ is the so-called Hodge laplacian. Self-adjointness of D yields that* $\ker D = \ker D^2$ *so that* $\ker D$ *coincides with the space $H = \oplus H^p$ of harmonic forms on M. It is a fundamental result from Hodge theory that, if M is compact, H^p is isomorphic to the p-th de Rham cohomology group,that is, to $H^p_{dR}(M) := \ker d_p / \operatorname{Im} d_{p-1}$, where $d_p : \wedge^p(T^*M) \to \wedge^{p+1}(T^*M)$.*

On the other hand, one can see that for $(x,\xi) \in T^*M$, $e \in (\wedge^*(T^*_xM))$,

$$\sigma_1(D)(x,\xi)e = \xi \wedge e - i_\xi e, \quad \sigma_0(D^2)(x,\xi) = -\|\xi\|^2.$$

where i_ξ is given by contraction with ξ. Hence, both D and D^2 are elliptic, and therefore Fredholm. In particular, we get that the spaces $H^p \cong H^p_{dR}(M)$ are finite dimensional.[2] *Note that, since D is self-adjoint,* $\operatorname{ind}(D) = 0$*. As we will see in the next two examples, we obtain operators with interesting indices if we consider gradings on $\wedge^*(T^*M)$.*

[2] The numbers $\beta_i := \dim H^p_{dR}(M)$ are called *Betti numbers.*

Example 3.2 *Consider the grading* $\wedge^*(T^*M) = \wedge^{even}(T^*M) \oplus \wedge^{odd}(T^*M)$*. Then* D *as in the previous example defines an operator*

$$D : C^\infty(\wedge^{even}(T^*M)) \to C^\infty(\wedge^{odd}(T^*M)),$$

which is again elliptic. Clearly, $\ker D$ *coincides with the even-dimensional harmonic forms, hence with* $\bigoplus_{p\ even} H^p_{dR}(M)$ *and, in the same way,* $\ker D^* = \bigoplus_{p\ odd} H^p_{dR}(M)$*, so that the Fredholm index is given by*

$$\operatorname{ind}(D) = \sum_{p\ even} \dim H^p_{dR}(M) - \sum_{p\ odd} \dim H^p_{dR}(M) = \chi(M),$$

the Euler characteristic of M*.*

In the even-dimensional case, the Atiyah-Singer index theorem reduces to the well-known fact that the Euler characteristic coincides with the Euler number (the Euler class of TM *computed on the fundamental class of the manifold).*

If $\dim M = 2$*, this result is known as the Gauss-Bonnet theorem and can be expressed by*

$$\int\int_M K dA = 2\pi\chi(M),$$

where is K *is the Gaussian curvature. One can also show using the (general) Atiyah-Singer index formula that the index of any differential operator on an odd-dimensional manifold is* 0*, generalizing the well-known fact that the Euler characteristic of an odd dimensional manifold is* 0*.*

Example 3.3 *Let now* $\dim M = 4k$*. Define a grading* $\wedge^*(T^*M) = \wedge^+(T^*M) \oplus \wedge^-(T^*M)$ *as follows: recall that the Hodge-* operator*

$$* : \wedge^p(T^*M) \to \wedge^{4k-p}(T^*M),\ *(e_1 \wedge ... \wedge e_p) = e_{p+1} \wedge ... \wedge e_{4k}$$

satisfies $*^2 = (-1)^p$*. If we let* $\omega_{\mathbb{C}}$ *be the complex volume element* $\omega_{\mathbb{C}} := (-1)^{k+\frac{p(p-1)}{2}} *$ *on* $\wedge^p(T^*M)$*, we have that* $\omega_{\mathbb{C}}^2 = 1$*. We let* $\wedge^\pm(T^*M)$ *be the* ± 1 *eigenspaces of* $\omega_{\mathbb{C}}$*. Consider again the operator* $D = d + d^*$*. In this case,* $\omega_{\mathbb{C}} D = -D\omega_{\mathbb{C}}$ *so that*

$$D : C^\infty(\wedge^+(T^*M)) \to C^\infty(\wedge^-(T^*M)).$$

is well defined. Again, D *is elliptic, hence Fredholm, and* $\ker D = \bigoplus_p H^p_+$*,* $\ker D^* = \bigoplus_p H^p_-$*, with* $H^p_\pm := H^p \cap C^\infty(\wedge^\pm(T^*M))$*. One can check that the Fredholm index is*

$$\operatorname{ind}(D) = \sum_p \dim H^p_+ - H^p_- = \dim H^{2k}_+ - \dim H^{2k}_- = \operatorname{sign}(M).$$

(Note that $\omega_{\mathbb{C}}$ is invariant on $H^p \oplus H^{4k-p}$ for $p < 2k$, so that the only contributions for the index that do not cancel out are the ones corresponding to H^{2k}). The signature of M, $\mathrm{sign}(M)$, is by definition, the signature of the bilinear form $(\phi, \psi) := \int_M \phi \wedge \psi$, $\phi, \psi \in H^{2k}$. Since $< \alpha, *\beta >= (\alpha, \beta)$, where $(\, , \,)$ denotes the inner product on $C^\infty(\wedge^*(T^*M))$, we see that H^{2k}_+ and H^{2k}_- are generated by $\{e_1, ..., e_m\}$ and $\{f_1, ..., f_n\}$, respectively, such that $< e_i, e_j >= \delta_{ij}$, $< f_i, f_j >= -\delta_{ij}$, $< e_i, f_j >= 0$, so that $\mathrm{sign}(M) = \dim H^{2k}_+ - \dim H^{2k}_- = \mathrm{ind}(D)$.

The index formula in this case was found by Hirzebruch, who showed that the signature is a topological invariant, and that it coincides with the $\widehat{L}$-genus of M:

$$\mathrm{sign}(M) = \widehat{L}(TM).$$

Example 3.4 *This example is the (twisted) complex analogue of Example 3.2. Let M be a complex manifold and $\wedge^{p,q}T^*M$ be the complex valued (p,q)-differential forms that is, the subspace of $\wedge^r T^*M$, $r = p + q$ generated by forms given locally by*

$$w = dz_{i_1} \wedge ... \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge ... \wedge d\overline{z}_{j_q},$$

where $z_1, ..., z_m$ are holomorphic coordinates. We have a differential operator

$$\overline{\partial} : \wedge^{p,q}T^*M \to \wedge^{p,q+1}T^*M, \quad \overline{\partial}(hw) \sum \frac{\partial h}{\partial \overline{z}_i} d\overline{z}_i \wedge w$$

with $h \in C^\infty(M)$.

Now let E be an holomorphic vector bundle over M, with an hermitian structure, and let $\wedge^{p,q}T^*M \otimes E$ be the complex valued (p,q)-differential forms with values in E. Then $\overline{\partial}$ can be extended to $\wedge^{p,q}T^*M \otimes E$ and $\overline{\partial}^*$ can be defined with respect to the induced hermitian structure on $\wedge^{p,q}T^*M \otimes E$. Consider $D = \overline{\partial} + \overline{\partial}^*$. We have as in Example 3.2 that D is elliptic, with $\ker D = \ker D^2$, the harmonic E-valued forms. Doulbeault's theorem now states that the spaces of E-valued harmonic (p,q)-forms is isomorphic to $H^{p,q}(M; E)$, the cohomology groups $\ker \overline{\partial}_{p,q} / \mathrm{Im}\, \overline{\partial}_{p,q-1}$. In particular, $H^{p,q}(M; E)$ is finite dimensional and we define the holomorphic Euler characteristic of E by $\chi(E) := \sum (-1)^q \dim H^{0,q}(M; E)$. (Note that $h^{p,q}(E) := \dim H^{p,q}(M; E)$ depend only on the complex structures of M and E, and so does $\chi(E)$.)

Considering a grading $\wedge^{0,q}T^*M \otimes E = \wedge^{0,even}T^*M \otimes E \oplus \wedge^{0,odd}T^*M \otimes E$, then $D : C^\infty(\wedge^{0,even}T^*M \otimes E) \oplus C^\infty(\wedge^{0,odd}T^*M \otimes E)$ is well defined and elliptic, and its Fredholm index is given by

$$\mathrm{ind}(D) = \sum (-1)^q \dim H^{0,q}(M; E) = \chi(E).$$

The Atiyah and Singer theorem states that

$$\chi(E) = \int_M ch(E) \cup Td(T_{\mathbb{C}}M)$$

with $ch(E)$ *the Chern character and* $Td(T_{\mathbb{C}}M)$ *the Todd class of* $T_{\mathbb{C}}M$ *(see §.6). It shows in particular that the Euler characteristic of* E *is indeed a topological invariant of* M *and* E*. The result above extends the Riemman-Roch-Hirzebruch theorem, known only for projective algebraic manifolds, to compact complex manifolds. In (real) dimension 2, with* E *a line bundle, it reduces to the Riemann-Roch theorem for Riemann surfaces.*

The examples given up to now are all instances of generalized Dirac operators (in the framework of Clifford modules). *The* Dirac operator will be considered in the next example.

Example 3.5 *Let* M *be a spin manifold of* $\dim M = 4k$ *and* S *be its complex spinor bundle, with* D *its associated Dirac operator (see [13]). As in Example 3.3, we have a volume element* $\omega_{\mathbb{C}} : S \to S$ *such that* $\omega_{\mathbb{C}}^2 = 1$*, and if we let* $S^{\pm}$ *be the* ± 1*- eigenspaces of* $\omega_{\mathbb{C}}$ *then* $D : C^{\infty}(S^+) \to C\infty(S^-)$ *is well defined. One has that* D *is elliptic, hence Fredholm. Atiyah and Singer showed that*

$$\operatorname{ind}(D) = \widehat{A}(M)$$

where $\widehat{A}(M)$ *is the* $\widehat{A}$*-genus of* M*. This is a priori a rational number and in general it is not an integer. The above formula shows that, however, for compact spin* $4k$*-manifolds that is indeed the case.*

In all the examples above, the formula for the Fredholm index was given depending on cohomology classes. The right framework for proving such formulas is, however, K-theory, and we will see in the next sections how this can be established. The step from K-theory to cohomology is then not so hard (§.6).

4 Symbol class

In this section, we define a class in K-theory associated to the symbol of a general elliptic pseudodifferential operator and show that the Fredholm index is well defined on such a class. We start with a short review of topological K-theory (see [1]).

Let X be a compact space. Then the set $V(X)$ of isomorphism classes of complex vector bundles over X is an abelian semigroup with direct sum. We let $K^0(X)$ be the Grothendieck group of $V(X)$, that is, the group of formal differences

$$K^0(X) = \{[E] - [F] \ : \ E, F \text{ vector bundles over } X\}$$

where $[E]$, $[F]$ are stable isomorphism classes, that is $[E] = [F]$ if and only if there exists G with $E \oplus G \cong F \oplus G$. We get that $[E] - [F] = [E'] - [F']$ if and only if there exist a vector bundle G such that

$$E \oplus E' \oplus G \cong F \oplus F' \oplus G.$$

In this way, we get a contravariant functor from the category of compact spaces to the category of abelian groups. As a trivial example, we see that, since two vector spaces are isomorphic if and only if have the same dimension, $K^0(\{x_0\}) = \mathbb{Z}$.

If X is a locally compact space, let X^+ be its one-point compactification. If $j : + \to X^+$ is the inclusion then $j^* : K^0(X^+) \to K^0(+) = \mathbb{Z}$ and we define

$$K^0(X) = \ker j^* \subset K^0(X^+).$$

The elements of $K^0(X)$ are now given by formal differences $[E]-[F]$ of vector bundles E and F over X that are trivial and isomorphic at infinity, that is, outside a compact subset of X. We again get a functor from the category of locally compact spaces with proper maps to abelian groups. It is easy to check that it is homotopy invariant.

Note that since every vector bundle over a compact space can be complemented,[3] we can always write an arbitrary element of $K^0(X)$ as $[E] - [\theta^n]$, where θ^n is the trivial n-dimensional bundle over X and E is trivial outside a compact.

Let $U \subset X$ be open. Then we have a map

$$j^* : K^0(U) \to K^0(X) \tag{6}$$

induced by the collapsing map $j : X^+ \to X^+/(X^+ - U)$. Also, given $Y \subset X$ closed,if $i : Y \to X$ is the inclusion, we have an exact sequence given by

$$K^0(X \backslash Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y). \tag{7}$$

Define now

$$K^{-n}(X) = K^0(X \times \mathbb{R}^n)$$

[3]In fact, every vector bundle over a not necessarily compact *manifold* has a complement in some trivial bundle - for each E, one always has a finite cover trivializing E.

The exact sequence (7) fits into an long exact sequence infinite to the left

$$\begin{aligned} ...K^{-2}(Y) &\xrightarrow{\delta} K^{-1}(X\backslash Y) \xrightarrow{j^*} K^{-1}(X) \xrightarrow{i^*} K^{-1}(Y) \\ &\xrightarrow{\delta} K^{0}(X\backslash Y) \xrightarrow{j^*} K^{0}(X) \xrightarrow{i^*} K^{0}(Y). \end{aligned} \tag{8}$$

where δ is the connecting map. The main feature of K-theory is the fundamental *Bott periodicity theorem* that states that there in fact only two K-groups: K^0 and K^{-1}.

Theorem 4.1 (Bott) $K^{-n+2}(X) \cong K^{-n}(X)$.

The above isomorphism is given by cup product with so called *Bott class* $\beta \in K^0(\mathbb{R}^2)$:

$$K^0(X \times \mathbb{R}^2) \to K^0(X), \ \alpha \to \alpha \cup \beta$$

where $\alpha \cup \beta = \pi_1^*\alpha \otimes \pi_2^*\beta$, with π_i, $i = 1, 2$, the projections onto the first and second component, respectively. The Bott class β is given by

$$\beta := [H^1] - [\theta]$$

H^1 the dual Hopft bundle over $P^1(\mathbb{C}) \cong \mathbb{R}^2 \cup \infty$ and θ the trivial bundle. For closed $Y \subset X$, using the isomorphism $K^{-2}(Y) \cong K^0(Y)$, the exact sequence (8) becomes a cyclic 6-term exact sequence.

For a locally compact space X, there is an alternative description of $K^0(X)$ that is of interest to index theory. An element $[E] - [F] \in K^0(X)$, with E, F vector bundles over X that are isomorphic (and trivial) on a neighborhood of infinity, represents a class $[E, F, \alpha]$, with $\alpha : E \to F$ a smooth homomorphism that is bijective outside a compact set. Two triples are said to be equivalent $[E, F, \alpha] = [E', F', \alpha']$ if and only if there are vector bundles G and H such that

$$(E, F, \alpha) \oplus (G, G, I_G) \cong (E', F', \alpha') \oplus (H, H, I_H)$$

that is, if $E \oplus G \cong E' \oplus H$, $F \oplus G \cong F' \oplus H$ and these isomorphisms behave well with respect to $\alpha \oplus I_G$ and $\alpha' \oplus I_H$. The set of equivalence classes of triples is a group under the direct sum (the identity is given by $[P, P, I_p]$) and the map

$$[E] - [F] \to [E, F, \alpha]$$

is well defined and an injective group homomorphism. We can see that this map is also surjective, at least when X is a manifold, since in this case any vector bundle over X can be complemented in some θ^n, (for the general case, see [1, 17]). We have then $[E, F, \alpha] = [H, \theta^n, \beta]$, with $H = E \oplus G$,

$F \oplus G = \theta^n$ and $\beta = \alpha \oplus I_G$. Moreover, we can assume that β is the identity outside a compact, so that H can be extended to X^+ and in this case, $[H] - [\theta^n] \in K^0(X)$.

Now let M be a compact manifold, E and F vector bundles over M, and P be an elliptic, order 0 pseudodifferential operator with symbol $\sigma : \pi^*E \to \pi^*F$. Since P is elliptic, σ is an isomorphism outside the zero-section, that is, outside a compact subset of T^*M, so that

$$[\sigma(P)] := [\pi^*E, \pi^*F, \sigma] \in K^0(T^*M)$$

is well defined. We call $[\sigma(P)]$ the *symbol class* of P.

In §.1 we saw that the Fredholm index is homotopy invariant, and in §.2 that, for elliptic pseudodifferential operators, it only depends on the symbol class (Theorem 2.3). Now we have the following.

Proposition 4.2 *The index of an elliptic pseudodifferential operator is well defined on the K-theory class of its symbol.*

Proof. If two operators P, Q have the same symbol class,

$$[\pi^*E, \pi^*F, \sigma(P)] = [\pi^*E', \pi^*F', \sigma(Q)],$$

then there exist G, H vector bundles over T^*M such that

$$(\pi^*E \oplus G, \pi^*F \oplus G, \sigma(P) \oplus I_G) \cong (\pi^*E' \oplus H, \pi^*F' \oplus H, \sigma(Q) \oplus I_H) \Leftrightarrow$$

$$(\pi^*(E \oplus G_0), \pi^*(F \oplus G_0), \sigma(P) \oplus \pi^* I_{G_0}) \cong (\pi^*(E' \oplus H_0), \pi^*(F' \oplus H_0), \sigma(Q) \oplus \pi^* I_{H_0}),$$

writing $G = \pi^*G_0$, $H = \pi^*H_0$, with G_0, H_0 vector bundles over M. If now I_{G_0} denotes the identity operator $C^\infty(M; G_0) \to C^\infty(M; G_0)$, then $P \oplus I_{G_0}$ is also an elliptic operator and clearly $\mathrm{ind}(P \oplus I_{G_0}) = \mathrm{ind}(P)$. We have also $\mathrm{ind}(Q) = \mathrm{ind}(Q \oplus I_{H_0})$. On the other hand it is easy to see, using the homotopy invariance of the index, that the indices of $P \oplus I_{G_0}$ and $Q \oplus I_{H_0}$ coincide, so that $\mathrm{ind}(P) = \mathrm{ind}(Q)$.

□

Now we check that, for a compact manifold M, $K^0(TM)$ is exhausted by symbol classes of pseudodifferential operators on M,[4]. Since the symbol map is surjective, it suffices to check that every class in $K^0(T^*M)$ is of the form $[\pi^*E, \pi^*F, \alpha]$, where α is positively homogeneous of degree 0 and an isomorphism outside a compact. (We identify here the categories of smooth and topological vector bundles.)

[4] This is not true if we consider only differential operators.

Let $[E, \theta^n, \beta] \in K^0(T^*M)$ such that β is the identity on $T^*M - K$, for some compact K. Note that there is an isomorphism $f : E \cong \pi^* E_0$, for $E_0 := E_{|M}$, and we can assume that f is the identity on $E_{|M-L} = (M - L) \times \mathbb{C}^n$, where $L \supset \pi(K)$ is open, relatively compact. We have then

$$[E, \theta^n, \beta] = [\pi^* E_0, \theta^n, \beta \circ f^{-1}],$$

where $\beta \circ f^{-1}$ is an isomorphism outside K (the identity on $TM - \pi^{-1}(L)$). If we deform $\beta \circ f^{-1}$ to be positively homogeneous, we get the symbol of a pseudodifferential operator of order 0.

We conclude then that the Fredholm index defines an homomorphism of K-groups:

$$\text{an-ind} : K^0(T^*M) \to \mathbb{Z}, \; [\sigma(P)] \mapsto \text{ind}(P). \tag{9}$$

The index formula computes this map depending on topological invariants of the manifold M and on the symbol class of P.

5 The index formula in K-theory

Let E be a complex vector bundle over a locally compact space X; then $K^0(E)$ is a $K^0(X)$-module, with $v \cup a := v \otimes \pi^*(a)$. A fundamental result by Thom states that $K^0(E)$ is K-oriented, that is, that there exists a class $\lambda_E \in K^0(E)$ that generates $K^0(E)$ as a $K^0(X)$-module:

$$\rho_E : K^0(X) \to K^0(E), \quad a \mapsto \lambda_E \cup a$$

is the Thom isomorphism. When E is the trivial bundle $E = \mathbb{C}$, this is the Bott isomorphism theorem. The class λ_E is induced by the complex by $\pi^* \wedge^* (E)$ defined as the exterior algebra of E (as a vector bundle over E). If X is compact, we have, in our alternative definition of $K^0(E)$, that

$$\lambda_E := [\pi^* \wedge^{even} (E), \pi^* \wedge^{odd} (E), \alpha] \in K^0(E),$$

where $\alpha_e(w) = e \wedge w - e^*(w)$, for $e \in E$, e^* its dual element, and $w \in \wedge^{even}(E_e)$.

Let M be a compact manifold and $i : M \to \mathbb{R}^n$ a smooth embedding, with normal bundle N. We denote also by $i : T^*M \to \mathbb{R}^{2n}$ the induced embedding on tangent bundles, which has normal bundle $N \oplus N$. Now, $N \oplus N$ can be given the structure of a complex bundle so that we have the Thom isomorphism

$$\rho : K^0(T^*M) \to K^0(N \oplus N). \tag{10}$$

On the other hand, recall that the normal bundle can be identified with an open tubular neighborhood of T^*M in $\mathbb{R}^{2n}$ so that we have a map

$$h : K^0(N \oplus N) \to K^0(\mathbb{R}^{2n}) \tag{11}$$

associated to the open inclusion (as in (6)). We define the *push-forward map* as

$$i_! : K^0(T^*M) \to K^0(\mathbb{R}^{2n}), \quad i_! := h \circ \rho. \tag{12}$$

This map is independent of the tubular neighborhood chosen. The transitivity of the Thom isomorphism yields that $(i \circ j)_! = i_! \circ j_!$, for two embeddings i, j. Note that considering the inclusion of a point $i : P \to \mathbb{R}^n$, we get that $i_! = \beta$, the Bott isomorphism. We have finally:

Theorem 5.1 (Atiyah-Singer) *Let P be an elliptic pseudodifferential operator on a compact manifold M without boundary, and let $\sigma \in K^0(T^*M)$ denote its symbol class. Then*

$$\mathrm{ind}(P) = \beta^{-1} \circ i_!(\sigma).$$

where $i_!$ is the push-forward map induced by an embedding $i : M \to \mathbb{R}^n$ and β denotes the Bott isomorphism.

(We can check directly that the left-hand side, the so-called topological index, is independent of the embedding i.)

Remark 5.2 *Note that the topological index is also well defined when M is not compact. In fact, one can also consider index maps on non-compact manifolds (see also the excision property (13) below). In this case, one computes the Fredholm index of operators that are multiplication at infinity, so that the symbols are constant on the fibers outside a compact and still define a K-theory class. In this context, the index theorem on $\mathbb{R}^n$ for elliptic operators that are multiplication at infinity is*

$$\mathrm{ind}(P) = \beta^{-1}(\sigma)$$

where β is the Bott isomorphism.

A proof of Theorem 5.1 can be found in [3]; we will sketch it below. It relies on two properties of the index map that characterize it uniquely:

1. if $M = \{x_0\}$ then ind is the identity;

2. if $i : M \to X$ is an embedding, then $\mathrm{ind}_X \circ i_! = \mathrm{ind}_M$.

In fact, if we have a family of maps $f_M : K^0(TM) \to \mathbb{Z}$ satisfying 1. and 2., and $i : M \to \mathbb{R}^n$ is an embedding, then the following diagram commutes:

$$\begin{array}{ccccc} K^0(TM) & \xrightarrow{i_!} & K^0(T\mathbb{R}^n) & \xrightarrow{\beta^{-1}} & K^0(\{pt\}) \\ {\scriptstyle f_M}\downarrow & & {\scriptstyle f_{\mathbb{R}^n}}\downarrow & & {\scriptstyle f_{\{pt\}}=id}\downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}). \end{array}$$

and we must have $f = \beta^{-1} \circ i_!$.

Now, while 1. is trivially checked for the index map (9), property 2. is indeed the core of this proof of the index theorem. Once we have set that the index is functorial with respect to push-forward maps, we reduce the computation of the index of arbitrary elliptic operators to those on spheres (in fact, on a point).

Note that for an open embedding $i : U \to M$, we get $i_! = h$, as in (11). Also, if $i : M \to V$ is the zero-section embedding in a complex vector bundle V, then $i_!$ coincides with the Thom isomorphism ρ_V. The proof of 2. is made in two steps: first one establishes *excision*, showing that, if i_1, i_2 are open embeddings $U \to M$ and, $h_1, h_2 : K^0(TU) \to K^0(TM)$ are the induced extension maps, then

$$\operatorname{ind} \circ h_1 = \operatorname{ind} \circ h_2. \tag{13}$$

Note that then the index is well defined also on open subsets $\operatorname{ind} : K^0(TU) \to \mathbb{Z}$, by picking a compactification of M[5]. We get by definition, $\operatorname{ind} \circ h = \operatorname{ind}$ and in this case

$$\operatorname{ind} \circ i_! = \operatorname{ind} \circ h \circ \rho = \operatorname{ind} \circ \rho.$$

We must study then the behavior of the index with respect to the Thom isomorphism. For this, Atiyah and Singer established the so called *multiplicativity property* of the index. In fact, the multiplicativity property will yield 2. when $i : M \to V$ is the zero section embedding in a *real* vector bundle over M, and the fact that the index is invariant with respect to the Thom isomorphism will stem as a consequence. In its simplest form, the multiplicativity property is

$$\operatorname{ind}^{M\times F}(ab) = \operatorname{ind}^M(a) \operatorname{ind}^F(b) \tag{14}$$

where $a \in K^0(TM)$, $b \in K^0(TF)$ and $ab \in K^0(T(M \times F))$ is defined through the external product. Ones need however a stronger version of (14) , which makes use of equivariant K-theory. Briefly, if G is a compact Lie group

[5]This map gives coincides with the Fredholm index for operators on U that are multiplication outside a compact.

acting on M, we consider G-vector bundles over M and define $K_G^0(M)$ in a similar way to the non-equivariant case. Most results follow trivially, and we have again a Bott isomorphism (this takes some extra care - see [17]) $\beta : K_G^0(X) \to K_G^0(X \times \mathbb{R}^n)$. Note that the K_G-group of a point is now $R(G)$, the representation ring of G. The index theorem (5.1) also holds in this case, where now we compute the index of G-equivariant operators and $\operatorname{ind} : K_G^0(TM) \to R(G)$.

Atiyah and Singer defined a product $K_G^0(TX) \times K_{G\times H}^0(TF) \to K_G^0(TY)$, with $Y = P \times_H F$ for some principal H-bundle P. The multiplicativity property now states that, for $a \in K_G^0(TX)$, $b \in K_{G\times H}^0(TF)$ such that $\operatorname{ind}_{G\times H}^F(b) \in R(G)$,

$$\operatorname{ind}_G^Y(ab) = \operatorname{ind}_G^M(a) \operatorname{ind}_{G\times H}^F(b). \tag{15}$$

The point is that if $j : P \to \mathbb{R}^n$ inclusion of a point, and $j_! : R(O(n)) \to K_{O(n)}(T\mathbb{R}^n)$, then $\operatorname{ind}_{O(n)} j_!(1) = 1$. Hence, if V is a real vector bundle over M, with $V = P \times_{O(n)} \mathbb{R}^n$, then for $a \in K_G^0(TM)$, and $b = j_!(1) \in K_{O(n)}(T\mathbb{R}^n)$ we get

$$\operatorname{ind}^V(ab) = \operatorname{ind}(a) \operatorname{ind}_{O(n)}(b) = \operatorname{ind}(a).$$

and $ab = i_!(a)$, with $i : M \to V$ the zero-section. Using excision, we get 2. in general.

The proof of the index theorem is then 'reduced' to checking the excision and multiplicativity properties for the Fredholm index, by finding suitable operators, together with showing that $\operatorname{ind}_{O(n)} j_!(1) = 1$.

At this point, we should note that there are many different approaches to prove the index theorem (we will see one in the context of noncommutative geometry in **??**). One of the advantages of the 'embedding proof' presented here is that it generalizes easily to families of operators (this is proved in [5]).

Remark 5.3 *Let $P = (P_t)_{t\in Y}$ be a continuous family of elliptic pseudodifferential operator $P_t : C^\infty(X, E) \to C^\infty(X, F)$, parametrized by some Hausdorff space Y, that is, such that in local representations the coefficients are jointly continuous in $t \in Y$ and $x \in X$.* [6] *If* $\dim\ker P_t$, *and* $\dim\operatorname{coker} P_t$ *are locally constant then* $\ker P$ *and* $\operatorname{coker} P$ *are vector bundles over Y and we get a class* $[\ker P] - [\operatorname{coker} P] \in K^0(Y)$. *In general, the dimensions will not be locally constant, but we can stabilize P so that we still get a well defined class in $K^0(Y)$; this will be, by definition, the (analytic) index of the family P. One*

[6] Actually, one considers more general families: we allow X and E, F to twist over Y, considering fiber bundles $\mathcal{X}$, with structure group $\operatorname{Diff}(X)$ and fiber X, and $\mathcal{E}$, $\mathcal{F}$ with structure groups $\operatorname{Diff}(E; X)$ and $\operatorname{Diff}(F; X)$ and fibers E and F (where $\operatorname{Diff}(E; X)$ are diffeomorphisms of E that carry fibers to fibers linearly).

can check that this index is invariant under homotopy, using a fundamental result by Atiyah that states that $K^0(Y) = [Y, \mathcal{F}]$. *Moreover, taking a map* $i : Y \times X \to Y \times \mathbb{R}^n$ *that restricts to embeddings* $i_t : X \to \mathbb{R}^n$, *for each* $t \in Y$, *we get a push-forward map* $i_! : K^0(Y \times TX) \to K^0(Y \times \mathbb{R}^{2n})$. *Using the Bott isomorphism* β_Y *to identify* $K^0(Y \times \mathbb{R}^{2n})$ *with* $K^0(Y)$, *one can then show that*

$$\mathrm{ind}(P) = \beta_Y^{-1} \circ i_!.$$

for P *a family of elliptic operators.*

6 The index formula in cohomology

We now give an equivalent formulation of Theorem 5.1 using cohomology; this was shown in [4]. For a locally compact space X, let $H^*(X)$ denote its rational cohomology with compact supports.[7]

To each complex vector bundle E over a manifold M, let $c(E) := c_1(E) + ... + c_n(E) \in H^{\mathrm{ev}}(M)$ denote its total Chern class. We can write formally $c(E) = \Pi_{k=1}^n (1 + x_k)$ such that $x_k \in H^2(M)$ and $c_k(E)$ is the symmetric function on the x_k's, and define

$$ch(E) := e^{x_1} + ...e^{x_n} = n + \sum_{j=1}^n x_j + ... \frac{1}{k!} \sum_{j=1}^n x_j^k + ...$$

We have $ch(E) \in H^{ev}(M)$ satisfying $ch(E \oplus F) = ch(E) + ch(F)$ and $ch(E \otimes F) = ch(E)ch(F)$. The Chern character is defined as the ring homomorphism

$$\mathrm{ch} : K^0(M) \to H^{ev}(M),\ [E] - [F] \mapsto ch(E) - ch(F).$$

If M is compact, ch defines a rational isomorphism $K^*(M) \otimes \mathbb{Q} \cong H^*(M)$.

Now let M and E be oriented, and let $\psi : H^*(M) \to H^*(E)$ denote the Thom isomorphism in cohomology, $\rho : K^*(M) \to K^*(E)$ denote the Thom isomorphism in K-theory. There is a class $\mu(E) \in H^{\mathrm{ev}}(M)$ such that for $a \in K^*(M)$,

$$\psi^{-1}\, \mathrm{ch}\, \rho(a) = \mathrm{ch}(a)\mu(E),$$

We find that $\mu(E) = (-1)^n\, \mathrm{Td}(\overline{E})^{-1}$, where $\overline{E}$ denotes the conjugate bundle of E and Td denotes the Todd class.

Finally, recall that for an oriented manifold M, the fundamental class is the homology class $[M] \in H_n(M; \mathbb{Z})$ defined by the orientation that generates

[7] Since we are dealing only with manifolds, it does not matter what cohomology theory we pick; one can think of real coefficients and de Rham cohomology.

$H_n(M;\mathbb{Z})$. In real coefficients, $u[M] = \int_M u$, for $u \in H^n(M;\mathbb{R})$. We give the tangent space TM the orientation of an almost complex manifold, identifying $T(TM) \cong \pi^*TM \oplus \pi^*TM \cong \pi^*TM \otimes \mathbb{C}$.

Theorem 6.1 (Atiyah-Singer) *Let P be an elliptic pseudodifferential operator on a compact manifold M without boundary, and let $\sigma \in K^0(TM)$ denote its symbol class, $\pi : TM \to M$ the projection. Then*

$$\operatorname{ind}(P) = (-1)^n \operatorname{ch}(\sigma) \cup \operatorname{Td}(T_{\mathbb{C}}M)[TM].$$

More suggestively, one can write, borrowing the notation from de Rham cohomology,

$$\operatorname{ind}(P) = (-1)^n \int_{TM} \operatorname{ch}(\sigma) \cup \operatorname{Td}(T_{\mathbb{C}}M).$$

The proof of the above formula can be found in [4], and goes roughly as follows. The main point is noting that for trivial even-dimensional bundles, the Thom isomorphisms and the Chern character do commute, so that we have that, for $v \in K^0(TP) = \mathbb{Z}$

$$\operatorname{ch} \beta(v)[T\mathbb{R}^n] = \psi \operatorname{ch}(v)[T\mathbb{R}^n] = \operatorname{ch}(v)[TP] = v,$$

writing the Thom isomorphism as $\psi_V(a)[V] = a[X]$, for $V \to X$ an oriented vector bundle. Hence, the inverse of the Bott isomorphism β is given by

$$\beta^{-1}(w) = \operatorname{ch}(w)[T\mathbb{R}^n],$$

for $w \in K^0(pt) = \mathbb{Z}$, and we have for $a \in K^0(TM)$, and $i : M \to \mathbb{R}^n$ an embedding,

$$\operatorname{ind}(a) = \beta^{-1} \circ i_!(a) = \operatorname{ch}(i_!(a))[T\mathbb{R}^n] = \operatorname{ch}(h \circ \rho(a))[T\mathbb{R}^n].$$

Using naturality of ch with respect to the extension map h, we get finally

$$\operatorname{ind}(a) = \operatorname{ch}(\rho(a))[TN] = \psi(\operatorname{ch}(a)\mu(TN))[TN] = \operatorname{ch}(a)\mu(TN)[TM].$$

Since $\mu(TN \oplus T(TM)) = \mu(T\mathbb{R}^n) = 1$, we have

$$\mu(TN) = \mu(T(TM))^{-1} = (-1)^n \operatorname{Td}(\pi^*TM \otimes \mathbb{C}),$$

where we identified $T(TM) \cong \pi^*TM \otimes \mathbb{C} \cong \overline{\pi^*TM \otimes \mathbb{C}}$. Atiyah and Singer's formula then follows.

For the so-called Hodge operator in Example 3.2, the right-hand side is just the Euler class; for the signature operator of Example 3.3 one gets Hirzebruch's L-genus, and the signature theorem follows. For the Dirac operator on spin manifolds, we get the $\widehat{A}$-genus. In particular, we get that the L-genus is an integer, and the same for the $\widehat{A}$- genus of spin manifolds.

For the Doulbeault operator of Example 3.4, the theorem above generalizes to compact manifolds Hirzebruch's Rieman-Roch theorem for projective algebraic varieties: the holomorphic Euler characteristic of a holomorphic bundle E is given by

$$\chi(M;E) = \mathrm{ch}(E)\,\mathrm{Td}(TM)[M].$$

7 The index in K-theory for C^*-algebras

In this section, we will place the index in the framework of noncommutative geometry. For a start, we have that the Fredholm index of elliptic pseudodifferential operators coincides with the connecting map in a long exact sequence in K-theory for C^*-algebras. We will see different forms of associating K-theory classes to elliptic operators, and define the Fredholm index using asymptotic morphisms and deformations of C^*-algebras.

We start with reviewing a few definitions and results from K-theory for C^*-algebras (see [8, 12]). It is well known, from Gelfand-Naimark theory, that for compact spaces X, Y, we have $X \cong Y$ iff $C(X) \cong C(Y)$; we will define a covariant functor K_0 from the category of C^*-algebras to that of abelian groups such that $K_0(C(X)) \cong K^0(X)$.

Note that since every vector bundle E over a compact space X can be complemented in a trivial bundle, we can always identify E with the image $p\theta^n$ of a homomorphism $p : \theta^n \to \theta^n$ such that $p^2 = p$, that is, of an idempotent map in the algebra $C(X) \otimes M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the $n \times n$ complex matrices. Conversely, given such a map, its image is locally trivial, hence it defines a vector bundle over X. In this respect, vector bundles over X can be replaced by projections on matrix algebras over $C(X)$.

Remark 7.1 *This leads to the Serre-Swan theorem that states that the category of vector bundles over a compact space X is equivalent to that of finitely generated projective $C(X)$-modules, associating to a vector bundle E the module of smooth sections $C^\infty(X;E)$ (which is finitely generated and projective since for a trivial bundle we get a finite rank module).*

One checks that two vector bundles are isomorphic, $p\theta^n \cong q\theta^m$, iff the associated idempotents are algebraically equivalent, that is, if there are u, v such that $p = uv$ and $q = vu$.

Let A be a C^*-algebra and assume for now that A has an identity. We will be considering projections in A, that is, self-adjoint idempotents. In this context, two projections p, q are algebraically equivalent iff there is $u \in A$ such that

$$p = u^*u, \quad q = uu^*;$$

such an u is said to be a partial isometry.[8] It is useful sometimes to use different equivalence relations on projections. Two projections p, q are unitarily equivalent, $p \sim_u q$ if there is an unitary u such that $q = upu^*$, and homotopic, $p \sim_h q$, if there is a continuous path of projections from p to q. We have

- $p \sim_h q \Rightarrow p \sim_u q \Rightarrow p \sim q$ in A;
- $p \sim q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$;
- $p \sim_u q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$.

Let $P_n(A)$ denote the set of projections in $M_n(A)$ and $P_\infty(A) := \lim_\rightarrow P_n(A)$, with inclusions $M_n(A) \rightarrow M_k(A)$, $n < k$ in the upper left corner. All three equivalence relations defined above coincide in $P_\infty(A)$. Take $P_\infty(A)/\sim$, the set of equivalence classes of projections; if

$$p \oplus q := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix},$$

direct sum is well defined on equivalence classes and we have an abelian semigroup. We define $K_0(A)$ to be the Grothendieck group associated to $P_\infty(A)/\sim$, so that elements of $K_0(A)$ are given by formal differences

$$\{[p] - [q] \ : \ p, q \in P_n(A), n \in \mathbb{N}\},$$

and $[p] = [q]$ iff there is $r \in P_n(A)$ with $p \oplus r \sim q \oplus r$, or equivalently $p \oplus 1_k \sim q \oplus 1_k$, where 1_k is the identity in $M_k(A)$. Clearly, $K_0(A) = K_0(M_n(A))$, for any $n \in \mathbb{N}$.

Every $*$-homomorphism $\phi : A \rightarrow B$, for unital A, B, induces a semigroup homomorphism $\phi_* : (P_\infty(A)/\sim) \rightarrow (P_\infty(B)/\sim)$. We obtain in this way a covariant functor from the category of unital C^*-algebras to the category of abelian groups. One easily checks that it is homotopy invariant.

Of course, if A is commutative, that is, $A = C(X)$ for some compact space X, then $K_0(A) = K^0(X)$. For instance, if $A = \mathbb{C}$, then $K_0(\mathbb{C}) = K_0(M_n(\mathbb{C})) = \mathbb{Z}$. An explicit isomorphism is induced by the map $p \mapsto$

[8] Any idempotent is algebraically equivalent to a projection

$\dim p\mathbb{C}^n$, for $p \in P_\infty(M_n(\mathbb{C}))$. For a separable Hilbert space H, we have also that $p \sim q$ iff $\dim pH^n = \dim qH^n$, for $p, q \in P_n(B(H))$ (identified with $P(B(H^n))$), so that $(P_\infty(B(H))/\sim) \cong \mathbb{N} \cup \infty$. Note that $n + \infty = m + \infty = \infty$, hence $n = m$ in $K_0(B(H))$, that is, we have $K_0(B(H)) = 0$.

To define K_0 also for non-unital algebras, we let A^+ be the unitization of the C^*-algebra A, that is, $A^+ := A \times \mathbb{C}$, with $(a, \alpha)(b, \beta) := (ab + \beta a + \alpha b, \alpha\beta)$ and unit $1_+ = (0, 1)$, so that $A^+ = \{a + \alpha 1_+ \ : \ a \in A, \alpha \in \mathbb{C}\}$. Let $\pi : A^+ \to \mathbb{C}$ be the projection; then

$$K_0(A) := \ker \pi^* \subset K_0(A^+).$$

The elements of $K_0(A)$ are now given by formal differences $[p] - [q]$, where $p, q \in P_n(A^+)$ are such that $p - q \in M_n(A)$. Noting that $[q] + [1_n - q] = [1_n]$, with 1_n the identity in $M_n(A^+)$, we have that $[p] - [q] = [p] - [1_n] + [1_n - q] = [p \oplus (1_n - q)] - [1_n]$, so that one can always write arbitrary elements of $K_0(A)$ as formal differences

$$K_0(A) = \{[p] - [1_n] \ : \ p \in P_n(A^+), \ p - 1_n \in M_n(A)\}.$$

A $*$-homomorphism $\phi : A \to B$ extends to an unit preserving $*$-homomorphism $\phi^+ : A^+ \to B^+$, and we get again a homotopy invariant functor from the category of C^*-algebras to that of abelian groups. The functor K_0 is half-exact: given an exact sequence $0 \to I \to A \to B \to 0$, we have that

$$K_0(I) \xrightarrow{\ \phi_*\ } K_0(A) \xrightarrow{\ \psi_*\ } K_0(B)$$

is exact. Also, the functor K_0 preserves direct limits, in the sense that

$$K_0(\varinjlim A_n) \cong \varinjlim K_0(A_n).$$

This yields an important stability property: let $\mathcal{K}$ denote the space of compact operators on some Hilbert space H; then we can write $\mathcal{K} = \lim_\to M_n(\mathbb{C})$ and we have that

$$K_0(\mathcal{K} \otimes A) = K_0(\varinjlim M_n(A)) \cong K_0(A).$$

In particular, $K_0(\mathcal{K}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$. An explicit isomorphism is given in this case $p \in P_n(\mathcal{K}) \mapsto \dim pH^n$ (note that a compact projection has finite rank).

Now we define the K_1-group: for a unitary C^*-algebra A, let $Gl_n(A)$ denote the group of invertible elements in $M_n(A)$, and $Gl_\infty(A) := \lim_\to Gl_n(A)$, with maps $Gl_n(A) \to Gl_k(A)$, $k > n$, given by $u \mapsto u \oplus 1_{k-n}$. We say that $u \sim v$ iff there is $k \in \mathbb{N}$ such that $u \oplus 1_{k-n} \sim_h v \oplus 1_{m-k}$, where $u \in Gl_n(A)$, $v \in Gl_m(A)$. For a an arbitrary C^*-algebra A, define now

$$K_1(A) := Gl_\infty(A^+)/\sim,$$

with $[u] + [v] := [u \oplus v]$. From the Whitehead lemma $[uv] = [vu] = [u \oplus v]$, so that $[uu^*] = [1_n] = 0$ and $K_1(A)$ is a group. We again get a half-exact, homotopy invariant functor from C^*-algebras to abelian groups, and one can check that it is also stable: $K_1(A \otimes \mathcal{K}) \cong K_1(A)$. If A is unital, $K_1(A) = Gl_\infty(A)/\sim$. We could have defined K_1 with unitaries: if we let $U_n(A)$ be the group of unitary elements in $M_n(A)$, and $U_\infty(A) := \lim_\to U_n(A)$, as above, then $U_n(A)$ is a retract of $Gl_n(A)$, through polar decomposition, so that $K_1(A) \cong U_\infty(A)/\sim$.

Since the unitary group of in $M_k(M_n(\mathbb{C}))$ is connected, for any k, n, we have that $K_1(M_n(\mathbb{C})) = 0$, for all $n \in \mathbb{N}$. As a consequence of stability, we have now $K_1(\mathcal{K}) \cong K_1(\mathbb{C}) \cong \mathbb{Z}$. Similarly, since $U_n(B(H))$ is connected, one has also that $K_1(B(H)) = 0$.

Letting $SA := C_0(\mathbb{R}, A)$ denote the suspension of A, one shows that $K_1(A) \cong K_0(SA)$. Noting that $C_0(\mathbb{R}, C_0(X)) \cong C_0(X \times \mathbb{R})$, we see that, in fact, for locally compact X, $K_1(C_0(X)) = K_0(X \times \mathbb{R}) = K^1(X)$.

Defining $K_n(A) := K_0(S^n A)$ we have that given an exact sequence $0 \to I \to A \to B \to 0$, there is a long exact sequence

$$\begin{aligned} ...K_2(B) \xrightarrow{\delta} K_1(I) \xrightarrow{i_*} K_1(A) \xrightarrow{\pi_*} K_1(B) \\ \xrightarrow{\delta} K_0(I) \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(B). \end{aligned} \tag{16}$$

where $\delta : K_{i+1}(B) \to K_i(I)$ is the so-called connecting map. For $[u] \in K_1(B)$, with $u \in Gl_n(B)$, one can always lift $u \oplus u^{-1}$ to an element $w \in Gl_{2n}(A)$ (that is, with $\pi(w) = u$). The map $\delta : K_1(B) \to K_0(I)$ is given by

$$\delta[u] := [w 1_n w^{-1}] - [1_n] \in K_0(I). \tag{17}$$

As in topological K-theory, the crucial result here is the Bott periodicity theorem that states that

$$K_2(A) = K_0(S^2 A) \cong K_0(A).$$

The group $K_0(C_0(\mathbb{R}^2))$ is generated by the Bott element

$$\beta := [p] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right], \text{ with } p(z) = \frac{1}{1 + z\bar{z}} \begin{pmatrix} z\bar{z} & z \\ \bar{z} & 1 \end{pmatrix},$$

identifying $\mathbb{R}^2$ with $\mathbb{C}$. (p corresponds to the tautological bundle H^1, generating $K^0(S^2)$). The Bott isomorphism can be written as $\beta_A : K_0(A) \to K_2(A)$ with

$$\beta(x) = \beta \times x,$$

where $[p] \times [q] := [p \otimes q] \in K_0(C(\mathbb{R}^2) \otimes A)$. As in the commutative case, the long exact sequence above becomes a cyclic 6-term exact sequence.

Consider now an Hilbert space H and the exact sequence

$$0 \longrightarrow \mathcal{K}(H) \longrightarrow B(H) \xrightarrow{\pi} B(H)/K(H) \longrightarrow 0 \qquad (18)$$

where $\mathcal{K}(H)$ are compact operators and $B(H)/K(H)$ is the Calkin algebra. Since $K_0(B(H)) = K_0(B(H)) = 0$ and $K_0(\mathcal{K}(H)) = \mathbb{Z}$, we have that the connecting map in the K-theory sequence induced by (18) is an isomorphism

$$\delta : K_1(B(H)/K(H)) \to \mathbb{Z}.$$

Now let $P \in B(H)$ be Fredholm; then it is invertible in $B(H)/K(H)$, hence it defines an element $[\pi(P)]_1$ in $K_1(B(H)/K(H))$. We have that the Fredholm index coincides with the connecting map in this sequence:

$$\mathrm{ind}(P) = \delta[\pi(P)]_1 \in K_0(\mathcal{K}) \cong \mathbb{Z}.$$

To check this, note first that we can always write $P = |P|V$, where $|P| = (P^*P)^{\frac{1}{2}} > 0$ and V is a partial isometry, so that V is Fredholm and $\mathrm{ind}(V) = \mathrm{ind}(P)$. One checks that $\pi(V)$ is a unitary and $[\pi(V)] = [\pi(P)]$. Now $\pi(V) \oplus \pi(V)^*$ lifts to $w = \begin{pmatrix} V & 1 - VV^* \\ 1 - V^*V & V^* \end{pmatrix}$, and from (17) we have

$$\delta[\pi(P)] = [w 1_n w^{-1}] - [1_n] = \left[\begin{pmatrix} VV^* & 0 \\ 0 & 1 - V^*V \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right].$$

Since VV^* and $1 - V^*V$ are the projections onto $\mathrm{Im}\, V$ and $\ker V$, respectively, so that $1 - VV^*$ is the projection onto $(\mathrm{Im}\, V)^\perp = \ker V^*$, we get then

$$\delta[\pi(P)] = \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 - V^*V \end{pmatrix}\right] - \left[\begin{pmatrix} 1 - VV^* & 0 \\ 0 & 0 \end{pmatrix}\right] = [1 - V^*V] - [1 - VV^*].$$

Under the isomorphism $K_0(\mathcal{K}(H)) \to \mathbb{Z}$, $[p] - [q] \mapsto \dim pH - \dim qH$, we have as claimed

$$\delta[\pi(P)] = \dim\ker V - \dim\ker V^* = \mathrm{ind}(V) = \mathrm{ind}(P).$$

More generally, naturality of the connecting map yields that for any subalgebra $\mathcal{E} \subset B(H)$ that fits into an exact sequence

$$0 \longrightarrow \mathcal{K}(H) \longrightarrow \mathcal{E} \longrightarrow C(X) \longrightarrow 0$$

(that is, such that the commutators in $\mathcal{E}$ are compact), the connecting map will give the Fredholm index. Now recall that for pseudodifferential operators of order 0 on a compact manifold M we do have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \overline{\Psi}(M) \xrightarrow{\sigma} C(S^*M) \longrightarrow 0,$$

and have now that for an elliptic pseudodifferential operator P, with invertible symbol $\sigma(P) \in C(S^*M)$,

$$\mathrm{ind}(P) = \delta[\sigma(P)]_1,$$

where $[\sigma(P)]_1 \in K_1(C(S^*M))$ is the class defined by $\sigma(P)$. This result also holds for operators acting on vector valued functions, where $[\sigma(P)] \in K_1(C(S^*M) \otimes \mathcal{M}_k(\mathbb{C})) = K_1(C^*M)$, and for operators in $\Psi(M;E)$, by complementing E in a trivial bundle.

The class $[\sigma(P)]_1 \in K_1(C(S^*M))$ relates to the symbol class as defined in §.4, $[\sigma(P)] \in K^0(T^*M) = K_0(C_0(T^*M))$, by the connecting map associated to the exact sequence $0 \to C_0(T^*M) \to C(B^*M) \to C(S^*M) \to 0$, where B^*M is the ball bundle, that is $[\sigma(P)] = \delta[\sigma(P)]_1$, $\delta : K_1(C(S^*M)) \to K_0(C_(T^*M))$.

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