II - REAL ANALYSIS

Contents

1 Measures

1.1 Jordan content in $\mathbb{R}^N$

Let I be an interval in $\mathbb{R}$. Then its 1-content is defined as $c_1(I) := b - a$ if I is bounded with endpoints a, b. If I is unbounded, we define $c_1(I) = +\infty$. More generally,

Definition 1.1.1. Let $R = I_1 \times ... \times I_N \subset \mathbb{R}^N$ be a N-rectangle, where $I_k \subset \mathbb{R}$ is an interval, $k = 1, ..., N$. Then the *N-content* of R is defined as

$$c_N(R) = c_1(I_1) \times ... \times c_1(I_N).$$

We always assume that if $c_1(I_k) = 0$ for some k, then $c_N(R) = 0$ (so the 2-content of a straightline in $\mathbb{R}^2$ is 0) and if $c_1(I_k) = \infty$ for some k, with $c_1(I_j) > 0$, $j \neq k$ then $c_N(R) = \infty$.

One important feature of the N-content (and as we shall see of measures of sets) is its *additivity*: if R_1, R_2 are *disjoint* rectangles then

$$c_N(R_1 \cup R_2) = c_N(R_1) + c_N(R_2).$$

This property gives us a way to extend the notion of content to finite unions of rectangles: we define

Definition 1.1.2. $\mathcal{U}(\mathbb{R}^N)$ is the class of sets given by finite unions of n-rectangles,

$\mathcal{E}(\mathbb{R}^N)$ is the class of sets given by finite unions of bounded n-rectangles, that is, bounded sets in $\mathcal{U}(\mathbb{R}^N)$. Elements of $\mathcal{E}(\mathbb{R}^N)$ are called *elementary* or *simple* sets.

Note that since $R \setminus S$ is a rectangle whenever R, S are rectangles, any finite union can assumed to be *disjoint*: if $U = \cup_{i=1}^{p} R_i$, then

$$U = \cup_{i=1}^{p} R_i', \quad R_1' = R_1,\ R_{k+1}' = R_{k+1} \setminus \cup_{i=1}^{k} R_i$$

where the rectangles R_k' are mutually disjoint. (This process of turning an arbitrary union into an union of disjoint sets will be used often.)

Definition 1.1.3. Let $U \in \mathcal{E}(\mathbb{R}^N)$ be such that $U = \cup_{i=1}^{p} R_i$, where R_i are disjoint rectangles. We define the *N-content* of U by

$$c_N(U) := \sum_{i=1}^{p} c_N(R_i).$$

Of course, one needs to show - see [NotesMR] - that the definition above does not depend on the partition of U into disjoint rectangles.

The class of simple sets is quite restrictive: e.g., balls and triangles are not simple sets. The point now is to consider sets that can be approximated, outer and inner, by simple sets. One can see that if $J \subset \mathbb{R}^N$ is bounded then there exist simple sets U, K such that $K \subset J \subset U$. Hence the following definition makes sense:

Definition 1.1.4. Let $J \subset \mathbb{R}^N$ be bounded. The *outer Jordan content* of J is defined as

$$\overline{c}_N(J) := \inf\{c_N(U) : U \supset J, U \in \mathcal{E}(\mathbb{R}^N)\}.$$

The *inner Jordan content* of J is defined as

$$\underline{c}_N(J) := \sup\{c_N(K) : K \subset J, K \in \mathcal{E}(\mathbb{R}^N)\}.$$

The bounded set J is said to be *Jordan measurable*, $J \in \mathcal{J}(\mathbb{R}^N)$, if $\underline{c}_N(J) = \overline{c}(J)$. In that case we define the *Jordan content* of J as

$$c_N(J) := \underline{c}_N(J) = \overline{c}(J).$$

Examples 1.1.5.

1. Finite sets are simple, hence Jordan measurable, and have content 0.

2. $D = \mathbb{Q} \cap [0, 1]$ not Jordan measurable, $\overline{c}(D) = 1$ and $\underline{c}(D) = 0$.

Useful criteria to show Jordan measurability, that relies mainly on the definition of sup and inf:

Proposition 1.1.6. *(i)* $J \in \mathcal{J}(\mathbb{R}^N)$ *if and only if for all* $\epsilon > 0$, *there exist simple sets* $U, K \in \mathcal{E}(\mathbb{R}^N)$ *such that*

$$K \subset J \subset U, \quad c_N(U \setminus K) < \epsilon$$

and in this case $c_N(J) \in]c_N(U) - \epsilon, c_N(K) + \epsilon[$.

(ii) $J \in \mathcal{J}(\mathbb{R}^N)$ *if and only if there exist simple sets* $U_n, K_n \in \mathcal{E}(\mathbb{R}^N)$ *such that*

$$K_n \subset J \subset U_n, \quad c_N(U_n \setminus K_n) \to 0$$

and in this case $c_N(J) = \lim c_N(K_n) = \lim c_N(U_n)$.

(iii) $J \in \mathcal{J}(\mathbb{R}^N)$ *with* $c_N(J) = 0$ *if and only if there is* $U \in \mathcal{E}(\mathbb{R}^N)$ *such that*

$$J \subset U, \quad c_N(U) < \epsilon$$

A set with $c_N(J) = 0$ is said to be a *null* set. Note that since $\underline{c}_N(J) \leq \overline{c}_N(J)$, any set with $\overline{c}_N(J) = 0$ is Jordan measurable (see also (iii)), in particular, if $N \in \mathcal{J}(\mathbb{R}^N)$ is a null-set and $J \subset N$ then J is Jordan measurable and $c_N(J) = 0$.

In (i), we can assume that U and K are open or closed without loss of generality (mainly because if U is simple, then also $intU$ and $\overline{U}$ are simple, and have the same content as U). In particular, taking K and open U closed in (i), we see that $U \setminus K$ covers the boundary ∂J of J and that $\partial J \in \mathcal{J}(\mathbb{R}^N)$ with $c_N(\partial J) = 0$. The converse is also true:

Proposition 1.1.7. *Let* $J \subset \mathbb{R}^N$ *be bounded. Then* $J \in \mathcal{J}(\mathbb{R}^N) \Leftrightarrow \partial J \in \mathcal{J}(\mathbb{R}^N)$ *and* $c_N(\partial J) = 0$. *In that case,* $c_N(J) = c_N(intJ) = c_N(\overline{J})$ *and*

(i) $c_N(J) = 0 \Leftrightarrow intJ = \emptyset$,

(ii) if $J \in \mathcal{E}(\mathbb{R}^N)$ *then* $c_N(J) = 0 \Leftrightarrow J$ *is finite.*

Note that a set with non-empty interior cannot be a null set, since in that case it contains a rectangle with positive content.

Properties of $\mathcal{J}(\mathbb{R}^N)$:

Proposition 1.1.8. *1. The class of* $\mathcal{J}(\mathbb{R}^N)$ *is a* semi-algebra:[1]

$$A, B \subset \mathcal{J}(\mathbb{R}^N) \ \Rightarrow A \cup B, A \setminus B \in \mathcal{J}(\mathbb{R}^N)$$

(and also $A \cap B \in \mathcal{J}(\mathbb{R}^N)$*).*

2. Let $A, B \in \mathcal{J}(\mathbb{R}^N)$. *The Jordan content* $c_N : \mathcal{J}(\mathbb{R}^N) \to [0, +\infty]$ *is:*

(i) Additive: if $A \cap B = \emptyset$, *then* $c_N(A \cup B) = c_N(A) + c_N(B)$.

(ii) Monotonic: if $A \subset B$, *then* $c_N(A) \leq c_N(B)$.

(iii) Subadditive: $c_N(A \cup B) \leq c_N(A) + c_N(B)$.

(iv) Invariant under translations and reflexions.

[1] it is not an algebra, since $\mathbb{R}^N \notin \mathcal{J}(\mathbb{R}^N)$.

3. *Products:* $A \in \mathcal{J}(\mathbb{R}^N)$, $B \in \mathcal{J}(\mathbb{R}^M)$ *then* $A \times B \in \mathcal{J}(\mathbb{R}^{N+M})$ *and* $c_{N+M}(A \times B) = c_N(A)c_M(B)$.

By induction, any union in the proposition above can be replaced by a finite union: $\mathcal{J}(\mathbb{R}^N)$ is closed for finite unions and intersections, and is finitely additive and subadditive.

We can show that (for any additive, non-negative set function) for $A, B \in \mathcal{J}(\mathbb{R}^N)$ (not necessarily disjoint):

$$c_N(A) + c_N(B) = c_N(A \cup B) + c_N(B \cap A). \tag{1}$$

Moreover, always have $c_N(A) = c_N(A \cap B) + c_N(A \setminus B)$.

Remark 1.1.9 (Riemann Integral)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative, bounded function and let

$$\Omega_f(E) := \{(x, y) : 0 \le y \le f(x), x \in E\}.$$

If $I = [a, b]$ is an interval, then

$$\overline{c_N}(\Omega_f(I)) = \overline{\int_a^b} f(x)dx = \inf S_D(f), \quad \underline{c_N}(\Omega_f(I)) = \underline{\int_a^b} f(x)dx = \sup s_D(f),$$

where $S_D(f)$, $s_D(f)$ are the upper and lowers Darboux sums relative to a decomposition D of I. Hence

$$f \text{ is Riemann integrable, } f \in \mathcal{R}(I) \quad \Leftrightarrow \quad \Omega_f(I) \in \mathcal{J}(\mathbb{R}^N).$$

Conversely, for $E \subset \mathbb{R}$, let χ_E be the characteristic, or indicator, function of E: $\chi_E(x) = 1$, $x \in E$, and $\chi_E(x) = 0, x \notin E$. Asume E is bounded, then

$$E \in \mathcal{J}(\mathbb{R}^N) \quad \Leftrightarrow \quad \chi_E \in \mathcal{R}(I),\ c_N(E) = \int_{\mathbb{R}} \chi_E.$$

The fact that there are countable sets that are not Jordan measurable yields an example of a sequence of Riemann integrable functions whose pointwise limit is not Riemann integrable: just take $E = \{p_k : k \in \mathbb{N}\} \notin \mathcal{J}(\mathbb{R}^N)$ and $E_n = \{p_k : k \le n\} \in \mathcal{J}(\mathbb{R}^N)$ (finite sets). Then $\chi_E = \lim \chi_{E_n}$ but χ_E not Riemann integrable on any interval.

We have that finite unions of Jordan measurable sets are always Jordan measurable, but this does not hold even if we take a countable union of points, as $\mathbb{Q} \cap [0, 1]$ shows. We would like to extend the definition of Jordan content to *countable* unions of Jordan measurable sets. This is made possible by the following fundamental result.

Theorem 1.1.10. *Let $A_n \in \mathcal{J}(\mathbb{R}^N)$, $n \in \mathbb{N}$, be disjoint. If $A = \cup_{n\in\mathbb{N}} A_n \in \mathcal{J}(\mathbb{R}^N)$ then*

$$c_N(A) = \sum_{n\in\mathbb{N}} c_N(A_n).$$

Proof. Let $A_n \in \mathcal{J}(\mathbb{R}^N)$, $n \in \mathbb{N}$, be disjoint and $A = \cup_{n\in\mathbb{N}} A_n \in \mathcal{J}(\mathbb{R}^N)$. Given $\epsilon > 0$, let K, K_n be closed and U, U_n be open such that

$$K \subset A \subset U, c_N(U \setminus K) < \epsilon, \qquad K_n \subset A_n \subset U_n, c_N(U \setminus K) < \frac{\epsilon}{2^n}.$$

We have then that $c_N(U) - \epsilon < c_N(A) < c_N(K) + \epsilon$ and $c_N(U_n) - \dfrac{\epsilon}{2^n} \leq c_N(A_n) \leq c_N(K_n) + \dfrac{\epsilon}{2^n}$, so that

$$\sum_{n\in\mathbb{N}} c_N(U_n) - \epsilon \leq \sum_{n\in\mathbb{N}} c_N(A_n) \leq \sum_{n\in\mathbb{N}} c_N(K_n) + \epsilon.$$

Now note that by Heine-Borel's theorem, K is compact, as it is closed and bounded. Since $\{U_n\}_{n\in\mathbb{N}}$ is an open cover, it has a finite subcover:

$$K \subset \cup_{n\in\mathbb{N}} U_n \quad \Rightarrow \quad K \subset \cup_{n=1}^{p} U_n.$$

It follows that

$$c_N(A) - \epsilon < c_N(K) \leq \sum_{n=1}^{p} c_N(U_n) \leq \sum_{n=1}^{\infty} c_N(A_n) + \epsilon$$

$$\Rightarrow \quad c_N(A) < \sum_{n=1}^{\infty} c_N(A_n) + 2\epsilon, \text{ for all } \epsilon > 0$$

hence $c_N(A) \leq \sum_{n=1}^{\infty} c_N(A_n)$.

Conversely, let again $A_n \in \mathcal{J}(\mathbb{R}^N)$, $n \in \mathbb{N}$, be disjoint and $A = \cup_{n\in\mathbb{N}} A_n \in \mathcal{J}(\mathbb{R}^N)$. Since, for all $n \in \mathbb{N}$, we have $A \supset \cup_{k=1}^{n} A_k$, by monotonicity and finite additivity:

$$c_N(A) \geq c_N\left(\cup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} c_N(A_k), \ \forall n \in \mathbb{N}$$

hence $c_N(A) \geq \sum_{k=1}^{\infty} c_N(A_k)$. □

A function satisfying the condition in the above theorem is said to be a *pre-measure*, a terminology that will be made clear in the following sections.

We now have a way of extending the Jordan content to countable unions of Jordan measurable, need to check that if $A = \cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} B_n$ for collections of measurable, disjoint, sets (A_n), (B_n) then since A_n is Jordan measurable and $A_n = \cup_{m=1}^{\infty} A_n \cap B_m$, we get from the previous result that

$$\sum_{n=1}^{\infty} c_N(A_n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_N(A_n \cap B_m) = \sum_{m=1}^{\infty} c_N(B_m),$$

where we apply to same reasoning to B_m to get the last equality.

Definition 1.1.11. We let $\mathcal{J}_\sigma(\mathbb{R}^N)$ denote the class of *countable unions of Jordan measurable* sets. Define the extended Jordan content $\tilde{c_N} : \mathcal{J}_\sigma(\mathbb{R}^N) \to [0, +\infty]$ such that if $A = \cup_{n\in\mathbb{N}} A_n$, with $A_n \in \mathcal{J}(\mathbb{R}^N)$ disjoint, then

$$\tilde{c_N}(A) := \sum_{n\in\mathbb{N}} c_N(A_n).$$

The previous theorem shows that the function defined above is indeed an extension of the Jordan content on $\mathcal{J}(\mathbb{R}^N)$, and we write in general $\tilde{c_N}$ as c_N. We let $\mathcal{E}_\sigma(\mathbb{R}^N)$ denote the class of countable unions of simple sets, that is, the class of countable unions of rectangles.

A set function λ is said to be σ-additive if

$$\lambda\left(\cup_{n\in\mathbb{N}} A_n\right) := \sum_{n\in\mathbb{N}} \lambda(A_n),\ A_n \text{ disjoint}$$

and σ-subadditive if

$$\lambda\left(\cup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \lambda(A_n).$$

Proposition 1.1.12. *The classes $\mathcal{J}_\sigma(\mathbb{R}^N)$ and $\mathcal{E}_\sigma(\mathbb{R}^N)$ are closed for countable unions and c_N is σ-additive and σ-subadditive on $\mathcal{J}_\sigma(\mathbb{R}^N)$ and $\mathcal{E}_\sigma(\mathbb{R}^N)$.*

Examples 1.1.13.

1. Any countable set is in $\mathcal{J}_\sigma(\mathbb{R}^N)$ and if $Q = \{q_n\}$ then

$$\tilde{c_N}(Q) = \tilde{c_N}\left(\cup_{n\in\mathbb{N}}\{q_n\}\right) = \sum_{n\in\mathbb{N}} c_N(\{q_n\}) = 0.$$

2. Let $D = [0, 1] \cap \mathbb{Q}$. Then $D \in \mathcal{J}_\sigma(\mathbb{R}^N)$, since it is countable, and $\tilde{c_N}(D) = 0$. By additivity, if $D^c := [0, 1] \cap \mathbb{R} \setminus \mathbb{Q} \in \mathcal{J}_\sigma(\mathbb{R}^N)$, then $\tilde{c_N}(D^c) = 1$.

   But $int(D^c) = \emptyset$, hence if $D^c = \cup A_n$, with $A_n \in \mathcal{J}(\mathbb{R}^N)$, then $int(A_n) = \emptyset$, hence $c_N(A_n) = 0$ and also $\tilde{c_N}(D^c) = 0$.

   We conclude that $D^c = [0, 1] \setminus D = [0, 1] \cap \mathbb{R} \setminus \mathbb{Q} \notin \mathcal{J}_\sigma(\mathbb{R}^N)$. In particular, $\mathcal{J}_\sigma(\mathbb{R}^N)$ is not closed for difference of sets, hence not an algebra.

   NOTE: In general, for $A \in \mathcal{J}_\sigma(\mathbb{R}^N)$, $\tilde{c_N}(A) = 0 \Leftrightarrow int(A) = \emptyset$. In particular, $A \in \mathcal{E}_\sigma(\mathbb{R}^N)$, $\tilde{c_N}(A) = 0 \Leftrightarrow A$ is countable.

3. Any open set is given by a countable union of rectangles, hence is σ-simple. Therefore $\mathcal{J}_\sigma(\mathbb{R}^N)$ contains all the open sets.

4. An open set $U \notin \mathcal{J}(\mathbb{R})$: let $\{q_n\}_{n\in\mathbb{N}} = [0, 1] \cap \mathbb{Q}$ and $\epsilon > 0$ be given. Define

$$U_n :=]q_n - \frac{\epsilon}{2^n}, q_n + \frac{\epsilon}{2^n}[, \qquad U := \cup_{n\in\mathbb{N}} U_n.$$

Then U is open, $U \in \mathcal{E}_\sigma(\mathbb{R}) \subset \mathcal{J}_\sigma(\mathbb{R})$. By subadditivity,

$$\tilde{c}(U) \leq \sum_{n\in\mathbb{N}} c(U_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n-1}} = 2\epsilon.$$

On the other hand, since $[0,1] \cap \mathbb{Q} \subset U$, the upper Jordan content $\bar{c}(U) \geq \bar{c}([0,1] \cap \mathbb{Q}) = 1$. If $U \in J(\mathbb{R})$, we would have

$$c(U) = \tilde{c}(U) < 2\epsilon, \qquad c(U) = \bar{c}(U) \geq 1.$$

For $\epsilon < 1/2$ this is a contradiction. Hence $U \notin J(\mathbb{R})$ for $\epsilon < 1/2$.

The following two examples will be used and often used as reference during the course.

Example 1.1.14 (Cantor set)**.** Let $I = [a, b]$. Define $T(I) := I \setminus \left]\frac{a+b}{2} - \frac{c(I)}{6}, \frac{a+b}{2} + \frac{c(I)}{6}\right[$ and $T(\cup_{k=1}^{p} I_k) := \sum_{k=1}^{p} T(I_k)$.

Let $F_0 = [a, b]$ and $F_n = T(F_{n-1})$, $n \in \mathbb{N}$. Then F_n is given by finite unions of closed sets, hence it is a closed, simple set and $c(F_n) \to 0$. Define

$$C(I) = \cap_{n\in\mathbb{N}} F_n.$$

Since $F_n \in \mathcal{E}(\mathbb{R}^N)$ and $C(I) \subset F_n$, with $c(F_n) \to 0$, it follows from Proposition (iii) that $C(I) \in \mathcal{J}(\mathbb{R})$ with $c(C(I)) = 0$. It is an uncountable set with $int(C(I)) = \emptyset$.

Topological properties:

– closed, hence compact
– nowhere dense $int(\overline{C(I)})) = int(C(I))) = \emptyset$
– perfect set: all points in $C(I)$ are limit points (no isolated points)
– totally disconnected.

Also have that $I \setminus C(I)$ is a countable union of open intervals, hence is in $\mathcal{E}_\sigma(\mathbb{R})$. But $C(I) \notin \mathcal{E}_\sigma(\mathbb{R})$ being an uncountable null set. Hence, $\mathcal{E}_\sigma(\mathbb{R}^N)$ is not closed for difference of sets, hence it is not an algebra.

Example 1.1.15 (Smith-Volterra-Cantor set)**.**

Borel Problem: Find a collection of sets $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^N)$ and $m_N : \mathcal{M} \to [0, +\infty]$ such that

(1) $\mathcal{M}$ is an algebra closed for countable unions and for $A_n \in \mathcal{M}$,

$$m_N\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} m_N(A_n).$$

(2) $\mathcal{M} \supset \mathcal{E}(\mathbb{R}^N)$ and $m_N(E) = c_N(E)$, for $E \in \mathcal{E}(\mathbb{R}^N)$.

Note that $\mathcal{M} \supset \mathcal{E}_\sigma(\mathbb{R}^N)$, in particular $\mathcal{M}$ contains all opens sets, as well as all closed sets in $\mathbb{R}^N$.

Note also that, since c_N is translation invariant, m_N is translation invariant on $\mathcal{E}_\sigma(\mathbb{R}^N)$. If we require that m_N is translation invariant on $\mathcal{M}$, we see now that such a collection $\mathcal{M}$ is necessarily *proper*, even though the existence on sets that cannot be measured by m_N relies on the axiom of Choice.

Example 1.1.16 (Vitali's set)**.** Define an equivalence relation on $[0,1[$ by $x \sim y \;\Leftrightarrow x - y \in \mathbb{Q}$. Let $V \subset [0,1[$ be such that the intersection of V with $[x]$ contains precisely one element (need Axiom of Choice). We claim that if $(\mathcal{M}, m_N)$ is a translation invariant solution for Borel's problem, then $V \notin \mathcal{M}$.

It follows from Vitali's example that:

If $(\mathcal{M}, m_N)$ is a translation invariant solution for Borel's problem then $\mathcal{M} \subsetneq \mathcal{P}(\mathbb{R}^N)$.

1.2 σ-Algebras and measure spaces

We now consider an arbitrary base space X.

Definition 1.2.1. Let $\mathcal{M} \subset \mathcal{P}(X)$. Then

(i) $\mathcal{M}$ is an *algebra* if $X \in \mathcal{M}$ and $A, B \in \mathcal{M} \Rightarrow A \cup B, A \setminus B \in \mathcal{M}$

(ii) $\mathcal{M}$ is a *σ-algebra* if it is an algebra and is closed with respect to *countable* unions:

$$A_n \in \mathcal{M}, n \in \mathbb{N} \;\Rightarrow\; \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}.$$

We have always $\emptyset, X$ are in $\mathcal{M}$ and that $\mathcal{M}$ is closed for countable intersections as well:

$$A_n \in \mathcal{M}, n \in \mathbb{N} \;\Rightarrow \bigcap_{n\in\mathbb{N}} A_n = X \setminus \left(\bigcup_{n\in\mathbb{N}} X \setminus A_n \right) \in \mathcal{M}.$$

Examples 1.2.2. 1. $\mathcal{P}(X)$ is the largest σ-algebra, $\{\emptyset, X\}$ is the smallest.

2. $\mathcal{J}(\mathbb{R}^N), \mathcal{E}(\mathbb{R}^N)$ semi-algebra, not closed for countable unions, hence not σ-algebras.

3. $\mathcal{J}_\sigma(\mathbb{R}^N)$, $\mathcal{E}_\sigma(\mathbb{R}^N)$: closed for countable unions but not algebras: not closed for difference of set. Hence, are not σ-algebras.

4. Let (X, τ) be a topological space. Then the collection τ of open sets in X is closed for countable unions, but fails to be an algebra, as it is not closed for complements.

Even if a given class $\mathcal{A}$ is not s σ-algebra, we can always consider the smallest σ-algebra that contains it. It is easy to see that the intersection of σ-algebras is still a σ-algebra, and $\mathcal{P}(X)$ is a σ-algebra containing any collection of sets $\mathcal{A}$. We define *the σ-algebra generated by $\mathcal{A}$* by

$$\mathcal{M}(\mathcal{A}) := \bigcap \mathcal{M},\ \mathcal{M}\ \sigma\text{-algebra}, \mathcal{A} \subset \mathcal{M}.$$

Definition 1.2.3. Let X be a topological space. The *Borel σ-algebra* $\mathcal{B}(X)$ is the σ-algebra generated by the class of opens sets. Sets in $\mathcal{B}(X)$ are called Borel sets, include all open sets, all closed sets, all countable unions of closed sets - such a set is called a $F\sigma$-set - and all countable intersections of open sets - such a set is called a G_δ-set.

When $X = \mathbb{R}^N$, it is easily seen that $\mathcal{B}(\mathbb{R}^N)$ is generated by open/closed/half- open rectangles.

Definition 1.2.4. Let $\mathcal{M}$ be a σ-algebra. A set function $\mu : \mathcal{M} \to [0, +\infty]$ is a *measure* on $\mathcal{M}$ if

(i) $\mu(\emptyset) = 0$,

(ii) μ is $\underline{\sigma\text{-additive}}$: for $A_n \in \mathcal{M}$, $n \in \mathbb{N}$, disjoint,

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

A *measure space* is a triple $(X, \mathcal{M}, \mu)$ where μ is a measure on $\mathcal{M}$. Sets in $\mathcal{M}$ are said to be *μ-measurable*. The measure is *finite* if $\mu(X) < \infty$ and *σ-finite* if there exist X_n with $\mu(X_n) < \infty$ and $X = \cup_{n\in\mathbb{N}} X_n$.

Note that $\mu(\emptyset) = 0 \Leftrightarrow \mu(A) < \infty$, for some $A \in \mathcal{M}$. Moreover, for $A, B \in \mathcal{M}$, not necessarily disjoint, we always have

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(B \cap A), \qquad \mu(A) = \mu(A \cap B) + \mu(A \setminus B). \tag{2}$$

Examples 1.2.5. 1. *Counting measure*: $\# : \mathcal{P}(X) \to [0, +\infty]$

$\#(E)$ = the number of elements of E, if E is finite, and $\#(E) = +\infty$, if E is infinite.

$\#$ is finite measure $\Leftrightarrow$ X is finite, and a σ-finite measure $\Leftrightarrow$ X is countable.

In particular, it is an example of non σ-finite measure on $\mathbb{R}$.

2. *Dirac measure* given $x_0 \in X$, define $\delta_{x_0} : \mathcal{P}(X) \to [0, \infty]$ by

$$\delta_{x_0}(E) = \begin{cases} 1, & \text{if } x_0 \in E, \\ 0, & \text{if } x_0 \notin E \end{cases}$$

3. *Dirac comb*: $\mu : \mathcal{P}(X) \to [0, +\infty]$ such that

$$\mu(E) = \#(E \cap \mathbb{Z}) = \sum_{n \in \mathbb{Z}} \delta_n(E).$$

   is a σ-finite measure.

   (NOTE: the sum of measures is always a measure -Exercise.)

4. *Probability measure*: $p : \mathcal{M} \to [0, 1]$ such that $p(X) = 1$, $\mathcal{M}$ is the space of events and $(X, \mathcal{M}, p)$ is called a *probability space*.

   For instance, take $p(E) = \dfrac{\#(E)}{\#(X)}$, X finite.

5. *Borel measures*: if X is a topological space, a Borel measure is a measure defined on σ-algebra of the Borel sets $\mathcal{B}(X)$, generated by the opens sets.

   In $\mathbb{R}^N$, the most important Borel measure is the *Lebesgue measure*, which is translation invariant and yields a solution to Borel's problem.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be increasing, define $\mu_f(]a, b[) := f(b) - f(a)$. Then μ is σ-additive on the σ-algebra generated by the open intervals, that is in $\mathcal{B}(\mathbb{R})$, so $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_f)$ is a measure space. The Lebesgue measure corresponds to the case $f(x) = x$.

   (The Dirac measure is a particular case, with $H(x) = 1, x \geq x_0$, $H(x) = 0$, $x < x_0$.)

7. *Haar measures*: invariant measures on locally compact topological groups, defined on Borel sets.

An additive, non-negative, function is always monotonic, as $\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$. Using monotonicity, we can see that σ-additivity implies σ-subadditivity. Moreover, a measure is always continuous with respect to monotonic sequences, in a sense made clear by the next result.

Proposition 1.2.6. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

1. *μ is σ-subadditive;*

2. *Let $E_n \in \mathcal{M}$, $n \in \mathbb{N}$ such that $\mu(E_1) < \infty$,*[2] *$E_{n+1} \subset E_n$ and $E = \cap_{n \in \mathbb{N}} E_n$ (write $E_n \searrow E$). Then*

$$\mu(E) = \lim \mu(E_n).$$

[2] $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \#)$, with $E_n = \{k \geq n\}$ then $\cap E_n = \emptyset$, $\mu(E_n) = \infty$.

3. *Let $E_n \in \mathcal{M}$, $n \in \mathbb{N}$ such that $E_n \subset E_{n+1}$ and $E = \cup_{n\in\mathbb{N}} E_n$ (write $E_n \nearrow E$). Then*

$$\mu(E) = \lim \mu(E_n).$$

Properties 2. and 3. are sometimes called *continuity from above* and *continuity from below*, respectively. (In [NotesR], Property 3. is called 'Lebesgue's monotone convergence theorem'.)

Proof. To see σ-subadditivity, let $A = \cup_{n=1}^{\infty} A_n$ and write $A = \cup_{n=1}^{\infty} A'_n$ with $A'_n = A_n \setminus \cup_{k=1}^{n-1} A_k$, then by σ-additivity and monotonicity

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A'_n) \le \sum_{n=1}^{\infty} \mu(A_n).$$

To prove 2., let $E_n \in \mathcal{M}$, $n \in \mathbb{N}$ such that $\mu(E_1) < \infty$, $E_{n+1} \subset E_n$ and $E = \cap_{n\in\mathbb{N}} E_n$ then E_1 can be written as a disjoint union

$$E_1 = E \cup \left(\cup_{k=1}^{\infty} E_k \setminus E_{k+1}\right)$$

hence

$$\mu(E_1) = \mu(E) + \sum_{k=1}^{\infty} \mu(E_k \setminus E_{k+1}) < \infty.$$

Since

$$\sum_{k=1}^{\infty} \mu(E_k \setminus E_{k+1}) = \lim \sum_{k=1}^{n} \mu(E_k) - \mu(E_{k+1}) = \mu(E_1) - \lim \mu(E_{n+1}),$$

and $\mu(E_1) < \infty$, it follows that $\mu(E) = \lim \mu(E_n)$.

To prove 3., let now $E_n \in \mathcal{M}$, $n \in \mathbb{N}$ such that $E_n \subset E_{n+1}$ and $E = \cup_{n\in\mathbb{N}} E_n$. Then E can be written as a disjoint union

$$E = \cup_{k=0}^{\infty} E_{k+1} \setminus E_k, \;\; E_0 = \emptyset$$

and in the same way

$$\mu(E) = \sum_{k=0}^{\infty} \mu(E_{k+1} \setminus E_k) = \lim \sum_{k=0}^{n} \mu(E_{k+1} \setminus E_k) = \lim \mu(E_{n+1}).$$

□

Let $(X, \mathcal{M}, \mu)$ be a measure space. Sets in $\mathcal{M}$ with $\mu(E) = 0$ are usually called *null sets* and play an important role in measure theory, as null sets are used as a means of approximation: a property that holds except on a null-set is said to hold *μ-almost everywhere*, μ-a.e. (and we often look for characterizations of measurable sets *minus* a null-set). Note that a countable union of null-sets is also a null-set, by σ-subadditivity.

Let $N \subset$ be a null set and $E \subset N$. If $E \in \mathcal{M}$, then $\mu(E) \le \mu(N) = 0$, hence E is also a null set. There is however no reason in general for $E \in \mathcal{M}$.

Definition 1.2.7. A measure space $(X, \mathcal{M}, \mu)$ is said to be *complete* if

$$E \subset N \in \mathcal{M},\ \mu(N) = 0 \Rightarrow E \in \mathcal{M},\ \mu(E) = 0.$$

Even if a given measure space is not complete, we can always form its *completion*, extending μ to a larger σ-algebra. Let $\mathcal{N} \subset \mathcal{M}$ be the collection of all null sets. Define

$$\overline{\mathcal{M}} := \{E \cup F : E \in \mathcal{M}, F \subset N, N \in \mathcal{N}\},$$

$$\overline{\mu} : \overline{\mathcal{M}} \to [0, +\infty],\ \ \overline{\mu}(E \cup F) := \mu(E).$$

First check that $\overline{\mu}$ is well-defined: if $E_1 \cup F_1 = E_2 \cup F_2$ then $E_1 \setminus E_2 \subset F_2 \subset N_2$ for some null set N_2, hence is also a null set (since $E_1 \setminus E_2 \in \mathcal{M}$) and $\mu(E_1) = \mu(E_1 \setminus E_2) + \mu(E_1 \cap E_2) = \mu(E_1 \cap E_2) = \mu(E_2)$, by the same reasoning with E_2.

Moreover, if $A \subset E \cup F \in \overline{\mathcal{M}}$ with $\overline{\mu}(E \cup F) = 0$, then E is a μ-null set, so A is a subset of a μ-null set and hence $A \in \overline{\mathcal{M}}$.

An alternative definition of $\mathcal{M}$ is the following:

$$A \in \overline{\mathcal{M}} \Leftrightarrow \text{there exist } U, K \in \mathcal{M} \text{ with } K \subset A \subset U \text{ and } \mu(U \setminus K) = 0,$$

which illustrates that sets in $\overline{\mathcal{M}}$ are precisely the ones that can be approximated by sets in $\mathcal{M}$, modulo a μ-null set. (Exercise.)

Theorem 1.2.8. *$(X, \overline{\mathcal{M}}, \overline{\mu})$ is a complete measure space and is the smallest complete extension of $(X, \mathcal{M}, \mu)$.*

1.3 Outer measures

Now we turn to the issue of defining measure spaces. One way of achieving this is to consider first outer approximations, which should be defined for any set, and then define measurability from there.

Definition 1.3.1. An *outer measure* is a set function $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ such that

(i) $\mu^*(\emptyset) = 0$,

(ii) μ^* is monotonic,

(iii) μ^* is <u>σ-subadditive</u>: for any $A_n \subset X$, $n \in \mathbb{N}$,

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \le \sum_{n\in\mathbb{N}} \mu^*(A_n).$$

So, we drop σ-additivity, requiring only the weaker σ-subadditivity ('approximation from the outside') but require on the other hand that μ^* is defined on the whole of $\mathcal{P}(X)$. Any measure defined on $\mathcal{P}(X)$ is also an outer measure, so the counting measure and the Dirac measures are outer measures.

Example 1.3.2. Outer Jordan content is subadditive but not σ-subadditive: take $D = \mathbb{Q} \cap [0,1]$, then $\overline{c_N}(D) = 1$, but $D = \{q_n\}$, and $\sum_{n\in\mathbb{N}} c_N(\{q_n\}) = 0$. Hence it is not an outer measure.

The following proposition gives a common way of obtaining outer measures:

Proposition 1.3.3. *Let $\mathcal{E} \subset \mathcal{P}(X)$ be such that $\emptyset, X \in \mathcal{E}$ and $\lambda : \mathcal{E} \to [0,\infty]$ such that $\lambda(\emptyset) = 0$. For $A \subset X$, let $\lambda^* : \mathcal{P}(X) \to [0,\infty]$ given by*

$$\lambda^*(E) = \inf\left\{\sum_{n=1}^{\infty} \lambda(E_n) : E_j \in \mathcal{E}, A \subset \cup_{n=1}^{\infty} E_n\right\}.$$

Then λ^* is an outer measure.

Now we want to associate a measure space to a given outer measure μ^*, that is, a σ-algebra $\mathcal{M}_{\mu^*}$ of measurable sets and a measure $\mu = \mu^*$ on $\mathcal{M}_{\mu^*}$, so μ^* is σ-additive on $\mathcal{M}_{\mu^*}$. Noting that a finitely additive function is σ-additive if and only if it is σ-subadditive, we want to *find a σ-algebra $\mathcal{M}_{\mu^*}$ where μ^* is additive.*

Definition 1.3.4. Let μ^* be an outer measure on X. A set $A \subset X$ is said to be *μ^*-measurable* if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \quad \text{for all } E \in \mathcal{P}(X).$$

We denote by $\mathcal{M}_{\mu^*}$ the class of all μ^*-measurable sets.

Note that we have always, by subadditivity, $\mu^*(E) \le \mu^*(E \cap A) + \mu^*(E \cap A^c)$. The next lemma shows that we get finite additivity of μ^* restricted to $\mathcal{M}_{\mu^*}$ (note that B can be any set).

Lemma 1.3.5. *Let $A \in \mathcal{M}_{\mu^*}$, $B \in \mathcal{P}(X)$. Then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B \setminus A)$. If $A \cap B = \emptyset$ then*

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Proof. Taking $E = A \cup B$ in the definition of μ^*-measurability, we get

$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c) = \mu^*(A) + \mu^*(B \cap A^c).$$

□

In fact, more is true, as we can prove in the same way that if $A \in \mathcal{M}_{\mu^*}$, $B, C \subset X$, with A, B disjoint, then

$$\mu^*(C \cap (A \cup B)) = \mu^*(C \cap A) + \mu^*(C \cap B).$$

We have then, by induction, that μ^* is (finitely) additive on any algebra contained in $\mathcal{M}_{\mu^*}$, hence σ-additive on any σ-algebra contained in $\mathcal{M}_{\mu^*}$.

Lemma 1.3.6. *$\mathcal{M}_{\mu^*}$ is a σ-algebra.*

Proof. It is closed for complements, by definition, and $\emptyset, X \in \mathcal{M}_{\mu^*}$. Let now $A, B \in \mathcal{M}_{\mu^*}$. We show that $A \cap B \in \mathcal{M}_{\mu^*}$, want to see that

$$\mu^*(E) \geq \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \cap B)^c)$$

for any $E \subset X$. Now $E \cap (A \cap B)^c = (E \cap A^c) \cup (E \cap A \cap B^c)$ (disjoint union), hence

$$\begin{aligned}\mu^*(E \cap (A \cap B)^c) + \mu^*(E \cap (A \cap B)) &\leq \mu^*(E \cap A^c) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A \cap B) \\ &= \mu^*(E \cap A^c) + \mu^*(E \cap A) = \mu^*(E)\end{aligned}$$

where we used μ^*-mensurability of B and A in the last equalities. Hence $\mathcal{M}_{\mu^*}$ is an algebra, in particular, closed for finite unions.

We show it is closed for countable unions. Let $A = \cup_{n \in \mathbb{N}} A_n$, with $A_n \in \mathcal{M}_{\mu^*}$, $n \in \mathbb{N}$, assume disjoint. Let $E \subset X$. For finite unions, we know that for any $n \in \mathbb{N}$,

$$\mu^*(E) = \mu^*(E \cap (\cup_{k=1}^n A_k)) + \mu^*(E \cap (\cup_{k=1}^n A_k)^c) \geq \mu^*(E \cap (\cup_{k=1}^n A_k)) + \mu^*(E \cap A^c)$$

(since $(\cup_{k=1}^n A_k)^c \supset A^c$). Moreover, by measurability of each A_n, using induction, one can see (Exercise) that

$$\mu^*(E \cap (\cup_{k=1}^n A_n)) = \sum_{k=1}^n \mu^*(E \cap A_k).$$

Letting $n \to \infty$, it follows

$$\mu^*(E) \geq \sum_{k=1}^\infty \mu^*(E \cap A_k) + \mu^*(E \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

by σ-subadditivity [3] . Since we always have, $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$, the measurability of A follows, and that finishes the proof. □

Given an outer measure μ^*, we have constructed a σ-algebra $\mathcal{M}_{\mu^*}$ where μ^* is σ-additive (in fact, we only needed additive), that is, so that the restriction of μ^* to $\mathcal{M}_{\mu^*}$ is a *true* measure.

Moreover, this measure is always *complete*. In fact *any set with* $\mu^*(A) = 0$ *is always measurable* : if $E \subset X$, then $\mu^*(E \cap A) \leq \mu^*(A) = 0$, so that

$$\mu^*(E) \leq \mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(A^c \cap E) \leq \mu^*(E).$$

Hence $\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E)$ and $A \in \mathcal{M}_{\mu^*}$. In particular, if $A \subset N$ where $N \in \mathcal{M}_{\mu^*}$ such that $\mu(N) = \mu^*(N) = 0$, then $\mu^*(A) \leq \mu^*(N) = 0$, so A is measurable and the space is complete.

We have proved:

[3] In fact, in this case we have σ-additivity: prove that $\mu^*(E \cap A) = \sum_{k=1}^\infty \mu^*(E \cap A_k)$

Theorem 1.3.7 (Caratheodory). *Let μ^* be an outer measure on X. Then there exists a complete measure space $(X, \mathcal{M}_{\mu^*}, \mu)$ such that*

$$\mu(E) = \mu^*(E), \quad \textit{for } E \in \mathcal{M}_{\mu^*}.$$

where $\mathcal{M}_{\mu^*}$ is as in 1.3.4.

The first application of Caratheodory's construction is to extend additive functions on algebras to measures on σ-algebras.

Definition 1.3.8. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, or a semi-algebra such that X is a countable union of elements of $\mathcal{A}$. A *premeasure* on $\mathcal{A}$ is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ such that $\mu^*(\emptyset) = 0$, and for $A_n \in \mathcal{A}$ disjoint, $n \in \mathbb{N}$, then

$$A = \cup_{n=1}^{\infty} A_n \in \mathcal{A} \;\Rightarrow\; \mu_0\,(A) \;=\; \sum_{n\in\mathbb{N}} \mu_0(A_n).$$

In particular, μ_0 is additive on $\mathcal{A}$. Now to each premeasure, we can associate an outer measure, according to

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu_0(A_n) : A_j \in \mathcal{A}, E \subset \cup_{n=1}^{\infty} A_n\right\}. \tag{3}$$

Proposition 1.3.9. *Let μ_0 be a premeasure on an algebra / semi-algebra*[4] *$\mathcal{A}$ and μ^* be defined as above, $\mathcal{M}_{\mu^*}$ the σ-algebra of μ^*-measurable sets. Then*

(i) $\mu^*_{|\mathcal{A}} = \mu_0$

(ii) $A \in \mathcal{M}_{\mu^*}$ *if and only if*

$$\mu_0(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c), \textit{ for all } B \in \mathcal{A},$$

(iii) $\mathcal{A} \subset \mathcal{M}_{\mu^*}$.

Proof. (i). Let $A \in \mathcal{A}$. It is clear from the definition that $\mu^*(A) \le \mu_0(A)$. Let $A \subset \cup A_n$, $A_n \in \mathcal{A}$. Then $A = \cup A \cap A_n \in \mathcal{A}$ hence, by the premeasure property,

$$\mu_0(A) = \sum \mu_0(A \cap A_n) \le \sum \mu_0(A_n).$$

Hence $\mu_0(A) \le \mu^*(A)$, so equality follows.

(ii) Let $A \subset X$ such that $\mu_0(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$, for all $B \in \mathcal{A}$,. For $E \subset X$ such that $E \subset \cup B_n, B_n \in \mathcal{A}$ we have $E \cap A \subset \cup(B_n \cap A)$, $E \cap A^c \subset \cup(B_n \cap A^c)$ hence

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \le \sum \mu^*(B_n \cap A) + \mu^*(B_n \cap A^c) = \sum \mu_0(B_n)$$

It follows $\mu^*(E) \le \mu^*(E \cap A) + \mu^*(E \cap A^c) \le \mu^*(E)$, so A is measurable.

(iii) is an easy consequence of (i) and (ii). □

[4]In this case, we assume that $X = \cup_{n\in\mathbb{N}} A_n$ for some collection $A_n \in \mathcal{A}$.

Theorem 1.3.10 (Hahn's extension theorem). *Let μ_0 be a premeasure on an algebra / semi-algebra $\mathcal{A}$ and $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. Then*

(i) μ_0 extends to a σ-additive function μ on $\mathcal{M}$ and $(X, \mathcal{M}, \mu)$ is a measure space.

(ii) If ν is also an extension of μ_0 to $\mathcal{M}$, then $\mu(E) = \nu(E)$, if $\mu(E) < \infty$, $E \in \mathcal{M}$. In particular, the extension is unique if μ_0 is σ-finite.

Proof. (i) is Caratheodory's extension restricted to $\mathcal{M}$, noting that if $\mathcal{A} \subset \mathcal{M}_{\mu^*}$, then also $\mathcal{M} \subset \mathcal{M}_{\mu^*}$ (as $\mathcal{M}_{\mu^*}$ is a σ-algebra).

As for (ii), if ν is another extension, then $\nu(E) \leq \mu^*(E) = \mu(E)$ as if $E \subset \cup A_i$, with A_i disjoint, then

$$\nu(E) \leq \nu(\cup A_i) = \sum \nu(A_i) = \sum \mu_0(A_i).$$

If $\mu(E) < \infty$, then for any $\epsilon > 0$, can take $E \subset A = \cup A_i$, $A_i \in \mathcal{A}$ disjoint such that $\mu(A \setminus E) < \epsilon$. Have $\mu(A) = \nu(A)$ by σ-additivity of μ, ν, as the measures coincide on $\mathcal{A}$. Hence

$$\mu(E) \leq \mu(A) = \nu(A) = \nu(E) + \nu(A \setminus E) \leq \nu(E) + \mu(A \setminus E) \leq \nu(E) + \epsilon, \ \forall \epsilon > 0$$

so $\mu(E) \leq \nu(E)$. If the space is σ-finite, any set can be written as a countable union of sets with finite measure, hence the two measures coincide. □

Note that in fact we always have a *complete* extension of μ_0 and $\mathcal{A}$ considering μ on the class $\mathcal{M}_{\mu^*}$ of μ^*-measurable sets.

Remark 1.3.11 (Inner measures). If μ^* is an outer measure obtained by extending a premeasure on an algebra, then we can define an *inner measure* by $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$. (Exercise [Fol] 1.4.19 - uses regularity)

1.4 Lebesgue measure

We are now back in $\mathbb{R}^N$ and will use the results in the previous section to provide a solution (in fact, two) to Borel's problem: we want to define a measure space $(\mathbb{R}^N, \mathcal{M}_N, m_N)$ such that $\mathcal{M}_N \supset \mathcal{E}(\mathbb{R}^N)$, that is, simple sets, and m_N extends the Jordan content:

$$m_N(E) = c_N(E), \quad \text{if } E \in \mathcal{E}(\mathbb{R}^N).$$

The outer Jordan content is subadditive but not σ-subadditive (just take $D = \mathbb{Q} \cap [0, 1] = \cup_{n\in\mathbb{N}}\{q_n\}$), hence Caratheodory's construction cannot be applied directly. Nevertheless, it follows from Theorem 1.1.10 that c_N is a premeasure on the semi-algebra $\mathcal{E}(\mathbb{R}^N)$: we saw in partcular that, if $A = \cup A_n \in \mathcal{E}(\mathbb{R}^N)$, with A_n disjoint, then

$$c_N(A) = \sum_{n\in\mathbb{N}} c_N(A_n).$$

From the results in the previous section, it induces an outer measure m_N^* and a σ-algebra where it becomes a measure.

Definition 1.4.1. The *Lebesgue outer measure* $m_N^* : \mathcal{P}(\mathbb{R}^N) \to [0, +\infty]$ is defined as

$$m_N^*(A) = \inf\{\sum_{n=1}^{\infty} c_N(R_n) : R_n \text{ bounded rectangle }, A \subset \cup_{n=1}^{\infty} R_n\}.$$

The *Lebesgue measurable sets* $\mathcal{L}(\mathbb{R}^N)$ are those sets A such that for any $E \subset \mathbb{R}^N$,

$$m_N^*(E) = m_N^*(E \cap A) + m_N^*(E \cap A^c).$$

The *Lebesgue measure* $m_N : \mathcal{L}(\mathbb{R}^N) \to [0, \infty]$ as

$$m_N(A) = m_N^*(A), \text{ if } A \in |\mathcal{L}(\mathbb{R}^N).$$

It is easy to see that m_N^* is indeed the outer measure induced by c_N as in (3). Note that the σ-algebra generated by $\mathcal{E}(\mathbb{R}^N)$ coincides with the σ-algebra generated by the collection of bounded rectangles, which coincides with the σ-algebra generated by the open sets, that is, with the Borel algebra $\mathcal{B}(\mathbb{R}^N)$.

It follows Proposition 1.3.9 that $A \in \mathcal{L}(\mathbb{R}^N)$ iff

$$c_N(R) = m_N^*(R \cap A) + m_N^*(R \cap A^c), \quad \forall R \text{ bounded rectangle.}$$

We summarize the results from the previous section:

(i) $\mathcal{L}(\mathbb{R}^N), \mathcal{B}(\mathbb{R}^N)$ are σ-algebras and m_N is a measure such that for any $E \in \mathcal{E}(\mathbb{R}^N)$,

$$m_N(E) = c_N(E).$$

(ii) m_N is the unique extension of c_N to $\mathcal{B}(\mathbb{R}^N)$ (as $\mathbb{R}^N$ is σ-finite).

(iii) $\mathcal{L}(\mathbb{R}^N)$ is a complete extension of $\mathcal{B}(\mathbb{R}^N)$ (will see that it is the completion).

Recalling Borel's problem of extending c_N we now have:

Theorem 1.4.2. $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), m_N)$ *and* $(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N), m_N)$ *are solutions to Borel's problem.*
$(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), m_N)$ *is the smallest solution, and* $(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N), m_N)$ *is a complete solution.*[5]

Note that completeness of $\mathcal{L}(\mathbb{R}^N)$ follows from the stronger crucial property, that we use often:

$$m_N^*(A) = 0 \Rightarrow A \in \mathcal{L}(\mathbb{R}^N).$$

Examples 1.4.3. 1. Countable sets

[5]it will follow from $\overline{\mathcal{B}(\mathbb{R}^N)} = \mathcal{L}(\mathbb{R}^N)$ that it is also the smallest complete solution.

2. All open sets and all the closed sets are in $\mathcal{B}(\mathbb{R}^N) \subset \mathcal{L}(\mathbb{R}^N)$, as are all sets of the form
$$G_\delta = \cap_{n\in\mathbb{N}} U_n,\ U_n \text{ open },\qquad F_\sigma = \cup_{n\in\mathbb{N}} K_n,\ K_n \text{ closed}.$$
Sets that are countable unions of closed sets are called *F-sigma* sets, and countable intersections of open sets are called *G-delta* sets.

   Also, the closure and the interior of any set are always measurable.

3. The set $U = \cup]q_n - \frac{\epsilon}{2^n}, q_n + \frac{\epsilon}{2^n}[$, where q_n are the rationals in $[0,1]$: open (not in $\mathcal{J}(\mathbb{R}^N)$)

4. Cantor and Volterra ('fat' Cantor): closed.

   Volterra's set shows that in $\mathcal{L}(\mathbb{R}^N)$, a set with positive measure can have empty interior, in fact be nowhere dense.

5. Cardinality: the Cantor set $C(I)$ and all subsets of Cantor are Lebesgue measurable (in fact, also Jordan measurable), as they are null sets. Since $C(I)$ is uncountable, it follows that
$$\#\mathcal{J}(\mathbb{R}^N) = \#\mathcal{L}(\mathbb{R}^N) = \#\mathcal{P}(\mathbb{R}).$$
On the other hand, the Borel sets are generated by a countable basis of open sets, and it can be proved that
$$\#\mathcal{B}(\mathbb{R}^N) = \#\mathcal{P}(\mathbb{N}) = \#(\mathbb{R})$$
So there are many more Lebesgue (and Jordan) measurable sets than Borel sets. It is not easy however to give an explicit description of a set in $\mathcal{L}(\mathbb{R}^N) \setminus \mathcal{B}(\mathbb{R}^N)$. We have, as we see below, $\mathcal{J}(\mathbb{R}^N) \subset \mathcal{L}(\mathbb{R}^N)$.

We give now two equivalent definitions of m_N^* (we take the extension of c_N to $\mathcal{E}_\sigma(\mathbb{R}^N)$ given by σ-additivity).

Proposition 1.4.4. *Let $\mathcal{E}_\sigma(\mathbb{R}^N)$ be the collection of countable unions of simple sets*
$$\begin{aligned} m_N^*(A) &= \inf\{c_N(E) : E \in \mathcal{E}_\sigma(\mathbb{R}^N), A \subset E\} \\ &= \inf\{c_N(U) : U \text{ is open }, A \subset U\}. \end{aligned}$$

Proof. We have $m_N^*(A) = \inf\{\sum_{n=1}^\infty c_N(E_n) : E_n \in \mathcal{E}(\mathbb{R}^N), A \subset \cup_{n=1}^\infty E_n\}$. The first equality follows noting that we can assume without loss of generality that E_n are disjoint and in this case if $E = \cup_{n=1}^\infty E_n \in \mathcal{E}_\sigma(\mathbb{R}^N)$, then $c_N(E) = \sum_{n=1}^\infty c_N(E_n)$.

As for the second, any open set $U \in \mathcal{E}_\sigma(\mathbb{R}^N)$, as it can be written as a countable union of (disjoint) rectangles, so
$$\inf\{c_N(E) : E \in \mathcal{E}_\sigma(\mathbb{R}^N), A \subset E\} \leq \inf\{c_N(U) : U \text{ is open }, A \subset U\}.$$

Moreover, given $E \in \mathcal{E}_\sigma(\mathbb{R}^N)$ and $\epsilon > 0$, can always find open $U \supset E$ such that $c_N(U) \leq c_N(E) + \epsilon$: if $c_N(E) < \infty$ and $E = \cup_{n\in\mathbb{N}} R_n$, take open rectangles $R'_n \supset R_n$ such that $c_N(R'_n \setminus R_n) < \epsilon/2^n$ and $U = \cup R'_n$. If $c_N(E) = \infty$ then any open $U \supset E$ also has infinite content. Hence, the reverse inequality holds, and equality follows. □

The first definition given above is quite similar in form to that of Jordan outer content: 'just' replace $\mathcal{E}(\mathbb{R}^N)$ by $\mathcal{E}_\sigma(\mathbb{R}^N)$. In particular, we can see that

$$\overline{c_N}(A) \geq m^*_N(A) \geq \underline{c_N}(A)$$

for any A hence if $A \in \mathcal{J}(\mathbb{R}^N)$ then $m^*_N(A) = c_N(A)$. Moreover,

Proposition 1.4.5. *$\mathcal{J}_\sigma(\mathbb{R}^N) \subset \mathcal{L}(\mathbb{R}^N)$ and $m_N = c_N$ on $\mathcal{J}_\sigma(\mathbb{R}^N)$*

Proof. We show that $\mathcal{J}(\mathbb{R}^N) \subset \mathcal{L}(\mathbb{R}^N)$, and so $m_N = c_N$ on $\mathcal{J}_(\mathbb{R}^N)$. Given $A \in \mathcal{J}(\mathbb{R}^N)$ then for any bounded rectangle R, $R \cap A, R \cap A^C \in \mathcal{J}(\mathbb{R}^N)$, by additivity

$$c_N(R) = c_N(R \cap A) + c_N(R \cap A^c) = m^*_N(R \cap A) + m^*_N(R \cap A^c).$$

Then $\mathcal{J}_\sigma(\mathbb{R}^N) \subset \mathcal{L}(\mathbb{R}^N)$, as $\mathcal{L}(\mathbb{R}^N)$ is a σ-algebra. Exercise: show m_N extends c_N. □

Not all Borel sets are Jordan measurable: saw in Example 1.1.13.4 an open non Jordan measurable set and we have seen that not all Jordan measurable are Borel measurable.

It follows from the second definition that if $A \in \mathcal{L}(\mathbb{R}^N)$ then

$$\begin{aligned} m_N(A) &= \inf\{c_N(U) : U \text{ is open }, A \subset U\} \\ &= \inf\{m_N(U) : U \text{ is open }, A \subset U\}. \end{aligned}$$

A Borel measure with the above property is said to be *(outer) regular*. Regularity means that the measure is completely determined by its values on open sets, by approximation.

This property, together with the fact that sets with 0 outer measure are measurable (completeness), give a number of useful characterizations of Lebesgue measurable sets.

Proposition 1.4.6. *$E \in \mathcal{L}(\mathbb{R}^N) \Leftrightarrow$*

(i) given $\epsilon > 0$, there is U open, with $E \subset U$ and $m^*_N(U \setminus E) < \epsilon \Leftrightarrow$

(ii) given $\epsilon > 0$, there is K closed, with $E \supset K$ and $m^*_N(E \setminus K) < \epsilon \Leftrightarrow$

(iii) given $\epsilon > 0$, there are U open, K closed, with $K \subset E \subset U$ and $m_N(U \setminus K) < \epsilon$.

Proof. (iii) is equivalent to (i) + (ii), while (ii) is equivalent to (i) taking complements. So we prove equivalence (i). Let $E \in \mathcal{L}(\mathbb{R}^N)$ and $\epsilon > 0$ be given.

If $m_N^*(E) < \infty$, then since $m_N^*(E) = \inf\{m_N(U) : U \text{ is open }, E \subset U\}$, we can take open $U \supset E$ such that $m_N(E) \le m_N(U) < m_N(E) + \epsilon$. Since $m_N(U) = m_N^*(E) + m_N^*(E \setminus U)$, by measurability of E, and $m_N^*(E) < \infty$, $m_N^*(U) < \infty$, it follows that $m_N^*(U \setminus E) < \epsilon$.

If $m_N^*(E) = \infty$, write $E = \cup E_n$, where $E_n = E \cap R_n$, with R_n bounded rectangles, disjoint such that $X = \cup R_n$. Then $m_N^*(E_n) < \infty$, and by the above there is open $U_n \supset E_n$ such that $m_N^*(U_n \setminus E_n) < \frac{\epsilon}{2^n}$. Let $U = \cup U_n$, open. Then $m_N^*(U \setminus E) < \epsilon$.

Conversely, if there is $U_n \supset E$ such that $m_N^*(U_n \setminus E) < \frac{1}{n}$, then letting $U = \cap U_n$, have

$$m_N^*(U \setminus E) \le m_N^*(U_n \setminus E) < \frac{1}{n} \ \Rightarrow m_N^*(U \setminus E) = 0 \ \Rightarrow U \setminus E \in \mathcal{L}(\mathbb{R}^N).$$

Hence $E = U \setminus (U \setminus E) \in \mathcal{L}(\mathbb{R}^N)$. □

The following consequence is very useful:

Corollary 1.4.7. *$E \in \mathcal{L}(\mathbb{R}^N) \Leftrightarrow$ if there exist U_n open, with $U_n \supset E$ such that $m_N^*(U_n \setminus E) \to 0$ and in that case $m_N(U_n) \to m_N(E)$.*

Recall that a F_σ set is a countable union of closed sets and a G_δ set is a countable intersection of open sets; F_σ and G_δ sets are in the Borel algebra.

Proposition 1.4.8. *$E \in \mathcal{L}(\mathbb{R}^N) \Leftrightarrow$ there are a F_σ set B and a G_δ set A such that $A \subset E \subset B$ and $m_N(B \setminus A) = 0$. In this case, $m_N(A) = m_N(B) = m_N(E)$.*

Proof. For each $n \in \mathbb{N}$, let K_n be closed, U_n be open, such that $K_n \subset E \subset U_n$ and $m_N(U_n \setminus K_n) < \frac{1}{n}$. Let $B = \cup K_n$ and $A = \cap U_n$, then $B \setminus A \subset U_n \setminus K_n$ for each n, and the result follows.

Conversely, if there are such A, B, then $E \setminus A \subset B \setminus A \in \mathcal{B}(\mathbb{R}^N) \subset \mathcal{L}(\mathbb{R}^N)$, hence by completeness, $E \setminus A \in \mathcal{L}(\mathbb{R}^N)$ and also $E = A \cup E \setminus A \in \mathcal{L}(\mathbb{R}^N)$. □

It follows straightaway from the definition of completion that:

Corollary 1.4.9. *$\mathcal{L}(\mathbb{R}^N)$ is the completion of the Borel algebra $\mathcal{B}(\mathbb{R}^N)$.*

Proposition 1.4.10. *$(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N), m_N)$ is the unique complete and regular (it is the largest regular and the smallest complete) solution to Borel's problem.*

Proof. Any complete solution contains $\mathcal{B}(\mathbb{R}^N)$, hence contains $\overline{\mathcal{B}(\mathbb{R}^N)} = \mathcal{L}(\mathbb{R}^N)$. Let now $(X, \mathcal{M}, \mu)$ be a regular solution to Borel's problem and $A \in \mathcal{M}$. By regularity, $\mu(A) = m_N^*(A)$ as $\mu = m_N = c_N$ on open sets. Let R be a bounded rectangle, then

$$c_N(R) = \mu(R) = \mu(A \cap R) + \mu(A^c \cap R) = m_N^*(A \cap R) + m_N^*(A^c \cap R)$$

hence $A \in \mathcal{L}(\mathbb{R}^N)$ and in this case $\mu(A) = \mu^*(A) = m_N(A)$. □

To finish this section, we now investigate some properties, of geometrical nature, of the Lebesgue measure that were known for the Jordan content: invariance, and behavior with respect to products.

Proposition 1.4.11. *1. Let $A \subset \mathcal{L}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. Then $A + x = \{a + x : a \in A\} \in \mathcal{L}(\mathbb{R}^N)$ and $m_N(A) = m_N(A + x)$.*

2. Let $A \subset \mathcal{B}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. Then $A + x = \{a + x : a \in A\} \in \mathcal{B}(\mathbb{R}^N)$ and $m_N(A) = m_N(A + x)$.

Proof. It follows from regularity and the fact that c_N is translation invariant that

$$m_N^*(A + x) = m_N^*(A).$$

Now for $E \subset X$, $x \in \mathbb{R}^N$, $E \cap (A + x) = ((E + (-x)) \cap A) + x$ hence

$$\begin{aligned} m_N^*(E \cap (A + x)) + m_N^*(E \cap (A^c + x)) &= m_N^*((E + (-x)) \cap A) + m_N^*((E + (-x)) \cap A^c) \\ &= m_N^*((E + (-x)) = m_N^*(E), \end{aligned}$$

by measurability of A.

As for the Borel case: let $\mathcal{A} = \{A \subset \mathbb{R}^N : A + x \in \mathcal{B}(\mathbb{R}^N), \forall x \in \mathbb{R}^N\}$. Then $\mathcal{A}$ contains the open sets and $\mathcal{A}$ is a σ-algebra, hence $\mathcal{A} \supset \mathcal{B}(\mathbb{R}^N)$. □

Moreover, the Lebesgue measure is also invariant with respect to unitary transformations [Foll Thm 2.44]

We have seen, when introducing Borel's problem, that no tranlsation invariant solution could be defined on the whole $\mathcal{P}(\mathbb{R}^N)$: the classical example is Vitali's set, which provides an example of a non- Lebesgue measurable set.

Examples 1.4.12. 1. Vitali's construction can be adapted so as to show that any A with $m_N^*(A) > 0$ contains a non-measurable subset.

2. Lebesgue measurable, not Borel measurable. follows form existence of non-measurable.

   Let $f : [0, 1] \to [0, 1]$ be Cantor function, that is $f(\sum x_i 3^{-i}) := \sum x_i 2^{-i}$ if $x_i \neq 1$, and f constant on subintervals with $x_j = 1$ for some (the smallest) j, $f(x) = \sum_{n=1}^{j} \sum x_i 2^{-i}$.

   Then f increasing, continuous, and $f(C) = [0, 1]$, where C is the Cantor set in $[0, 1]$. (Also known as 'devil's staircase'). Such a function maps Borel sets to Borel sets.

   Let $V \subset [0, 1]$ be Vitail's set and take $E \subset C$ such that $E = f^{-1}(V)$. Then E is Lebesgue measurable (even Jordan measurable), as a subset of the Cantor set, but not Borel measurable, as in this case $f(E) = V$ would be as well.

Behavior with respect to products: see first

Lemma 1.4.13. *Let $A \subset \mathbb{R}^N$ and $B \subset \mathbb{R}^M$, then $m^*_{N+M}(A \times B) \leq m^*_N(A) m^*_M(B)$.*

In particular, it follows that if $m^*_N(A) = 0$ (or $m^*_M(B) = 0$) then $m^*_{N+M}(A \times B) = 0$, hence $A \times B \in \mathcal{L}(\mathbb{R}^{N+M})$.

Proposition 1.4.14.

1. *Let $A \in \mathcal{L}(\mathbb{R}^N)$ and $B \in \mathcal{L}(\mathbb{R}^M)$. Then $A \times B \in \mathcal{L}(\mathbb{R}^{N+M})$ and $m_{N+M}(A \times B) = m_N(A) m_M(B)$.*
2. *Let $A \in \mathcal{B}(\mathbb{R}^N)$ and $B \in \mathcal{B}(\mathbb{R}^M)$. Then $A \times B \in \mathcal{B}(\mathbb{R}^{N+M})$ and $m_{N+M}(A \times B) = m_N(A) m_M(B)$.*

2 Integral

2.1 Measurable functions

We now take maps between measure spaces, only interested in maps that 'respect' the measures. Measurable functions are the morphims in the category of measure spaces.

Definition 2.1.1. $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is said to be $(\mathcal{M}, \mathcal{N})$-*measurable* if $f^{-1}(A) \in \mathcal{M}$, for all $A \in \mathcal{N}$.

Note that $\{f^{-1}(A) : A \in \mathcal{N}\}$ is always a σ-algebra. It is easy to see that the composition of measurable functions is measurable on the respective spaces.

Proposition 2.1.2. *If $\mathcal{N}$ is the σ-algebra generated by some class $\mathcal{E}$ then $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is $(\mathcal{M}, \mathcal{N})$-measurable if $f^{-1}(A) \in \mathcal{M}$, for all $A \in \mathcal{E}$.*

Proof. $\{E \subset Y : f^{-1}(E) : A \in \mathcal{M}\}$ is a σ-algebra and contains $\mathcal{E}$, hence contains $\mathbb{N}$. □

Corollary 2.1.3. *If X, Y are topological spaces, then any continuous function is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.*

Proof. The σ-algebras $\mathcal{B}_X, \mathcal{B}_Y$ are generated by open sets in X, Y. □

We will be mostly considering functions $f : X \to \mathbb{R}$ (or $\overline{\mathbb{R}}$ or $\mathbb{C}$), for some fixed measure space $(X, \mathcal{M}, \mu)$. In that case, if f is $(\mathcal{M}, \mathcal{B}(\mathbb{R}))$-measurable, we simply say that f is $\mathcal{M}$-measurable, or just measurable. Note that since $\mathcal{B}(\mathbb{R})$ is generated by open or closed rays, we have:

Corollary 2.1.4. *$f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable $\Leftrightarrow f^{-1}(]a, \infty[) \in \mathcal{M}$, for all $a \in \mathbb{R} \Leftrightarrow f^{-1}([a, \infty[) \in \mathcal{M}$, for all $a \in \mathbb{R} \Leftrightarrow f^{-1}(]\infty, a[) \in \mathcal{M}$, for all $a \in \mathbb{R} \Leftrightarrow f^{-1}(]\infty, a]) \in \mathcal{M}$, for all $a \in \mathbb{R}$.*

Examples 2.1.5.

1. A constant function is always measurable.

2. Let $E \subset X$ and χ_E be its characteristic function, that is, such that $\chi_E(x) = 1, x \in X$, $\chi_E(x) = 0, x \notin X$. Then χ_E is $\mathcal{M}$-measurable iff $E \in \mathcal{M}$.

Proposition 2.1.6. *Let $f, g : X \to \mathbb{R}$ be $\mathcal{M}$-measurable, $c \in \mathbb{R}$. Then $f + g$, cf, fg are $\mathcal{M}$-measurable.*

The previous result also holds for $f : X \to \overline{\mathbb{R}}$, under the usual convention $0 \cdot \infty = 0$, and assuming that there are no indeterminate signs in $f + g$. Moreover, the (pointwise) limit of measurable functions is always measurable. In fact:

Theorem 2.1.7. *Let $f_n : X \to \overline{\mathbb{R}}$ be $\mathcal{M}$-measurable functions. Then*

$$g_1(x) = \sup_{n\in\mathbb{N}}\{f_n(x)\}, \qquad G(x) = \limsup f_n(x),$$

$$h_1(x) = \inf_{n\in\mathbb{N}}\{f_n(x)\}, \qquad H(x) = \liminf f_n(x)$$

are $\mathcal{M}$-measurable. If $\lim f_n(x) = f(x)$ exists, then f is $\mathcal{M}$-measurable.

Proof. For $a \in \mathbb{R}$, we have

$$g_1^{-1}(]a, +\infty[) = \{x : \sup f_n(x) > a\} = \cup_{n\in\mathbb{N}}\{x : f_n(x) > a\} = \cup_{n\in\mathbb{N}} f_n^{-1}(]a, +\infty[),$$

$$h_1^{-1}(] - \infty, a[) = \{x : \inf f_n(x) < a\} = \cup_{n\in\mathbb{N}}\{x : f_n(x) < a\} = \cup_{n\in\mathbb{N}} f_n^{-1}(] - \infty, a[).$$

Since f_n is measurable, $n \in \mathbb{N}$, $f_n^{-1}(]a, +\infty[) \in \mathcal{M}$ and $f_n^{-1}(] - \infty, a[) \in \mathcal{M}$, hence g_1, h_1 are measurable.

As for G, write $g_k(x) = \sup_{n\geq k} f_n(x)$, then g_k is measurable by the above and decreasing, so

$$G(x) = \limsup f_n(x) = \lim_{k\to\infty} g_k(x) = \inf g_k(x)$$

is measurable, as g_k is measurable. For H we use a similar argument. If $\lim f_n(x) = f(x)$, then $\lim f_n(x) = \limsup f_n(x) = \liminf f_n(x)$ is $\mathcal{M}$-measurable. □

Moreover, noting that if f is measurable and $g = f$ μ-a.e. (that is except on a μ-null set), then g is also measurable (Exercise), then one can see that if f_n measurable,

$$f_n(x) \to f(x) \ \mu - a.e \Rightarrow f \text{ is measurable .}$$

(take the characteristic function of the set where $f_n(x) \to f(x)$).

It also follows that for f, g $\mathcal{M}$-measurable functions, $\max(f, g)$ and $\min(f, g)$ are also $\mathcal{M}$-measurable. Given $f : X \to \mathbb{R}$, define

$$f^+(x) = \max(f(x), 0), \qquad f^-(x) = \max(-f(x), 0).$$

Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Corollary 2.1.8. *f is $\mathcal{M}$-measurable if, and only if, f^+, f^- are $\mathcal{M}$-measurable, and in this case, $|f|$ is also $\mathcal{M}$-measurable.*

Now we show that any measurable function is always the limit of 'simpler' measurable functions.

Definition 2.1.9. A function $s : X \to \mathbb{R}$ is said to be *simple* if $Im(s)$ is finite and s is measurable or, equivalently, if s can be written as

$$s = \sum_{k=1}^{p} a_k \chi_{E_k}$$

where $E_k = s^{-1}(a_k)$ is measurable, $a_1, ..., a_p \in \mathbb{R}$.

It is clear that if s, t simple functions, $c \in \mathbb{R}$, then $s + t$, cs, st are also simple functions.

The keypoint now is that *any measurable function can be approximated by simple functions.* Let $f : X \to [0, \infty]$ be given. Take a finite partition of $[0, \infty]$, $\mathcal{P} = \{0 = y_0 < y_1 < ... < y_n\}$ and write

$$F(\lambda) = f^{-1}(]\lambda, +\infty] = \{x : f(x) > \lambda\}$$

and define $E_k = F(y_k) \setminus F(Y_{k+1})$, $E_n = F(y_n)$. To such a $\mathcal{P}$ we associate the simple function

$$s(x) = \begin{cases} y_k, & \text{if } x \in E_k \\ 0, & \text{if } x \notin \cup E_k. \end{cases}$$

Then $0 \leq s(x) \leq f(x)$ and $f(x) - s(x) \leq y_{k+1} - y_k$ if $x \in E_k$.

Theorem 2.1.10. $(X, \mathcal{M}, \mu)$ *measure space.*

(i) Let $f : X \to [0, +\infty]$ be measurable. Then there exist simple functions s_n with $0 \leq s_n \leq s_{n+1}$ and $s_n(x) \to f(x)$, $x \in X$.

(ii) Let $f : X \to \mathbb{R}$ be measurable. Then there exist simple functions s_n with $|s_n| \leq |s_{n+1}| \leq |f|$ and $s_n(x) \to f(x)$, $x \in X$.

Proof. (i) Take $\mathcal{P}_n = \{0, \frac{1}{2^n}, ..., \frac{k}{2^n}, ..., \frac{2^{2n}}{2^n} = 2^n\}$ and s_n the simple function associated to $\mathcal{P}_n$ as above, so that $f(x) \in]\frac{k}{2^n}, \frac{k+1}{2^n}[\Rightarrow s_n(x) = \frac{k}{2^n}$. Then $\mathcal{P}_{n+1} \supset \mathcal{P}_n$, hence $s_n \leq s_{n+1}$, and for all $x \leq 2^n$

$$0 \leq f(x) - s_n(x) \leq \frac{1}{2^n} \Rightarrow \lim s_n(x) = f(x).$$

(ii) Write $f = f^+ - f^- = \lim s_n$, where $s_n = s_n^+ - s_n^-$ and s_n^+, s_n^- are given as in (i) with respect to f^+, f^-, respectively. Then $|f| = f^+ + f^- = \lim(s_n^+ + s_n^-) = \lim |s_n|$. □

Interesting consequence:

Proposition 2.1.11. *$(X, \mathcal{M}, \mu)$ measure space, $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If f is $\overline{\mathcal{M}}$-measurable, then $f = g$ $\overline{\mu}$-a.e, for some μ-measurable function g.*

Proof. If $f = \chi_E$ with $E \in \overline{\mathcal{M}}$ then, by definition of completion, can take $A \in \mathcal{M}$ such that $\overline{\mu}(E) = \mu(A)$, hence $\chi_E = \chi_A$ $\overline{\mu}$-a.e., so the result holds for simple function, as a (finite) union of $\overline{\mu}$-null sets is $\overline{\mu}$-null.

For the general case, take s_n simple $\overline{\mu}$-measurable, with $s_n(x) \to f(x)$ and t_n simple μ-measurable such that, for each $n \in \mathbb{N}$, $t_n = s_n$ $\overline{\mu}$-a.e. Let $A_n = \{x : s_n(x) \neq t_n(x)\}$, then $\cup A_n$ is $\overline{\mu}$-null, and $t_n(x) \to f(x)$ on $X \setminus \cup A_n$. Take now $\cup A_n \subset N \in \mathcal{M}$ with $\mu(N) = 0$. Then $g = \lim \chi_{X \setminus N} t_n$ is $\mathcal{M}$-measurable and $f = g$ μ-a.e. □

It follows that any Lebesgue measurable function $f : \mathbb{R}^N \to \mathbb{R}$ coincides m_N-a.e with a Borel measurable function.

2.2 The Lebesgue integral

Let $(X, \mathcal{M}, \mu)$ be a measure space. We first define the integral for non-negative functions.

Definition 2.2.1. Let $s : X \to [0, +\infty]$, $s = \sum_{k=1}^{p} a_k \chi_{E_k}$ be a simple function. Then

$$\int_X s d\mu := \sum_{k=1}^{p} a_k \mu(E_k).$$

Let $f : X \to [0, +\infty]$ be $\mathcal{M}$-measurable. Then define

$$\int_X f\, d\mu := \sup \left\{ \int_X s d\mu : 0 \leq s \leq f, s \text{ simple} \right\}.$$

If $E \in \mathcal{M}$, then $\int_E f d\mu := \int_X \chi_E f\, d\mu$.

Proposition 2.2.2. *Let f, g be non-negative, $\mathcal{M}$-measurable functions, $c \geq 0$, then*

(i) $\int_X cf\, d\mu = c \int_X f\, d\mu$,

(ii) $f \leq g \Rightarrow \int_X f\, d\mu \leq \int_X g\, d\mu$

(iii) *if $E, F \in \mathcal{M}$, $E \subset F$, then* $\int_E f\, d\mu \leq \int_F f\, d\mu$.

(iv) $\int_X f\, d\mu = 0 \Leftrightarrow f = 0$ *μ-a.e.*

If f, g are simple functions:

(v) $\int_X (f + g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$,

(vi) The function $\lambda : \mathcal{M} \to [0, +\infty]$ such that $\lambda(E) := \int_E f\,d\mu$ is a measure.

Proof. Easy to see that (i) and (ii) hold for simple functions, hence, taking the sup, also hold for f measurable. (iii) is a consequence of (ii).

As for (iv): if $f = \sum_{k=1}^{p} a_k \chi_{E_k}$ is simple, $a_k \geq 0$, then

$$\int_X f\,d\mu = \sum a_k \mu(E_k) = 0 \Leftrightarrow a_k = 0 \vee \mu(E_k) = 0, \forall k$$

and the equivalence is proved in this case. Now if $f = 0$ a.e. and s is any simple function such that $0 \leq s \leq f$, then $s = 0$ a.e. so $\int_X s\,d\mu = 0$, hence $\int_X f\,d\mu = 0$. Conversely, if $\mu(\{x : f(x) > 0\} > 0$, then writing $\{x : f(x) > 0\} = \cup_{n\in\mathbb{N}} E_n$, with $E_n = \{x : f(x) > \frac{1}{n}\} \in \mathcal{M}$, we have that $\mu(E_n) > 0$ for some n, hence from (ii) and (iii),

$$\int_X f\,d\mu \geq \int_{E_n} f\,d\mu > \frac{1}{n}\mu(E_n) > 0.$$

For (v) and (vi), we assume (for now) that $f = \sum_{k=1}^{p} a_k \chi_{E_k}$ and $g = \sum_{j=1}^{m} b_j \chi_{F_j}$, where E_k are disjoint, and F_j are disjoint. Then

$$\begin{aligned}\int_X (f+g)\,d\mu &= \sum_{k,j} (a_k + b_j)\mu(E_k \cap F_j) = \sum_{k,j} a_k \mu(E_k \cap F_j) + \sum_{k,j} b_j \mu(E_k \cap F_j)\\ &= \sum_k a_k \mu(E_k) + \sum_j b_j \mu(F_j) = \int_X f\,d\mu + \int_X g\,d\mu.\end{aligned}$$

Finally, we prove (vi): have $\lambda(\emptyset) = 0$, let $A_n \in \mathcal{M}$ disjoint, $A = \cup_{n\in\mathbb{N}} A_n$. Then

$$\begin{aligned}\lambda(A) &= \int_A f\,d\mu = \sum_{k=1}^{p} a_k \mu(E_k \cap A) = \sum_{k=1}^{p} \sum_{n\in\mathbb{N}} a_k \mu(E_k \cap A_n)\\ &= \sum_{n\in\mathbb{N}} \sum_{k=1}^{p} a_k \mu(E_k \cap A_n) = \sum_{n\in\mathbb{N}} \int_{A_n} f\,d\mu = \sum_{n\in\mathbb{N}} \lambda(A_n).\end{aligned}$$

Hence λ is σ-additive, hence a measure. □

In fact, (v) and (vi) also hold for general measurable, non-negative functions. This will follow from the next fundamental result, one of the cornerstones of Lebesgue's integration theory, that will allow us, in particular, to define the integral as a sup over a countable set:

Theorem 2.2.3 (Monotone convergence / Beppo-Levi). *Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_n : X \to [0, +\infty]$ be $\mathcal{M}$-measurable, such that $f_n(x) \leq f_{n+1}(x)$ and $f(x) = \lim f_n(x)$. Then f is $\mathcal{M}$-measurable and*

$$\int_X f\,d\mu = \lim \int_X f_n, d\mu.$$

Proof. □

Taking simple functions $s_n \nearrow f$, as in Theorem 2.1.10, we have then that, for $E \in \mathcal{M}$,

$$\int_E f\,d\mu = \lim \int_E s_n\,d\mu = \sup \int_E s_n\,d\mu.$$

Monotone convergence also holds for decreasing sequences (g_n), provided we assume that $\int_X g_1\,d\mu < \infty$: just take the increasing, non-negative sequence $f_n = g_1 - g_n$. Moreover, a.e. convergence is sufficient:

Corollary 2.2.4. *Let $f_n : X \to [0, +\infty]$ be $\mathcal{M}$-measurable, such that $f_n(x) \nearrow f(x)$ μ-a.e. Then f is $\mathcal{M}$-measurable and*

$$\int_X f\,d\mu = \lim \int_X f_n, d\mu.$$

Proof. Let $E = \{x : f_n(x) \nearrow f(x)\}$, $\mu(E^c) = 0$. Then $\chi_E(x) f_n(x) \nearrow \chi_E(x) f(x)$ for all $x \in X$, hence by Monotone convergence, and by Proposition 2.2.2 (iv),

$$\int_X f\,d\mu = \int_X \chi_E f\,d\mu = \lim \int_X \chi_E f_n, d\mu = \lim \int_X f_n, d\mu.$$

□

As a consequence of the Monotone Convergence theorem we can generalize the additivity of the integral from simple functions to measurable functions, which moreover holds also for *infinite sums*:

Proposition 2.2.5. *Let $f_n : X \to [0, +\infty]$ be $\mathcal{M}$-measurable. Then*

(i) $\int_X (f_1 + f_2)\,d\mu = \int_X f_1\,d\mu + \int_X f_2\,d\mu$.

(ii) $\int_X \left(\sum_{n=1}^\infty f_n\right)\,d\mu = \sum_{n=1}^\infty \int_X f_n\,d\mu$.

Proof. (i) Take $s_n \nearrow f_1$, $t_n \nearrow f_2$ as in Theorem 2.1.10, then $s_n + t_n \nearrow f_1 + f_2$. From the Monotone Convergence theorem:

$$\int_X (f_1 + f_2)\,d\mu = \lim \int_X (s_n + t_n)\,d\mu = \lim \int_X s_n\,d\mu + \int_X s_n\,d\mu = \int_X f_1\,d\mu + \int_X f_2\,d\mu.$$

(ii) It follows from (i) that for any $k \in \mathbb{N}$,

$$\int_X \left(\sum_{n=1}^k f_n\right) d\mu = \sum_{n=1}^k \int_X f_n\,d\mu.$$

Since $\sum_{n=1}^k f_n \nearrow \sum_{n=1}^\infty f_n$, we have, again from Monotone convergence, that

$$\int_X \left(\sum_{n=1}^\infty f_n\right) d\mu = \lim_k \int_X \left(\sum_{n=1}^k f_n\right) d\mu = \lim_k \sum_{n=1}^k \int_X f_n\,d\mu = \sum_{n=1}^\infty \int_X f_n\,d\mu.$$

□

Examples 2.2.6. 1. Improper Riemann integrals: let e.g $f(x) = \frac{1}{x^\alpha}$, $f : [0, \infty[\to [0, \infty]$, $\alpha > 0$ and m be the lebesgue measure. Then f is Lebesgue measurable, as it is continuous, and $f_n = f\chi_{]1/n,n[} \nearrow f$, hence

$$\int_0^\infty f dm = \lim \int_{1/n}^n x^\alpha \, dx.$$

2. Consider again the Lebesgue measure on $\mathbb{R}$. Then $f_n = \chi_{]n,n+1[}$ is measurable, $f_n(x) \to 0$, for all x, but $\lim \int_{\mathbb{R}} f_n \, dm = 1 \neq \int_{\mathbb{R}} \lim f_n \, dm$.

3. As in 2., let now $f_n = n\chi_{]0,\frac{1}{n}[}$. Then $\lim \int_{\mathbb{R}} f_n \, dm = 1 \neq \int_{\mathbb{R}} \lim f_n \, dm = 0$.

Even if we cannot interchange integral with limit in general, as examples 2. and 3. show, we have always:

Theorem 2.2.7 (Fatou's Lemma). *$f_n : X \to [0, +\infty]$ be $\mathcal{M}$-measurable. Then*

$$\int_X \liminf f_n \, d\mu \leq \liminf \int_X f_n \, d\mu.$$

Proof. Let $h_k(x) = \inf_{n \geq k} f_n(x)$, so that $h_k \nearrow \liminf f_n$. By Monotone Convergence,

$$\int_X \liminf f_n \, d\mu = \lim \int_X h_k \, d\mu = \sup_k \int_X h_k \, d\mu.$$

Since $h_k \leq f_n$, for $n \geq k$, we have

$$\int_X h_k \, d\mu \leq \int_X f_n \, d\mu, \ n \geq k \ \Rightarrow \ \int_X h_k \, d\mu \leq \inf_{n \geq k} \int_X f_n \, d\mu.$$

It follows that

$$\sup_k \int_X h_k \, d\mu \leq \sup_k \left(\inf_{n \geq k} \int_X f_n \, d\mu \right) = \liminf \int_X f_n \, d\mu$$

and the result is proved. □

Again, a similar result holds also for lim sup, assuming that $\int_X f_n \, d\mu < K < \infty$, for all $n \in \mathbb{N}$. Monotone convergence can be proved from Fatou's lemma (easy Exercise).

Now we consider measurable functions $f : X \to \overline{\mathbb{R}}$ Recall that f is measurable iff f^+, f^- are measurable, non-negative functions.

Definition 2.2.8. Let $f : X \to \overline{\mathbb{R}}$ be $\mathcal{M}$-measurable. If $\int_X f^+ \, d\mu < \infty$ or $\int_X f^- \, d\mu < \infty$, then

$$\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$

We say that f is *integrable* if $\int_X |f| \, d\mu < \infty$, or equivalently, if $\int_X f^+ \, d\mu < \infty$ and $\int_X f^- \, d\mu < \infty$.

Examples 2.2.9. 1. $\mu = \delta_{x_0}$ then f is integrable iff $f(x_0) \neq \infty$ and $\int_X f d\delta_{x_0} = f(x_0)$.

2. $\mu = \#$ counting measure. Then $f : \mathbb{N} \to \mathbb{R}$ integrable iff $\sum |f(n)| < \infty$ and

$$\int_{\mathbb{N}} f d\# = \sum f(n).$$

3. $\mu = m_N$ the Lebesgue measure on $\mathbb{R}^N$. The integral coincides with the Riemann integral over bounded rectangles.

   In $\mathbb{R}$ this can be proved noting that the upper and least sums coincide with the integral of suitable simple functions and using the Monotone or Dominated convergence theorem. We will see this equivalence in the next section in a different way.

Proposition 2.2.10. *Let f, g be integrable functions, $c \in \mathbb{R}$, then*

(i) cf *is integrable and* $\int_X cf\, d\mu = c \int_X f\, d\mu$,

(ii) $f + g$ *is integrable and* $\int_X (f + g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$,

(iii) $\left|\int_X f\, d\mu\right| \leq \int_X |f|\, d\mu$,

(iv) $\mu(\{x : f(x) = \pm\infty\}) = 0$,

(v) $\{x : f(x) > 0\}$ *is σ-finite.*

(vi) $\int_E f\, d\mu = \int_E g\, d\mu$, *for all* $E \in \mathcal{M} \Leftrightarrow \int_X |f - g|\, d\mu = 0 \Leftrightarrow f = g$ *μ-a.e.*

Proof. For (i), $\int_X |cf| d\mu = |c| \int_X |f| d\mu < \infty$ hence cf is integrable and the equality of integrals follows noting that $(cf)^+ = cf^+$, $(cf)^- = cf^-$, if $c \geq 0$ and $(cf)^+ = -cf^-$, $(cf)^- = -cf^+$, if $c < 0$.

For (ii), note that $\int_X |f + g| d\mu \leq \int_X |f| d\mu + \int_X |g| d\mu < \infty$ so $f + g$ is integrable. Writing $h = f + g$, have $h^+ - h^- = f^+ - f^- + g^+ - g^- \Leftrightarrow h^+ + f^- + g^- = h^- + f^+ + g^+$ hence

$$\int_X (h^+ + f^- + g^-) d\mu = \int_X (h^- + f^+ + g^+) d\mu$$

and using additivity of the integral for non-negative functions, the result follows (as all the integrals involved are finite).

For (iii), writing $f = f^+ - f^-$:

$$\left|\int_X f\, d\mu\right| = \left|\int_X f^+ d\mu - \int_X f^-\, d\mu\right| \leq \left|\int_X f^+ d\mu\right| + \left|\int_X f^-\, d\mu\right| = \int_X |f|\, d\mu.$$

(iv): If $\mu(\{x : f(x) = \pm\infty\}) > 0$, then $\int_X |f|\, d\mu \geq \int_{\{x : f(x)=\pm\infty\}} |f|\, d\mu = \infty$.

For (v), can write $\{x : f(x) > 0\} = \cup_n \{x : f(x) > \frac{1}{n}\}$ and $\mu(\{x : f(x) > \frac{1}{n}\} < \infty$, by a similar argument.

Finally (vi): the second equivalence follows from Proposition 2.2.2 (iv). If $f = g$ μ-a.e. then $|f - g| = 0$ μ-a.e., hence $\int_E |f - g|\, d\mu = 0$. Conversely, let $h = f - g$ and assume that $\int_E h\, d\mu = 0$ for all $E \in \mathcal{M}$ (by (v) can assume that f, g are finite, possibly changing in a set of measure zero). If $\mu(\{x : h(x) \neq 0\}) > 0$ then, with $E^+ = \{x : h^+(x) > 0\}$, $E^- = \{x : f^-(x) > 0\}$, we have, in the first case, $\mu(E^+) > 0$ or $\mu(E^-) > 0$. Hence,

$$\int_{E^+} h\, d\mu = \int_{E^+} h^+\, d\mu > 0$$

which is a contradiction. The second case is similar. Hence $h = 0 \Leftrightarrow f = g$ μ-a.e $\square$

It follows from (vi) that we can change the values of a given integrable function on a set of measure zero without changing the integral. (In particular, we can always assume that an integrable function is finite, possibly changing the values on a set of measure zero). We say that two integrable functions f and g are *equivalent* if $f = g$, μ-a.e. and let

$$L^1_\mu(X) = \{[f] : f \text{ is integrable }\},$$

where $[f]$ denotes the equivalence class of f. We often identify a function f with its equivalence class. Note that

$$d(f, g) := \int_x |f - g|\, d\mu$$

makes $L^1_\mu(X)$ a metric space (in fact, a normed space).

Theorem 2.2.11 (Dominated Convergence)**.** *Let $f_n \in L^1_\mu(X)$ such that $|f_n| \leq g$ μ-a.e. with $g \in L^1_\mu(X)$. If $f_n \to f$ μ-a.e., then $f \in L^1_\mu(X)$ and*

$$\int_X f\, d\mu = \lim \int_X f_n, d\mu.$$

Proof. From $|f_n| \leq g$, we have $g - f_n \geq 0$ and $g + f_n \geq 0$ and $g - f_n$, $g + f_n$ are measurable. From Fatou's lemma:

$$\int_X (g + f) d\mu = \int_X \lim(g + f_n) d\mu \leq \liminf \int_X (g + f_n) d\mu = \int_X g\, d\mu + \liminf \int_X f_n\, d\mu$$

$$\int_X (g - f) d\mu = \int_X \lim(g - f_n) d\mu \leq \liminf \int_X (g - f_n) d\mu = \int_X g\, d\mu - \limsup \int_X f_n\, d\mu$$

Hence, as g is integrable,

$$\limsup \int_X f_n\, d\mu \leq \int_X f d\mu \leq \liminf \int_X f_n\, d\mu$$

In particular, $f \in L^1_\mu(X)$. Since we have always $\limsup \int_X f_n \, d\mu \geq \liminf \int_X f_n \, d\mu$ we conclude that

$$\int_X f d\mu = \limsup \int_X f_n \, d\mu = \liminf \int_X f_n \, d\mu = \lim \int_X f_n \, d\mu.$$

□

As a first application, we give conditions such that a not necessarily non-negative series can be integrated term by term:

Proposition 2.2.12. *Let $f_n \in L^1_\mu(X)$ such that $\sum_{n=1}^\infty \int_X |f_n| \, d\mu < \infty$. Then $\sum_{n=1}^\infty f_n$ converges μ-a.e. to $f \in L^1_\mu(X)$ and*

$$\int_X f d\mu = \int_X \sum_{n=1}^\infty f_n \, d\mu = \sum_{n=1}^\infty \int_X f_n \, d\mu.$$

Proof. From Monotone Convergence,

$$\int_X \sum_{n=1}^\infty |f_n| \, d\mu = \sum_{n=1}^\infty \int_X |f_n| \, d\mu < \infty,$$

hence $g := \sum_{n=1}^\infty |f_n| \in L^1_\mu(X)$. In particular, g is finite μ-a.e., so $\sum_{n=1}^\infty f_n$ converges μ-a.e. and $\left|\sum_{n=1}^k f_n\right| \leq g$, for all $k \in \mathbb{N}$. It then follows from Dominated Convergence that $f = \sum_{n=1}^\infty f_n \in L^1_\mu(X)$ and

$$\int_X f d\mu = \lim_k \int_X \sum_{n=1}^k f_n \, d\mu = \lim_k \sum_{n=1}^k \int_X f_n \, d\mu = \sum_{n=1}^\infty \int_X f_n \, d\mu.$$

□

Another useful application of the Dominated Convergence theorem has to do with *parametric integrals*:

Theorem 2.2.13. *Let $f : X \times [a, b] \to \mathbb{R}$ be such that $f(\cdot, t) \in L^1_\mu(X)$ for all $t \in [a, b]$. Let*

$$F(t) = \int_X f(x, t) \, d\mu(x).$$

(i) If $f(x, \cdot)$ is continuous, for μ-a.e. x, and there is $g \in L^1_\mu(X)$ such that for all $t \in [a, b]$, $|f(x, t)| \leq g(x)$ μ-a.e., then F is continuous.

(ii) If $\frac{\partial f}{\partial t}$ exists and there is $g \in L^1_\mu(X)$ such that for all $t \in [a, b]$, $\left|\frac{\partial f}{\partial t}(x, t)\right| \leq g(x)$ μ-a.e., then F is differentiable and

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x, t) \, d\mu(x).$$

Proof. Let $t_0 \in [a, b]$ and $t_n \to t_0$ (if $t_0 = a, b$ take $t_n \geq a$ or $t_n \leq b$). Let $f_n(x) = f(x, t_n)$. Then $|f_n| \leq g$ μ-a.e and $f_n(x) \to f(x, t_0)$. By dominated convergence

$$\lim_{t\to t_0} F(t) = \lim_n F(t_n) = \lim_n \int_X f_n \, d\mu = \int_X \lim f_n \, d\mu = F(t_0).$$

The proof of (ii) is similar, taking now $g_n(x) = \frac{f(x,t_n)-f(x,t_0)}{t_n-t_0}$. □

2.3 Product spaces

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. We want to consider the integral on $X \times Y$ and relate it with the integral in X and in Y. First we need to define a measure space on $X \times Y$.

Definition 2.3.1. The *product σ-algebra* $\mathcal{M} \otimes \mathcal{N}$ is the σ-algebra generated by $A \times B$, with $A \in \mathcal{M}$, $B \in \mathcal{N}$.

It is the smallest σ-algebra such that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are measurable. Moreover, $f.Z \to X \times Y$ is measurable iff $\pi_X \circ f$ and $\pi_y \circ f$ are measurable.

We want now to define a measure on $\mathcal{M} \otimes \mathcal{N}$ that is somehow the product of μ and ν.

Examples 2.3.2. 1. $\mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}^M) = \mathcal{B}(\mathbb{R}^{N+M})$.

By Proposition 1.4.14, $\mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}^M) \subset \mathcal{B}(\mathbb{R}^{N+M})$, and equality follows as $\mathcal{B}(\mathbb{R}^{N+M})$ is generated by $M + N$-rectangles. For $A \in \mathcal{B}(\mathbb{R}^N)$, $B \in \mathcal{B}(\mathbb{R}^M)$, have

$$m_{N+M}(A \times B) = m_N(A) m_N(B)$$

and m_{N+M} will be the product measure.

2. $\mathcal{L}(\mathbb{R}^N) \otimes \mathcal{L}(\mathbb{R}^M) \subsetneq \mathcal{L}(\mathbb{R}^{N+M})$.

Inclusion follows again from Proposition 1.4.14. In this case equality fails as $\mathcal{L}(\mathbb{R}^{N+M})$ is complete but $\mathcal{L}(\mathbb{R}^N) \otimes \mathcal{L}(\mathbb{R}^M)$ is not complete.

Lemma 2.3.3. *Let $A \times B = \cup_{j\in\mathbb{N}} A_j \times B_j$ where $A, A_j \in \mathcal{M}$ and $B, B_j \in \mathcal{N}$ disjoint. Then*

$$\mu(A)\nu(B) = \sum_j \mu(A_j)\nu(A_j).$$

Proof. We write

$$\mu(A)\nu(B) = \int_Y \mu(A)\chi_B(y) \, d\nu(y) = \int_Y \left(\int_X \chi_A(x)\chi_B(y) \, d\mu(x) \right) d\nu(y).$$

Now note that

$$\chi_A(x)\chi_B(y) = \chi_{A\times B}(x,y) = \sum_j \chi_{A_j\times B_j}(x,y) = \sum_j \chi_{A_j}(x)\chi_{B_j}(y).$$

Therefore;

$$\int_X \chi_A(x)\chi_B(y)\,d\mu(x) = \int_X \sum_j \chi_{A_j}(x)\chi_{B_j}(y)\,d\mu(x) = \sum_j \int_X \chi_{A_j}(x)\chi_{B_j}(y)\,d\mu(x) = \sum_j \mu(A_j)\chi_{B_j}(y),$$

where we used the σ-additivity of the integral of non-negative, measurable functions. Hence, again by σ-additivity,

$$\mu(A)\nu(B) = \sum_j \mu(A_j)\int_Y \chi_{B_j}(y)\,d\nu(y) = \sum \mu(A_j)\nu(B_j).$$

□

Let now $\mathcal{E}$ be the collection of finite unions of $\mathcal{M}\times\mathcal{N}$-rectangles, then $\mathcal{E}$ is an algebra. For $E = \cup_{i=1}^p A_i \times B_i \in \mathcal{E}$, with A_j disjoint and B_j disjoint, define (check is well-defined)

$$\lambda(E) = \sum_{i=1}^p \mu(A_i)\nu(B_i).$$

Lemma 2.3.4. *λ is a premeasure on $\mathcal{E}$.*

Proof. We checked above that λ is σ-additive in the class of $\mathcal{M}\times\mathcal{N}$-rectangles. □

Finally:

Theorem 2.3.5. *There exists a complete measure space $(X\times Y, \mathcal{K}, \rho)$ such that $\mathcal{M}\otimes\mathcal{N}\subset\mathcal{K}$ and $\rho(E) = \lambda(E)$ for $E\in\mathcal{E}$, in particular,*

$$\rho(A\times B) = \mu(A)\nu(B),\ A\in\mathcal{M}, B\in\mathcal{N}.$$

Proof. Follows from Hanh's extension theorem - using Caratheodory's construction. □

Definition 2.3.6. The *product measure* $\mu\otimes\nu$ is the restriction of ρ to $\mathcal{M}\otimes\mathcal{N}$.

Note that we have in particular (recalling the definition of outer measure induced by a premeasure):

$$\mu\otimes\nu(E) = \inf\left\{\sum \mu(A_n)\nu(B_n) : E\subset \cup_{n\in\mathbb{N}} A_n\times B_n\right\}.$$

We are now ready to relate the integral on $X\times Y$ with respect to the measure $\mu\otimes\nu$ with the integrals in X and Y and prove the fundamental Fubini-Lebesgue's theorem.

Definition 2.3.7. *Sections:* let $E \subset X \times Y$ and $f : X \times Y \to \mathbb{R}$. Then we define

$$E_x = \{y \in Y : (x, y) \in E\}, \quad E^y = \{x \in X : (x, y) \in E\},$$

$$f_x : Y \to \mathbb{R}, \quad f^y : X \to \mathbb{R}, \quad f_x(y) = f^y(x) = f(x, y).$$

In particular, $(\chi_E)_x = \chi_{E_x}$, $(\chi_E)^y = \chi_{E^y}$.

Proposition 2.3.8. *(i)* $E \in \mathcal{M} \otimes \mathcal{N} \Rightarrow E_x \in \mathcal{N}$, $E^y \in \mathcal{M}$;

(ii) f *is* $\mathcal{M} \otimes \mathcal{N}$*-measurable* $\Rightarrow f_x$ *is* $\mathcal{N}$*-measurable and* f^y *is* $\mathcal{M}$*-measurable.*

Remark 2.3.9. Unless $\mathcal{M} = \mathcal{P}(X)$, $\mathcal{N} = \mathcal{P}(Y)$, the space $\mathcal{M} \otimes \mathcal{N}$ is not complete: if e.g $A \subset Y$ is not in $\mathcal{N}$, then taking a non-empty μ-null set $N \subset X$, have $N \times A \subset N \times Y$ and $(\mu \otimes \nu)(N \times Y) = 0$ but $N \times A \notin \mathcal{M} \otimes \mathcal{N}$: it it were then for $x \in N$, the section $(N \times A)_x = A \in \mathcal{N}$.

For the Lebesgue measure, we have then that $\mathcal{L}(\mathbb{R}^N) \otimes \mathcal{L}(\mathbb{R}^M) \subsetneq \mathcal{L}(\mathbb{R}^{N+M})$. In fact, $\mathcal{L}(\mathbb{R}^{N+M}) = \overline{\mathcal{L}(\mathbb{R}^N) \otimes \mathcal{L}(\mathbb{R}^M)}$.

Hence the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are well-defined. Want to show that we can obtain $\mu \otimes \nu(E)$ by integrating the first function on X or integrating the second function on Y. We see this first on $\mathcal{E}$.

Lemma 2.3.10. *If* $E \in \mathcal{E}$ *then* $x \mapsto \nu(E_x)$ *is* $\mathcal{M}$*-measurable,* $y \mapsto \mu(E^y)$ *is* $\mathcal{N}$*-measurable and*

$$(\mu \otimes \nu)(E) = \int_X \nu(E_x)\, d\mu = \int_Y \mu(E^y)\, d\nu.$$

To show that the lemma above holds in the class $\mathcal{M} \otimes \mathcal{N}$, it would suffice to see now that the class of sets where the conclusions hold is a σ-algebra. Instead we show that it is a *monotone class*: a collection $\mathcal{A}$ is a monotone class if $E_i \in \mathcal{A}$, $E_i \nearrow E$ or $E_i \searrow E$ then $E \in \mathcal{A}$. Any σ-algebra is a monotone class (and if $\mathcal{A}$ is an algebra, then $\mathcal{A}$ is monotone class iff it is a σ-algebra). In general, the σ-algebra generated by some collection $\mathcal{A}$ coincides with the smallest monotone class that contains $\mathcal{A}$ ([Fol] 2.35)

The next results can be regarded as Fubini's theorem for sets:

Theorem 2.3.11. *Let* $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *be* σ*-finite measure spaces. If* $E \in \mathcal{M} \otimes \mathcal{N}$ *then* $x \mapsto \nu(E_x)$ *is* $\mathcal{M}$*-measurable,* $y \mapsto \mu(E^y)$ *is* $\mathcal{N}$*-measurable and*

$$(\mu \otimes \nu)(E) = \int_X \nu(E_x)\, d\mu = \int_Y \mu(E^y)\, d\nu.$$

Proof. Let $\mathcal{F}$ be the class of sets $E \subset X \times Y$ such that the conclusions of the theorem hold for E. Then $\mathcal{F} \supset \mathcal{E}$. We now prove it is a monotone class, which proves that it contains $\mathcal{M} \otimes \mathcal{N}$.

Let $E_n \in \mathcal{F}$ such that $E_n \nearrow E = \cup E_n$. Then, for each $y \in Y$, $f_n(y) = \mu(E_n^y)$ is $\mathcal{N}$-measurable and $f_n \nearrow f = \mu(E^y)$. Then by Monotone Convergence, f is $\mathcal{N}$-measurable and

$$\int_Y \mu(E^y)\,d\nu = \lim \int_Y \mu(E_n^y)\,d\nu = \lim(\mu \otimes \nu)(E_n) = (\mu \otimes \nu)(E)$$

where in the last step we used Monotone convergence for sets / continuity from below. In a similar way, we can show

$$\int_X \nu(E_x)\,d\mu = (\mu \otimes \nu)(E),$$

so $E \in \mathcal{F}$. Let now $E_n \in \mathcal{F}$ such that $E_n \searrow E = \cap E_n$. If for all $x \in X$, $y \in Y$,

$$\int_Y \mu((E_1)^y)\,d\nu < \infty, \quad \int_X \mu((E_1)_x)\,d\mu < \infty,$$

we can proceed in the same way, and use Monotone Convergence for decreasing sequences and sets to get $E \in \mathcal{F}$.

In particular, the result if proved when both μ and ν are finite measures: in this case $\mathcal{F}$ is a monotone class that contains $\mathcal{E}$, hence by the remarks above, it contains $\mathcal{M} \otimes \mathcal{N}$.

In the σ-finite case, write $X \times Y = \cup X_j \times Y_j$, where X_j, Y_j have finite measure and $X_j \times Y_j \nearrow X \times Y$. Then the lemma holds for $E \cap (X_j \times Y_j)$ and an application of Monotone Convergence yields the result also for E. $\square$

Note that the equality in the previous theorem can be written as

$$\int_{X\times Y} \chi_E\,d\mu \otimes \nu = \int_X \left(\int_Y (\chi_E)_x(y)\,d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X (\chi_E)^y(x)\,d\mu(x) \right) d\nu(y).$$

Theorem 2.3.12 (Fubini-Tonelli-Lebesgue). *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces.*

1. *(Tonelli) If $f : X \times Y \to [0, +\infty]$ is $\mu \otimes \nu$-measurable then*

$$g(x) = \int_Y f_x\,d\nu, \quad h(y) = \int_X f^y\,d\mu$$

are measurable and

$$\int_{X\times Y} f\,d\mu \otimes \nu = \int_Y \left(\int_X f_x(y)\,d\nu(y) \right) d\mu(x) = \int_X \left(\int_Y f^y(x)\,d\nu(y) \right) d\mu(x).$$

2. *(Fubini) If $f \in L^1_{\mu\otimes\nu}(X\times Y)$ then $f_x \in L^1_\nu(Y)$ μ-a.e., $f^y \in L^1_\mu(X)$ ν-a.e., $g \in L^1_\mu(X)$, $h \in L^1_\nu(Y)$ and also*

$$\int_{X\times Y} f\, d\mu\otimes\nu = \int_Y \left(\int_X f_x(y)\, d\nu(y) \right) d\mu(x) = \int_X \left(\int_Y f^y(x)\, d\nu(y) \right) d\mu(x).$$

Proof. In the previous theorem, we saw that the (1) is true for characteristic functions, and it follows also for non-negative, measurable simple functions. Hence, if f is $\mu\otimes\nu$-measurable, let f_n be simple such that $f_n \nearrow f$. Then $g_n \nearrow g$, $h_n \nearrow h$ so g, h are measurable, and Monotone Convergence yields the result for f.

For 2., assuming now that f is integrable, an application of 1. to $|f|$ yields that g and h are finite a.e., that is, $f_x \in L^1_\nu(Y)$ μ-a.e., $f^y \in L^1_\mu(X)$ ν-a.e., and also that $g \in L^1_\mu(X)$, $h \in L^1_\nu(Y)$, as the iterated integrals of $|f|$ are finite.

Equality of iterated integrals in this case now follows from an application of 1. to f^+, f^-. □

Examples 2.3.13. 1. Let $X = Y = \mathbb{N}$, $\mu = \nu = \#$. Then Fubini's theorem states that if the series $\sum_{n\in\mathbb{N}, m\in\mathbb{N}} a_{n,m}$ is absolutely convergent then

$$\sum_{n,m\in\mathbb{N}} a_{n,m} = \sum_n \sum_m a_{n,m} = \sum_m \sum_n a_{n,m}.$$

2. Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}([0,1])$ and $\mu = m$, $\nu = \#$ (not σ-finite). Let D be the diagonal in $X\times Y$ and $f = \chi_D$. Then

$$\int_{[0,1]} \int_{[0,1]} \chi_D(x,y)\, d\#\, dm = 1, \quad \int_{[0,1]} \int_{[0,1]} \chi_D(x,y)\, dm\, d\# = 0,$$

and

$$\int_{[0,1]\times[0,1]} \chi_D(x,y)\, d(m\otimes\#) = \infty.$$

As we have noted, the space $\mathcal{M}\otimes\mathcal{N}$ is in general not complete. We can give a version of Tonelli/Fubini in the complete case, in particular, for $\mathcal{L}(\mathbb{R}^{N+M})$, recalling that any measurable function on the completion coincides a.e. with a function on $\mathcal{M}\otimes\mathcal{N}$ ([Foll] 2.39).

Remark 2.3.14 (Equivalent definition of integral)**.** Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $\mu\otimes m$ be the product measure on $\mathcal{M}\times\mathcal{B}(\mathbb{R})$, m the Lebesgue measure.

Then $f : X \to [0, +\infty]$ is $\mathcal{M}$-measurable if, and only if, $\Omega_f(X)$ is $\mu\otimes m$-measurable, for any $E \in \mathcal{M}$, where $\Omega_f(X) = \{(x,y) : 0 \le y \le f(x)\}$, and in this case

$$\int_X f d\mu = (\mu\otimes m)(\Omega_f(X)).$$

This result can be checked directly if f is simple, and by Monotone convergence, for integrals and measures, we see that if f is measurable then $\Omega_f(X)$ is measurable and the integral coincides with $(\mu \otimes m)(\Omega_f(X))$. Conversely, note that sets $f^{-1}()\lambda, \infty[$, $\lambda > 0$ coincide with sections of $\Omega_f(X)$.

For arbitrary functions, we get then that $f : X \to \mathbb{R}$ is integrable iff $\Omega_f^+(X) = \{(x,y) : 0 \le y \le f(x)\}$, $\Omega_f^-(X) = \{(x,y) : 0 \ge y \ge f(x)\}$, are $\mu \otimes m$-measurable and in this case

$$\int_X f d\mu = (\mu \otimes m)(\Omega_f^+(X)) - (\mu \otimes m)(\Omega_f^-(X)).$$

In particular, we see that the Lebesgue integral on $\mathbb{R}^N$ coincides with the Riemann integral, when defined, as the Lebesgue measure generalizes Jordan content.

2.4 Differentiation of measures

We consider now a more general class of measures that are not necessarily non-negative.

Definition 2.4.1. Let $\mathcal{M}$ be a σ-algebra on X and $\nu : \mathcal{M} \to \mathbb{R}$. Then ν is a *signed measure* if $\nu(\emptyset) = 0$, ν attains at most one of the values $+\infty$ or $-\infty$ and $\nu(\cup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \nu(A_n)$, for any disjoint $A_n \in \mathcal{M}$. If ν is finite, we say ν is a *real measure*.

We will call from now on the non-negative measures considered so far *positive measures*. Any positive (finite) measure is a signed (real) measure. If μ is given as the difference of two finite, positive, measures, then μ is a real measure. Note that, in general, a signed measure is not monotonic.

The main reason we are interested in real measures here is that if μ is a positive measure and $f : X \to \mathbb{R}$ is μ-integrable, then the *indefinite integral*

$$\lambda(E) := \int_E f \, d\mu$$

is a real measure. In this case we write $f = \frac{d\lambda}{d\mu}$ and call it the generalized derivative of λ with respect to μ. A fundamental question in integration theory is to determine which measures admit such a representation. Note that if $\mu(E) = 0$ then $\lambda(E) = 0$, and we will see that this property is also sufficient to show that λ has a generalized derivative.

We first study the structure of real measures.

Definition 2.4.2. Let ν be a real or signed measure.

- A is *ν-positive* if $\nu(E) \ge 0$, for all $E \subset A \Leftrightarrow$ if $\nu(E \cap A) \ge 0$, for all $E \in \mathcal{M}$.
- A is *ν-negative* if $\nu(E) \le 0$, for all $E \subset A$

- A is *ν-null* if $\nu(E) = 0$, for all $E \subset A$

Note that a ν-null set has measure 0, but not all measure 0 sets are ν-null (unless the measure is monotonic). Moreover, ν is monotonic when restricted to the class of ν-positive or ν-negative sets:

Proposition 2.4.3. *(i) P is ν-positive, $Q \subset P$, then Q is ν-positive and $\nu(Q) \leq \nu(P)$;*

(ii) N is ν-negative, $L \subset P$, then L is ν-negative and $\nu(L) \geq \nu(N)$.

(iii) if P_n are ν-positive, then $P = \cup_{n\in\mathbb{N}} P_n$ is also ν-positive and $\nu(P) \geq \nu(P_n)$

Theorem 2.4.4 (Hahn's decomposition)**.** *Let ν be a real measure. Then there exist a pair (P, N) with ν-positive set P and a ν-negative set N such that $X = P \cup N$, $P \cap N = \emptyset$. If (P', N') is another such pair then $P \setminus P'$ and $P' \setminus P$ are ν-null sets.*

Examples 2.4.5.
$\nu = \int_x f d\nu$ then can take $P = \{x : f(x) \geq 0\}$ and $N = \{x : f(x) \leq 0\}$.
$\nu = \delta_1 - \delta_{-1}$ then $\nu(\{-1, 1\}) = 0$ but it is not a ν-null set.

Definition 2.4.6. Let ν be a real or signed measure and (P, N) be a Hahn's decomposition for ν. Then

- the *positive variation* of ν is $\nu^+(E) := \nu(E \cap P)$;
- the *negative variation* of ν is $\nu^-(E) := -\nu(E \cap N)$;
- the *total variation* of ν is $|\nu|(E) := \nu^+(E) + \nu^-(E)$.

We can always write

$$\nu(E) = \nu^+(E) - \nu^-(E), \ E \in \mathcal{M}.$$

It is easy to check that $\nu^\pm$, $|\nu|$ are positive measures, finite if ν is finite. Moreover $\nu^\pm$ do not depend on the Hahn decomposition taken: if (P', N') is another such decomposition, then $\nu(P \setminus P') = \nu(P' \setminus P) = 0$, hence

$$\nu(E \cap P) = \nu(E \cap (P \setminus P')) + \nu(E \cap P \cap P') = \nu(E \cap P \cap P') = \nu(E \cap P').$$

In fact, (Exercise)

$$\nu^+(E) = \sup\{\nu(E \cap A) : A \in \mathcal{M}\}, \quad \nu^-(E) = -\inf\{\nu(A \cap E) : A \in \mathcal{M}\}$$

and a set E is ν-positive / ν-negative / ν-null iff $\nu^-(E) = 0$ / $\nu^+(E) = 0$ / $|\nu|(E) = 0$, respectively.

Definition 2.4.7. Let ν, ν' be signed, or real, measures. Then ν and ν' are said to be *mutually singular*, $\nu \perp \nu'$, if there are $A, B \in \mathcal{M}$ such that $X = A \cup B$, $A \cap B = \emptyset$ and B is ν-null, A is ν'-null.

Note that if $\nu \perp \nu'$ as above, then $\nu(E) = \nu(E \cap A)$ and $\nu'(E) = \nu'(E \cap B)$. The *support* of a measure ν is the smallest $S \in \mathcal{M}$ such that $\nu(E) = \nu(E \cap S)$. Then $\nu \perp \nu'$ iff ν and ν' have disjoint supports. We proved:

Theorem 2.4.8 (Jordan's decomposition). *Let ν be a signed measure. Then there exist unique positive measures ν^+, ν^- such that*

$$\nu = \nu^+ - \nu^-, \quad \nu^+ \perp \nu^-.$$

If ν is real then $\nu^\pm$, $|\nu|$ are finite and bounded.

We can now define integration with respect to signed measures: let $L^1_\nu(X) = L^1_{\nu^+}(X) \cap L^1_{\nu^-}(X)$ and

$$\int_X f\, d\nu := \int_X f\, d\nu^+ - \int_X f\, d\nu^-.$$

Definition 2.4.9. Let μ be a positive measure and ν be signed measure. We say that ν is *absolutely continuous* with respect to μ, $\nu << \mu$, if

$$\mu(E) = 0 \Rightarrow |\nu|(E) = 0.$$

Easy to check that $\nu << \mu \Leftrightarrow |\nu| << \mu \Leftrightarrow \nu^+ << \mu$ and $\nu^- << \mu$.

Lemma 2.4.10. *If $\nu << \mu$ and $\nu \perp \mu$ then $\nu = 0$.*

Proof. Let A, B be such that $X = A \cup B$ and $\mu(A) = |\nu|(B) = 0$. Then by absolute continuity, $|\nu|(A) = 0$, hence $|\nu| = 0$ and $\nu = 0$. □

Examples 2.4.11.
1. The delta measure δ_0 is not absolutely continuous with respect to the Lebesgue measure m, as $m(\{0\}) = 0$ and $\delta(\{0\}) = 1$. In fact $\delta_0 \perp m$.
2. If $f \in L^1_\mu(X)$, the *indefinite integral*

$$\lambda(E) := \int_E f\, d\mu$$

is a real measure and $\lambda << \mu$.

Theorem 2.4.12 (Radon-Nikodym-Lebesgue). *Let ν be a σ-finite signed measure and μ be a σ-finite positive measure. Then there exists a unique pair (λ, ρ) of σ-finite signed measures such that*

$$\nu = \lambda + \rho, \quad \lambda << \mu, \quad \rho \perp \mu.$$

Moreover, there is $f \in L^1_\mu(X)$, unique μ-a.e., such that

$$\lambda(E) = \int_E f\, d\mu.$$

Proof. Uniqueness of the decomposition follows from the previous lemma. Also, if $f, g : X \to \mathbb{R}$ are μ-integrable such that

$$\lambda(E) = \int_E f\, d\mu = \int_E g\, d\mu, \ \forall E \in \mathcal{M}$$

then $f = g$ μ-a.e., so f and g represent the same element in $L^1_\mu(X)$.

1) To prove that such a decomposition exists, we first assume that ν and μ are both *finite positive measures*.[6] Let

$$\mathcal{F} = \{f : X \to [0, \infty] : \int_E f d\mu \le \nu(E), \forall E \in \mathcal{M}\}.$$

Then $\mathcal{F} \neq \emptyset$ and $f, g \in \mathcal{F} \Rightarrow h = \max\{f, g\} \in \mathcal{F}$, hence $\max\{f_1, ..., f_n\} \in \mathcal{F}$ if $f_1, ..., f_n \in \mathcal{F}$.

Now let $a = \sup\{\int_X f\, d\mu : f \in \mathcal{F}\}$. Then $a \le \nu(X) < \infty$. Let $f_n \in \mathcal{F}$ such that $\int_X f_n\, d\mu \to a$ and $g_n = \max\{f_1, ..., f_n\} \in \mathcal{F}$ increasing. Let

$$f = \sup\{f_n\} = \lim g_n.$$

By Monotone convergence,

$$\int_E f\, d\mu = \lim \int_E g_n\, d\mu \le \nu(E)$$

so $f \in \mathcal{F}$. Moreover, $\int_X f\, d\mu = a$, since $f \ge f_n$, hence $\int_X f\, d\mu \ge \int_X f_n\, d\mu$, for all $n \in \mathbb{N}$, so it follows that $\int_X f\, d\mu \ge \lim \int_X f_n\, d\mu = a$.

Let now $\lambda(E) := \int_E f\, d\mu$ and $\rho := \nu - \lambda$, so ρ is a positive, finite, measure. The proof is finished in the positive, finite case if we show that $\rho \perp \mu$.

2) Now for σ-finite, positive measures: write $X = \cup A_j$ with A_j disjoint, and $\mu(A_j) < \infty$, $\nu(A_j) < \infty$. Let $\mu_j(E) := \mu(E \cap A_j)$, $\nu_j(E) := \nu(E \cap A_j)$. Then μ_j, ν_j are positive, finite measures, so by what we just proved

$$\nu_j = \lambda_j + \rho_j, \quad \rho_j \perp \mu_j, \quad \lambda_j(E) = \int_E f_j\, d\mu_j.$$

Then (λ, ρ) is a suitable decomposition for ν, with $\lambda = \sum \lambda_j$, $\rho = \sum \rho_j$.

3) For real measures, apply the results just proved to f^+, f^-. □

The pair (λ, ρ) is called the *Lebesgue decomposition* for ν with respect to μ. We now have:

[6] In this case, λ and ρ are also positive, in particular, always have $\lambda(E) \le \nu(E)$.

Theorem 2.4.13 (Radon-Nikodym). *Let ν be a σ-finite signed measure and μ be a σ-finite positive measure. Assume $\nu << \mu$. Then there exists unique $f \in L^1_\mu(X)$ such that*

$$\nu(E) = \int_E f \, d\mu.$$

Proof. Since $\nu << \mu$, the pair $(\nu, 0)$ is a Lebesgue decomposition. The result follows from Radon-Nikodym-Lebesgue and uniqueness of decomposition. □

Example 2.4.14. $X = [0,1]$, $\mathcal{M} = \mathcal{B}([0,1])$, $\nu = m$ the Lebesgue measure, and $\mu = \#$. Then $m << \#$, but there is no function f such that $m(E) = \int_{[0,1]} f \, d\#$.

We have then that the class of measures given by an indefinite integral with respect to some given positive measure μ coincide with the absolutely continuous measures with respect to μ.

Definition 2.4.15. If $\nu << \mu$ then we define the *generalized derivative* of ν with respect to μ as the unique

$$\frac{d\nu}{d\mu} \in L^1_\mu(X) \text{ s.t. } \nu(E) = \int_E \frac{d\nu}{d\mu} \, d\mu.$$

Proposition 2.4.16. *Let ν be a σ-finite real measure and μ, λ σ-finite positive measures, $\nu << \mu$, $\mu << \lambda$.*

(i) If ν' is also σ-finite real measure with $\nu' << \mu$ then

$$\frac{d(\nu + \nu')}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\nu'}{d\mu}.$$

(ii) For all $g \in L^1_\nu(X)$:

$$\int_X g \, d\nu = \int_X g \frac{d\nu}{d\mu} \, d\mu.$$

(iii) Have also $\nu << \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}, \ \lambda - a.e.$$

(iv) If $\lambda << \mu$ then

$$\frac{d\lambda}{d\mu}\frac{d\mu}{d\lambda} = 1, \ \mu - a.e.$$

For the remainder of this section, we outline how the Fundamental Theorem of Calculus for Riemann integrals can be generalized in the Lebesgue setting, using the results seen so far.

We consider now Borel measures on $\mathbb{R}$. (MISSING)

2.5 L^p-spaces

Definition 2.5.1. Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p < \infty$. The space $L^p_\mu(X)$ is the space of equivalence classes of $\mathcal{M}$-measurable functions $f : X \to \overline{\mathbb{R}}$ such that

$$\int_X |f(x)|^p d\mu \ < \ \infty,$$

where two functions are equivalent if they coincide μ-a.e. For $f \in L^p_\mu(X)$ we define

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu \right)^{\frac{1}{p}}.$$

The space $L^\infty_\mu(X)$ is the space space of equivalence classes of $\mathcal{M}$-measurable functions $f : X \to \overline{\mathbb{R}}$ such that $\inf\{\sup g : f = g \ \mu - a.e.\} < \infty$, and define

$$\|f\|_\infty = \inf\{\sup g : f = g \ \mu - a.e.\}.$$

For $p = 1$, we saw that $L^1_\mu(X)$ is a vector space and it is easy to check that $\|\cdot\|_1$ is a norm.

Theorem 2.5.2 (Holder's inequality)**.** *Let $1 \leq p \leq \infty$ and $f \in L^p_\mu(X)$ and $g \in L^q_\mu(X)$, with $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1_\mu(X)$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

If $p = 2$, this is the Cauchy-Schwartz inequality.

Theorem 2.5.3 (Minkowsky's inequality)**.** *Let $f, g \in L^p_\mu(X)$. Then $f + g \in L^p_\mu(X)$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

It then follows that $L^p_\mu(X)$ is a normed space, for $1 \leq p \leq \infty$. For $p = 2$, we get an inner product space:

$$< f, g >:= \int_X fg \, d\mu.$$

Theorem 2.5.4 (Dominated Convergence)**.** *Let $f_n \in L^p_\mu(X)$, $1 \leq p < \infty$ such that $|f_n| \leq g$ μ-a.e. with $g \in L^p_\mu(X)$. If $f_n \to f$ μ-a.e., then $f \in L^p_\mu(X)$ and*

$$\int_X f \, d\mu = \lim \int_X f_n, d\mu.$$

Proposition 2.5.5. *Let $f_n \in L^p_\mu(X)$ such that $\sum_{n=1}^\infty \|f_n\|_p < \infty$. Then $\sum_{n=1}^\infty f_n$ converges μ-a.e. to $f \in L^p_\mu(X)$ and*

$$\int_X f d\mu = \int_X \sum_{n=1}^\infty f_n \, d\mu = \sum_{n=1}^\infty \int_X f_n \, d\mu.$$

Proposition 2.5.6. *Let V be a normed space. Then V is complete iff any absolutely convergent series is convergent.*

Theorem 2.5.7 (Riesz-Fischer)**.** *If $1 \leq p < \infty$, then $L^p(X)$ is complete, hence a Banach space. If $p = 2$, it is a Hlbert space.*

Proposition 2.5.8. *Let $1 \leq p \leq \infty$. The class of simple function $s = \sum a_j \chi_{E_j}$, with $\mu(E_j) < \infty$, is dense in $L^p_\mu(X)$.*

In $\mathbb{R}$ (or $\mathbb{R}^N$), simple functions can be approximated by continuous functions: it follows from Urysohn's lemma and the definition of Lebesgue measurability that given $E \in \mathcal{L}(\mathbb{R})$ there exist K compact and U open such that $K \subset E \subset U$ and $m(U \setminus K) < \epsilon$, so that there exists $f \in C_c(\mathbb{R})$, continuous with compact support, such that $\chi_K \leq f \leq \chi_U$, hence

$$m(\{x : f(x) \neq \chi_E\} < \epsilon.$$

Together with density of simple functions this yields:

Proposition 2.5.9. *Continuous functions with compact support are dense in $L^p_\mu(\mathbb{R})$.*