Contents

Notas de Apoio às Aulas Teóricas de FTAR

Prof. Responsável: Catarina Carvalho

1^{o} Semestre de 2015/2016

1 Topological spaces

1.1 Metric spaces

Neighborhoods
Continuous functions
Convergence sequences
Cauchy sequences

1.2 Topologies and basis

1.3 Continuous functions

1.4 Subspace topologies

rel metric

1.5 Product topologies

rel metric

1.6 Quotient spaces

Definition 1.1. We say that Y *is a quotient of* X (with respect to p) if $p : X \to Y$ is surjective and the topology on Y is such that $U \subset Y$ open, if, and only if $p^{-1}(U)$ is open in X. This topology is called a *quotient topology* on Y (induced by p), and p is called a *quotient map*.

Note that since p is surjective, we always have $p(p^{-1}(B)) = B$, for all $B \subset Y$. We say that a subset $A \subset X$ is *saturated* if $p^{-1}(p(A)) = A$ (iff $x \in A \Rightarrow f^{-1}(f(x)) \subset A$ iff $A = p^{-1}(B)$, for some $B \subset Y$.) Then the definition of quotient topology is equivalent to the following: $U \subset Y$ is open if and only if $U = p(V)$, for some saturated open set $V \subset X$. In particular, if p is also injective, then it is a homeomorphism (any set is saturated).

Useful to have ways of checking if a map is quotient:

Proposition 1.2. *Let $f : X \to Y$ be continuous and surjective. Then if f is an open / closed map, f is a quotient.*

(We will see later that if X is compact and Y is Hausdorff, then any continuous map is closed, hence any continuous surjective map is a quotient map.)

Example 1.3. Let $f : [0,1] \to S^1$ be given by $f(x) = (cos 2\pi x, \operatorname{sen} 2\pi x)$. Then f is surjective, continuous and closed, hence a quotient map.

Given a topology on X and a surjective map $f : X \to Y$, the quotient topology on Y (induced by f) is the *smallest / coarsest* topology on Y such that f is continuous. It is said to be the final topology on Y with respect to f. The following proposition is a general property of such final topologies: it says that a map is continuous on a quotient of X if it induces a continuous map on X.

Proposition 1.4. *Let $p : X \to Y$ be a quotient map and $g : Y \to Z$. Then g is continuous if, and only if, $g \circ p : X \to Z$ is continuous.*

Now assume we are given a map $f : X \to Z$ and that Y is a quotient of X, induced by p. Then $f = \tilde{f} \circ p$ for some $\tilde{f} : Y \to Z$ if, and only if, f is constant on $\{p^{-1}(y)\}$, for any $y \in Y$. We say that f induces a map on the quotient, or f descends to the quotient.

Proposition 1.5. *Let $p : X \to Y$ be a quotient map and $f : X \to Z$ be such that f is constant on $\{p^{-1}(y)\}$, for any $y \in Y$. Then $f = \tilde{f} \circ p$ for some $\tilde{f} : Y \to Z$ and*

(i) f continuous iff $\tilde{f}$ continuous;

(ii) f is a quotient map iff $\tilde{f}$ is a quotient map.

<ins>Equivalence relations</ins>: Let X be a topological space and suppose we are given an equivalence relation $\sim$ on X. Let X^* be the set of equivalence classes of elements of X

$$X^* = \{[x] \ : \ x \in X\}.$$

Then, there is a surjective map $p : X \to X^*$, $x \mapsto [x]$ and we can endow X^* with the quotient topology. In this case, $U \subset X^*$ is open iff $\cup_{[x]\in U}[x] = \{y \in X \ : \ y \in [x], [x] \in U\}$ is open in X.

Examples 1.6. 1. $X = [0,1]$ with the equivalence $0 \sim 1$, $x \sim x$, $x \in]0,1[$. Then $X^* \cong S^1$.

2. $X = [0,1] \times [0,1]$: cylinder, Mobius band, torus, Klein bottle (do two quotients)

3. If $A \subset X$, then $x \sim y$ if and only if $x, y \in A$. Then the space *X with A collapsed to a point* is defined as $X/A := X^* = X \setminus A \cup [A]$ with the quotient topology. Examples: cone over X, suspension.

Now we see that any quotient space comes in fact from an equivalence relation. First note that giving an equivalence relation on X is the same as giving a partition of X into disjoint subsets: it is clear that any two equivalence classes are disjoint (if $z \sim y$ and $z \sim x$ then $x \sim y$); conversely, if $X = \cup A_i$ with $A_i \cap A_j = \emptyset$, $i \neq j$, then can define $x \sim y$ iff $x, y \in A_i$ for some i and easy to check it is equivalence relation and A_i's are the equivalence classes.

Let $f : X \to Y$ be a surjective map. Then clearly the sets $f^{-1}(y)$, $y \in Y$, are disjoint and form a partition of X, hence define an equivalence relation:

$$x \sim y \text{ iff } f(x) = f(y).$$

Denoting by X^* the set of equivalence classes, we get a bijection $\tilde{f} : X^* \to Y$, that is, f descends to the quotient space X^* and the induced map is injective. We know from Proposition 1.5 that $\tilde{f}$ is continuous / quotient map iff f is continuous / quotient map.

Theorem 1.7. *Let $f : X \to Y$ be a quotient map and $X^* = \{f^{-1}(y) : y \in Y\}$. Then $Y \cong X^*$.*

Proof. Let $\tilde{f} : X^* \to Y$, $[x] \mapsto f(x)$ be the bijection defined above. Then $\tilde{f}$ is a quotient map and it is injective, hence a homeomorphism. □

(In fact, if the map $\tilde{f}$ is a homeomorphism, then necessarily f is quotient map, as $f = \tilde{f} \circ p$, where $p : X \to X^*$ such that $p(x) = [x]$ and the composite of quotient maps is a quotient map.)

Examples 1.8.

restriction to subspaces
product
hausdorff

2 Connected spaces

Definition 2.1. Let X be a topological space. A *separation* for X is a pair of non-empty open sets (U, V) such that $X = U \cup V$ and $U \cap V = \emptyset$.

Connectedness is a topological property:

Proposition 2.2. *Let $f : X \to Y$ be continuous and X connected. Then $f(X)$ is connected.*

Corollary 2.3. *If $f : X \to Y$ is a homeomorphism, then X connected if, and only if, Y is connected.*

3 Countability axioms

Definition 3.1. Let X be a topological space

- X is said to be *first countable* if for any $x \in X$, there is a countable basis for the neighborhoods of x.
- X is said to be *second countable* if there is a countable basis for the topology on X.

Of course, second countable $\Rightarrow$ first countable. First countability is a mild assumption, while having a countable basis gives much more information about a space.

A large class of first countable spaces are *metric spaces*: just take the balls at x with radius $1/n$, $n \in \mathbb{N}$ (or rational). We proved the following proposition for metric topologies:

Proposition 3.2. *Let X be first countable, $A \subset X$, Y a topological space. Then*

(i) $x \in \overline{A}$ *if, and only if, there is a sequence $x_n \in A$, $n \in \mathbb{N}$, such that $x_n \to x$.*

(ii) $f : X \to Y$ *is continuous if, and only if, for any $x_n \to x$, $x_n, x \in X$, then $f(x_n) \to f(x)$.*

(Note that if we are given a countable basis at x, then there is always a *decreasing* countable basis: just take $B'_1 = B_1$, $B'_2 = B_1 \cap B_2$, ..., $B'_n = \cap_{i=1}^n B_n$ - show it is basis).

A limit point is not necessarily the limit of some sequence, as the following examples show.

Examples 3.3. 1. $\mathbb{R}^{\mathbb{N}}$ the space of real sequence with the box topology and take $A = \{x = (x_n) : x_n > 0\}$, show that the zero sequence is in $\overline{A}$ but there is no sequence in A converging to 0. Hence $\mathbb{R}^{\mathbb{N}}$ the space of real sequence with the box topology is not first countable, and also not metrizable.

2. $\mathbb{R}^J$ with J uncountable, with the product topology, take A to be the subset of all sequences $x = (x_\alpha)$ with $x_\alpha = 1$ for all but finite values of α, then the zero sequence is $\overline{A}$ but there is no sequence in A converging to 0 (since J is uncountable, for any sequence $x_n = (x_{n\alpha}) \in A$ there must $\alpha \in J$ such that $x_{n\alpha} = 1$ for all $n \in \mathbb{N}$).

   Hence $\mathbb{R}^J$ with J uncountable is not first countable in the product topology, and also not metrizable.

Now we see more examples.

Examples 3.4. 1. Discrete spaces always first countable (are metrizable): $\{x\}$ is a basis at x, but second countable if, and only if, countable.

2. $\mathbb{R}^n$ is first and second countable:
$$]p_1, q_1[\times \ldots \times]p_n, q_n[, \quad p_i, q_i \in \mathbb{Q},\ i = 1, \ldots, n$$
is a countable basis for the standard topology.

3. An uncountable set with the co-finite / co-countable topology is not first countable: if $\mathcal{B}$ is a basis, $x \in X$, have $\{x\} = \cap_{B \in \mathcal{B}} B$ (it is T1 - see next section), if $\mathcal{B}$ is countable then $X \setminus \{x\}$ is countable union of finite/countable sets, hence countable.

4. $\mathbb{R}^2$ with the topology given by slotted discs: a basic neighborhhod of $x \in \mathbb{R}^2$ is given by $\{x\} \cup B$ with B an open ball with straight lines through x removed: not first countable.

5. $\mathbb{R}^{\mathbb{N}}$ the space of real sequences, with the product topology, is second countable: a basis is given by sets of the form ΠU_n such that $U_{n_i} =]p_i, q_i[,\ p_i, q_i \in \mathbb{Q},\ i = 1, \ldots, k$, and $U_n = \mathbb{R}$, $n \neq n_i$.

6. $\mathbb{R}^{\mathbb{N}}_\rho$ the space of real sequences, with the uniform topology, is not second countable: $\{0, 1\}^{\mathbb{N}}$ is an uncountable discrete subset, as $\overline{\rho}(x, y) = 1$ for any sequences $x \neq y$, hence $B_\rho(x, 1) = \{x\}$ is open. It is first countable, as it is a metric space.

Proposition 3.5. *(i) subspaces of first / second countable spaces are first / second countable;*

(ii) countable *products of first / second countable spaces are first / second countable (in product topology).*

(iii) open, continuous images of first / second countable spaces are first / second countable.

Proof. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ be a countable basis (at $x \in A$ or for the whole topology of X). Then $\{B \cap A : B \in \mathcal{B}\}$ is also a basis.

Let $X = \Pi_{n \in \mathbb{N}} X_n$ and $\mathcal{B}_n$ be a basis for X_n (at $x_n \in X_n$ or for the whole topology of X_n). Then the collection of open sets of the form ΠU_n such that $U_{n_i} \in \mathcal{B}_{n_i}$ for $i = 1, \ldots, p$ and $U_n = X_n$, $n \neq n_i$ forms a basis.

Let $f : X \to Y$ be a continuous, open map. Then if $\mathcal{B}$ is a basis, also $\{f(B) : B \in \mathcal{B}\}$ is a basis. □

Moreover, one can prove that a product of first / second countable spaces is first / second countable if, and only if, it is a *countable* product. It follows from (iii) that countability axioms are topological properties, ie, preserved by homeomorphisms (it is however not true in general that the continuous image of a first / second countable space is first / second countable).

It is easy to check that in a space with a countable basis:

(i) discrete[1] subsets must be countable;

(ii) collections of disjoint open sets must be countable.

Recall that a subspace $D \subset X$ is *dense* if $\overline{D} = X$.

Proposition 3.6. *If X has a countable basis, then*

(1) X has a countable dense subset,

(2) any open cover of X has a countable subcover.

Proof. Let $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ be a countable basis. To prove (1), pick $x_n \in B_n$ and let $D = \{x_n : n \in \mathbb{N}\}$. Then for any $x \in X$ and any neighborhood U of x, there is $B_n \subset U$, hence $D \cap U \neq \emptyset$ and $x \in \overline{D}$, that is D is dense.[2]

To prove (2), let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X and for each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{U \in \mathcal{U} : B_n \subset U\}$. If $\mathcal{U}_n \neq \emptyset$, pick $U_n \in \mathcal{U}_n$ and let $\mathcal{U}' = \{U_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$. Then for $x \in X$, exists $U \in \mathcal{U}$ and $B_n \in \mathcal{B}$ such that $x \in B_n \subset U$, so U_n is defined and $x \in B_n \subset U_n$, and we conclude that $\mathcal{U}'$ covers X.

□

A space X that satisfies (1) is said to be *separable* and a space X that satisfies (2) is said to be *Lindeloff*. These conditions are usually weaker than being second countable, for instance $\mathbb{R}_l$ is separable and Lindeloff but not second countable. However, if we a have a metric:

Proposition 3.7. *If X is metrizable, then*

$$\textit{second countable} \Leftrightarrow \textit{separable} \Leftrightarrow \textit{Lindeloff.}$$

Proof. If X is separable, and D is a dense countable set, then $\mathcal{B} = \{B(a, 1/n) : a \in D\}$ is countable and is a basis for the topology of X: for any open U, $x \in U$, there is $n \in \mathbb{N}$ such that $B(x, 1/n) \subset U$ and there is $a \in B(x, 1/(2n))$, as D is dense. Then $B(a, 1/(2n)) \subset U$.

If X is Lindeloff, take a countable subcover of the collection $B(x, 1/n)$, show it is a basis. □

[1]A subset $A \subset X$ is discrete if the induced subspace topology is discrete, ie, $\{x\}$ is open in A, $x \in A$.

[2]This provides a way of constructing a dense subset from any basis: just pick one element from each basic set.

If X compact, it is Lindeloff. If it is compact and metrizable, then it is second countable (can see directly: since for any $n \in \mathbb{N}$ it can be covered by a finite number of ball of radius $1/n$. The family of all those balls is a countable basis).

Proposition 3.8. *(i) open subspaces of separable spaces are separable; closed subspaces of Lindeloff are Lindeloff.*

(ii) countable *products of separable spaces are separable (in product topology).*

(iii) continuous images of separable/ Lindeloff spaces are separable/ Lindelof.

Examples 3.9. 1. $\mathbb{R}_l$

– first countable: $[x, x+q[$, $q \in \mathbb{Q}$ is a basis at $x \in \mathbb{R}$

– not second countable: if $\mathcal{B}$ is a basis, let $B_x \in \mathcal{B}$ such that $B_x \subset [x, x+1[$. Then $x \neq y \Rightarrow B_x \neq B_y$, as $x = \min B_x \neq \min B_y$, hence $\mathcal{B}$ is uncountable.

– separable: $\mathbb{Q}$ is dense;

– Lindeloff.

2. $\mathbb{R}_l \times \mathbb{R}_l$: first countable and separable (as $\mathbb{R}_l$ is, not second countable (as $\mathbb{R}_l \times \{0\}$ is a subspace), not Lindeloff:

   Let $L = \{(x, -x) : x \in \mathbb{R}\}$ then L is closed and $L \cap [x, b[\times[-x, c[= \{(x, -x)\}$ (ie, the subspace topology in L is discrete; since L uncountable, $\mathbb{R}_l^2$ not second countable). Cover $\mathbb{R}_l^2$ by $\mathbb{R}_l^2 \setminus \mathbb{R}$ and $[a, b[\times[-a, c[$, $a, b, c \in \mathbb{R}$, then there is no countable subcover, hence not Lindeloff.

   (Also shows: L closed, not separable, and $L \subset \mathbb{R}_l^2$ separable.)

3. The ordered square $I_0^2 = [0,1] \times [0,1]$ with the dictionary order:

   – it is Lindeloff - compact; but $[0,1]\times]0,1[$ not Lindeloff (open), since the collection $\{x\}\times]0,1[$, $x \in [0,1]$, has no countable subcover

   – not separable

   – not second countable: the collection of disjoint open sets $\{x\}\times]0,1[$, $x \in [0,1]$ is uncountable (or $\{x\} \times \{1/2\}$ is a discrete uncountable set).

4 Separation axioms

Definition 4.1. Let X be a topological space, $x, y \in X$.

- *X is T1* : if for $x, y \in X$, $x \neq y$ there exist neighborhoods U of x and V of y such that $y \notin U$ and $x \notin V$.

- X *is Hausdorff* (or T2) if for $x, y \in X$, $x \neq y$ there exist neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

- X *is regular* if X is T1 [3] and for $x \in X$, $B \subset X$ closed with $x \notin B$, there exist open sets U, V such that $x \in U$, $B \subset V$ and $U \cap V = \emptyset$.

- X *is normal* if X is T1 and for $A, B \subset X$ closed with $A \cap B = \emptyset$, there exist open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Always have:

$$\text{normal} \Rightarrow \text{regular} \Rightarrow \text{Hausdorff} \Rightarrow \text{T1}$$

but the reverse implications NOT true in general (Hausdorff, not regular: $\mathbb{R}_K$; T1, not Hausdorff: $\mathbb{R}_K/K$; regular, not normal: $\mathbb{R}_l^2$, or $\mathbb{R}^J$, J uncountable)

Remark 4.2. 1. In T1 spaces: $x \notin \overline{\{y\}}$, all $x \neq y$.

$$\begin{aligned} X \text{ is T1 } &\Leftrightarrow \{x\} \text{ is closed, } x \in X \\ &\Leftrightarrow \{x\} = \cap_{x \in U \text{ open}} U = \cap_{B \in \mathcal{B}_x} B,\ \mathcal{B}_x \text{ basis of nbhds of } x. \end{aligned}$$

In T1 spaces, x is limit point of A iff any nbhd of x intersects A in *infinitely* many points. (If $F = (A \setminus \{x\}) \cap U$ is finite, hence closed, then $X \setminus F$) is open nbhd of x that does not intersect A.)

2. In Hausdorff spaces, there is *unicity* of limits of sequences and of extensions of continuous functions defined on dense subsets. Moreover:

$$X \text{ is Hausdorff} \Leftrightarrow \Lambda = \{(x, x) : x \in X\} \text{ is closed in } X \times X$$

(note that $U \cap V = \emptyset$ iff $(U \times V) \cap \Lambda = \emptyset$, so X is Hausdorff iff $(x, y) \notin \overline{\Lambda}$ for $x \neq y$) that is, $\overline{\Lambda} \subset \Lambda$.

Examples 4.3. 1. X infinite with the cofinite topology is $T1$ but not Hausdorff.[4]

2. $\mathbb{R}_K$ Hausdorff, not regular: K is closed cannot be separated from 0.

3. $\mathbb{R}_l$ is normal, $\mathbb{R}_l^2$ not normal (but regular).

4. if J uncountable $\mathbb{R}^J$ not normal (hard to prove: see Ex. 32.9 [Munkres] - see that $\mathbb{N}^J$ not normal)

[3] In $\mathbb{R}$ with topology given by $]-\infty, a[$, $a \in \overline{\mathbb{R}}$ (is T0, not T1, not Hausdorff) any two closed sets intersect

[4] It is the smallest $T1$ topology on X - check.

The following result gives equivalent definitions of regular and normal spaces, that we will use quite often:

Proposition 4.4. *Let X be a T1 space.*

(i) X is regular $\Leftrightarrow$ $x \in X$, U nbhd of x, there exists nbhd W of x such that $\overline{W} \subset U$.

(ii) X is normal $\Leftrightarrow$ $A \subset X$ closed, $U \supset A$ open, there exists open $W \supset A$ such that $\overline{W} \subset U$.

Proof. (i) U nbhd of $x \Leftrightarrow B = X \setminus U$ is closed, $x \notin B$; W nbhd of x such that $\overline{W} \subset U \Leftrightarrow$ the open $V = X \setminus \overline{W} \supset X \setminus U = B$ and $U \cap V = \emptyset$.

□

Proposition 4.5. *1) <u>Products</u>: if $\forall \alpha \in J$, X_α is T1 / Hausdorff / regular, then $X = \Pi_{\alpha \in J} X_\alpha$ is T1 / Hausdorff / regular.*

2) <u>Subspaces:</u> if X is T1 / Hausdorff / regular and $A \subset X$, then A is T1 / Hausdorff / regular.

For normal spaces, not true: $\mathbb{R}_l^2$ not normal, $\mathbb{R}_l$ normal so 1) fails and $[0,1]^J$ normal, $]0,1[^J$ not normal, so 2) fails. However, if $A \subset X$ is closed and X is normal, then A is normal. In fact, the image of a normal space under a closed continuous map is normal.

It is not true in general that images under continuous functions of Hausdorff / regular spaces are Hausdorff / regular, not even open or closed images. However, it is easy to see that any of the separation axioms are topological, in that they are preserved by homeomorphisms.

Even if regular is usually weaker that normal, if we add a countability axiom then the two notions coincide.

Theorem 4.6. *Let X have a countable basis (ie, second countable). Then X regular $\Leftrightarrow$ X normal.*

Proof. Need to show that X regular and second countable then X is normal. Let $\mathcal{B}$ be a countable basis for X and $A, B \subset X$ be closed.

For each $x \in A$, by regularity, can take a neighborhood U of x such that $\overline{U} \cap B = \emptyset$. Since any such U contains a basis element and $\mathcal{B}$ is countable, we obtain a countable covering of X by sets U_n such that $\overline{U_n} \cap B = \emptyset, n \in \mathbb{N}$.

We can do the same for the set B to obtain a covering of B by open sets V_n such that $\overline{V_n} \cap A = \emptyset, n \in \mathbb{N}$. Let

$$U = \cup_{n \in \mathbb{N}} U_n, \quad V = \cup_{n \in \mathbb{N}} V_n.$$

Then U, V open, and $U \supset A$, $V \supset B$, but U and V need not be disjoint. Now let

$$U_n' = U_n \setminus \cup_{k=1}^n \overline{V_k}, \quad V_n' = V_n \setminus \cup_{k=1}^n \overline{U_k}$$

and $U' = \cup U_n'$, $V = \cup V_n'$. Then U', V' open, $U' \supset A$, $V' \supset B$ and $u' \cap V' = \emptyset$. □

Note that we proved in fact that if X is Lindeloff and regular, then it is normal.

Theorem 4.7. *Let X be compact Hausdorff. Then X is normal.*

Proof. We have seen that in a Hausdorff space compact (disjoint) sets are separated by open sets. Since any closed subset of X is compact, as X is compact, normality follows. □

Corollary 4.8. *Any locally compact Hausdorff space is regular.*

Proof. It is a subspace of a normal, hence regular, space hence it is regular. (Alternatively, any locally compact Hausdorff has a basis of neighborhoods with compact closure: given $x \in X$ and U a neighborhood of x, there exists open B, with $x \in B$ and $\overline{B} \subset U$, ie., X is regular.) □

A locally compact Hausdorff space needs not be normal: e.g., take $]0,1[^J \subset [0,1]^J$ with J uncountable. Then $]0,1[^J$ is locally compact Hausdorff as it is open in $[0,1]^J$, which is compact Hausdorff (hence normal). But $]0,1[^J \cong \mathbb{R}^J$ not normal.

Theorem 4.9. *Let X be metrizable. Then X is normal.*

Proof. Let d be a metric inducing the topology on X, and $A, B \subset X$ be closed, $A \cap B = \emptyset$. Then for each $a \in A$ and $b \in B$, since $A \cap \overline{B} = \emptyset$ and $B \cap \overline{A} = \emptyset$, we can find $\epsilon_a > 0$ and $\epsilon_b > 0$ such that $B(a, \epsilon_a) \cap B = \emptyset$ and $B(b, \epsilon_b) \cap A = \emptyset$. Let

$$U = \cup_{a \in A} B(a, \epsilon_a/2), \quad V = \cup_{b \in B} B(b, \epsilon_b/2).$$

Then U, V open, $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$: for $a \in A$, $b \in B$, $d(a,b) \geq \max(\epsilon_a, \epsilon_b)$, but if there were $z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$ then

$$d(a,b) \leq d(a,z) + d(z,b) < \frac{\epsilon_a + \epsilon_b}{2} < \max(\epsilon_a, \epsilon_b).$$

□

Examples 4.10.

1. $\mathbb{R}^n$ is normal.
2. Any discrete space is normal - metrizable.
3. $\mathbb{R}^{\mathbb{N}}$ is normal - metrizable (any countable product of metrizable spaces is metrizable).
4. $\mathbb{R}^J$ with the uniform topology is normal (not with product!)

5 Urysohn's Lemma and applications

5.1 Urysohn's Lemma

One of the most important features of normal spaces is that normality is the suitable condition to prove the very useful Urysohn's Lemma.

Theorem 5.1 (Urysohn's Lemma). *Let X be a normal space, $A, B \subset X$ closed and disjoint. Then there exists continuous*

$$f : X \to [0,1] \quad s.t. \quad f(A) = 0,\ f(B) = 1.$$

(Of course, $[0,1]$ can be replaced by any interval $[a,b]$.) Before we get to the proof, note that, for a T1 space, the existence of such a function f clearly yields normality: $U = f^{-1}([0,\epsilon[$ and $V = f^{-1}(]1-\epsilon,1]$ are open disjoint (if $\epsilon \leq 1/2$) sets that contain A and B, respectively.

To prove this theorem, we have to use the the topology on X to construct 'from scratch' such a continuous function f. The main idea is to construct a sequence of ordered neighborhoods of A that will be the level sets of f.

We will use the alternative characterization of normal spaces: in a normal space, given A closed and $U \supset A$ open, there exists an open $V \supset A$ such that $\overline{V} \subset U$.

Proof. We outline the proof (see the details in [Munkres]).

Step 1) First we construct a sequence U_q, for $q \in [0,1] \cap \mathbb{Q}$ such that U_q is open and

$$p < q \Rightarrow \overline{U_p} \subset U_q.$$

We do this as follows: let $U_1 = X \setminus B$ and by normality, take U_0 open such that $A \subset U_0$ and $\overline{U_0} \subset U_1$. Let $P = \{q_n : n \in \mathbb{N}, q_n \in [0,1] \cap \mathbb{Q}\}$, for instance $P = \{1, 0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, ...\}$.

Then since $\overline{U_0} \subset U_1$, $\overline{U_0}$ closed, U_1 open, we can take $U_{\frac{1}{2}}$ open such that $\overline{U_0} \subset U_{\frac{1}{2}}$ and $\overline{U_{\frac{1}{2}}} \subset U_1$. We then proceed to define $U_{\frac{1}{3}}$ in between U_0 and $U_{\frac{1}{2}}$, then $U_{\frac{2}{3}}$ in between $U_{\frac{1}{2}}$ and U_1, etc.

- U_0 open such that $A \subset U_0$ and $\overline{U_0} \subset U_1$;
- $U_{\frac{1}{2}}$ open such that $\overline{U_0} \subset U_{\frac{1}{2}}$ and $\overline{U_{\frac{1}{2}}} \subset U_1$;
- $U_{\frac{1}{3}}$ open such that $\overline{U_0} \subset U_{\frac{1}{3}}$ and $\overline{U_{\frac{1}{3}}} \subset U_{\frac{1}{2}}$;
- $U_{\frac{2}{3}}$ open such that $\overline{U_{\frac{1}{2}}} \subset U_{\frac{1}{3}}$ and $\overline{U_{\frac{2}{3}}} \subset U_1$;
- ...

We can show by induction that if we have a family U_q defined for the first n elements of P then also $U_{q_{n+1}}$ is well-defined: just take its immediate predecessor p and sucessor q in $P_n = \{q_0, q_1, ..., q_n\}$. Since $p < q$, $\overline{U_p} \subset U_q$; let $U_{q_{n+1}}$ such that $\overline{U_p} \subset U_{q_{n+1}}$ and $\overline{U_{q_{n+1}}} \subset U_q$. Then $\overline{U_p} \subset U_q$ for all $p < q$, for all $p, q \in P_{n+1}$, that is, for all $p, q \in \mathbb{Q} \cap [0, 1]$.

For convenience, we define $U_q = \emptyset$ for $q \in \mathbb{Q}$, $q < 0$, and $U_q = X$, for $q \in \mathbb{Q}$, $q > 1$. Then $\overline{U_p} \subset U_q$ for all $p < q, p, q \in \mathbb{Q}$.

Step 2) For any $x \in X$, the set $\{p \in \mathbb{Q} : x \in U_p\}$ is non-empty and bounded from below, hence we can define

$$f(x) = \inf\{p \in \mathbb{Q} : x \in U_p\}.$$

Then $f(A) = 0$, and $f(B) = 1$.

Step 3) Show that f is continuous. Note that

(1) $x \in \overline{U_r} \Rightarrow f(x) \leq r$: since if $x \in \overline{U_r}$ then $x \in U_s$, for $s > r$, hence $f(x) \leq r$;

(2) $x \notin U_r \Rightarrow f(x) \geq r$: since for $s < r$, $U_s \subset U_r$, hence $x \notin U_s$, $s < r$ and $f(x) \geq r$.

Let $x_0 \in X$ such that $f(x_0) \in]a, b[$. Take $p, q \in \mathbb{Q}$ such that $x_0 \in]p, q[\subset]a, b[$. Then, $p < f(x_0) < q$ yields that $x_0 \in U_q \setminus \overline{U_p}$ (by (2), $f(x) < q \Rightarrow x \in U_q$ and by (1), $f(x) > p \Rightarrow x \notin \overline{U_p}$). Moreover,

$$x \in U_q \Rightarrow f(x) \leq q, \quad x \notin \overline{U_p} \Rightarrow f(x) \geq p.$$

It follows that for any $x \in U = U_q \setminus \overline{U_p}$ open, $f(x) \in [p, q]$ hence $f(U) \subset]a, b[$ and f is continuous. □

Corollary 5.2. *Let X be a normal space, A closed and $U \supset A$ open. Then there exists continuous*

$$f : X \to [0, 1] \quad s.t. \quad f(A) = 1,\ f(x) = 0, x \notin U.$$[5]

Proof. Just take $B = X \setminus U$ in Urysohn's lemma. □

It follows from Urysohn's Lemma that X is normal $\Leftrightarrow$ X is T1 and for $A, B \subset X$ closed, there is $f : X \to [0, 1]$ continuous such that $f(A) = 0$ and $f(B) = 1$.

[5] f is said to have *support* in U.

5.2 Completely regular spaces

Definition 5.3. *X is completely regular* if X is T1 and for $x \in X$, $B \subset X$ closed with $x \notin B$, there if $f : X \to [0,1]$ continuous such that $f(x) = 0$, $f(B) = 1$.

A completely regular space is regular: given $x \in X$ and A closed with $x \notin A$, let $U = f^{-1}([0,1/2[$ and $V = f^{-1}(]1/2,1])$. We always have

$$\text{normal} \Rightarrow \text{completely regular} \Rightarrow \text{regular}$$

but the reverse implications are also NOT true in general: $\mathbb{R}_l^2$ is completely regular (being the product of completely regular spaces) but not normal. Examples of regular spaces that are not completely regular are more sophisticated (see Tychonoff 'cork screw' example, or [Munkres] Ex. 33.11).

This new axiom is better behaved than normality:

Proposition 5.4. *1) Products: if $\forall \alpha \in J$, X_α is completely regular, then $X = \Pi_{\alpha\in J} X_\alpha$ is completely regular.*

2) Subspaces: if X is completely regular and $A \subset X$, then A is completely regular.

It follows that any subspace of a normal space is completely regular. In particular, any subspace of a compact, Hausdorff space is completely regular. In particular, any locally compact Hausdorff space is completely regular. We see now that completely regular spaces can always be embedded in a compact Hausdorff space, which leads to a characterization of spaces that have a compactification, and is a step in proving Urysohn's metrization theorem.

First note that we can give equivalent definition of a completely regular space using neighborhoods, instead of closed sets: X is completely regular if, and only if, X is T1 and given $x_0 \in U$ open, there exists a continuous function $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(x) = 0$, $x \notin U$.

Theorem 5.5 (Embedding theorem). *Let X be T1. If there is a family of continuous functions $f_\alpha : X \to [0,1]$ such that for $x_0 \in U$ open, there exists α such that $f_\alpha(x_0) > 0$ and $f_\alpha(X \setminus U) = 0$, then*

$$F : X \to [0,1]^J, \quad F(x) = (f_\alpha(x))_{\alpha\in J}$$

is an embedding.

Proof. First, F is continuous: in the product topology F is continuous iff $\pi_\alpha \circ F = f_\alpha$ is continuous for all $\alpha \in J$.

Then F is injective: if $x \neq y$, take a neighborhood U of x such that $y \notin U$ (X is T1). Then there is $\alpha \in J$ such that $f_\alpha(x) > 0$ and $f_\alpha(y) = 0$, hence $(F(x))_\alpha \neq (F(y))_\alpha \Rightarrow F(x) \neq F(y)$.

Now see that F is open: let $U \subset X$ be open, $x_0 \in U$ and $z_0 = F(x_0) \in F(U)$. We show that there is a neighborhhod of z_0 contained in $F(U)$, hence $F(U)$ is open.

Take α such that $f_\alpha(x_0) > 0$, ie, $(z_0)_\alpha > 0$, and $f_\alpha(X \setminus U) = 0$. Let $W = \pi_\alpha^{-1}(]0, 1]) \cap F(X)$, open in $F(X)$. Then $z_0 \in W$ and

$$z \in W \Leftrightarrow \pi_\alpha(z) > 0 \wedge z \in F(X) \Rightarrow z = F(x) \wedge \pi_\alpha(F(x)) = f_\alpha(x) > 0 \Rightarrow x \in U.$$

Hence, $W \subset F(U)$. □

Since the conditions of the embedding theorem are equivalent to X being completely regular, and by Tychonoff's theorem, $[0, 1]^J$ is compact, it follows:

Corollary 5.6. *X is completely regular if, and only if, X is homeomorphic to a subspace of a compact, Hausdorff space.*

In particular, any metrizable (hence normal) space embeds in a compact, Hausdorff space.

Remark 5.7. We saw in section 6 that a compactification of a space X is a compact Hausdorff space Y such that $X \subset Y$ and $\overline{X} = Y$. There can be many compactifications to a given (necessarily completely regular) space. We saw that if X is locally compact, then there is a minimal compactification, adding one point 'at infinity'.

Now any embedding $h : X \to Z$ with Z compact, Hausdorff, defines a compactification of X: take $X_0 = h(X) \cong X$ and $Y_0 = \overline{h(X)}$, which is compact, Hausdorff (being closed). Let A be a set such that there is a bijection $k : A \to Y_0 \setminus X_0$ and define $Y = X \cup A$. Then define $H : Y \to Y_0$ such that $H(x) = h(x), x \in X$, $h(x) = k(x), x \in A$. If we endow Y with the topology given by U open iff $H(U)$ open, we get that H is a homeomorphism, Y is compact Hausdorff and $X \subset Y$ is a subspace. Moreover, $H(\overline{X}) = \overline{H(X)} = \overline{h(X)} = Y_0$, hence $\overline{X} = Y$. (Note that $H : Y \to Z$ is an embedding that extends h.)

The results above show that

X is completely regular iff it has a compactification.

5.3 Urysohn's Metrization theorem

We will now see Urysohn's metrization theorem. Recall that a *countable* product of metric spaces is always metrizable: if $X = \Pi_{n \in \mathbb{N}} X_n$ and $\overline{d_n}(a) = \min\{d_n(a), 1\}$ is the standard bounded metric associated to the metric d_n on X_n, then we can define a metric (inducing the topology) on X for instance by

$$D(x, y) := \sup_{n \in \mathbb{N}} \frac{\overline{d_n}(x_n, y_n)}{n}.$$

Theorem 5.8 (Urysohn's metrization theorem)**.** *Let X be regular and have countable basis. Then X is metrizable.*

(Note that metrizable ⇒ regular, but not necessarily second countable. eg discrete uncountable space, or the space of sequence $\mathbb{R}^{\mathbb{N}}$ in the uniform topology.)

Proof. We will construct an embedding of X in $[0,1]^{\mathbb{N}}$ with the product topology, which yields that X, being a subspace of a metrizable space, is also metrizable.

We showed in Theorem 4.6 that a regular, second countable space is normal. We will use Urysohn's lemma, and the countable basis, to construct a countable family satisfying the conditions of the Embedding theorem.

Let $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ be a countable basis for the topology on X. Let $x_0 \in X$, U a neighborhood of x_0 and $B_n \in \mathcal{B}$ such that $x_0 \in B_n \subset U$. By regularity, we can take B_m such that $\overline{B_m} \subset B_n \subset U$. Now apply Urysohn's lemma (in fact, Corollary 5.2) to $\overline{B_m} \subset B_n$ to get that there is a continuous function

$$g_{n,m} : X \to [0,1], \quad g_{n,m}(\overline{B_m}) = 1,\ g_{n,m}(X \setminus B_n) = 0,$$

in particular, $g_{n,m}(x_0) = 1$ and $g_{n,m}(x) = 0$, $x \notin U$.

We obtain in this way a countable family of function satisfying the conditions of the Embedding theorem. Hence X can be embedded in $[0,1]^{\mathbb{N}}$, and therefore it is metrizable. □

Corollary 5.9. *X regular and second countable ⇔ X metrizable and separable .*

Proof. Recall that if X is metrizable, then second countable ⇔ separable. □

Corollary 5.10. *Let X be compact Hausdorff. Then X metrizable ⇔ second countable.*

Example 5.11. Recall that a *n-manifold M* is a Hausdorff, second countable space where each point has a neighborhood diffeomorphic to an open subset of $\mathbb{R}^n$.

Then, since the closed ball in $\mathbb{R}^n$ is compact (Heine-Borel's theorem), we can show that M is locally compact, hence regular. It follows that any n-manifold is metrizable.

Another nice application of Urysohn's Lemma is a short proof that any compact n-manifold can be embedded in $\mathbb{R}^N$, for some N (see section 36 - [Munkres]).

5.4 Tietze extension theorem

Our final application of Urysohn's lemma is an extension theorem. Extending continuous functions is a useful tool in many applications of topology. A consequence of Urysohn's lemma is the following important theorem, that says that in normal spaces, real functions on a *closed* subset can be always extended to the whole space.[6]

[6]It is clear that on open sets that is not always possible, e.g, $f(x) = 1/x$ on $\mathbb{R}^+$.

Theorem 5.12 (Tietze extension theorem)**.** *Let X be a normal space, $A \subset X$ be closed. Then*

(i) $f : A \to [a, b]$ continuous can be extended to $g : X \to [a, b]$ continuous;

(ii) $f : A \to \mathbb{R}$ continuous can be extended to $g : X \to \mathbb{R}$ continuous.

In fact, this theorem is equivalent to Urysohn's lemma, in that, in one direction, we use it in the proof, and conversely, if (i) or (ii) holds then $f(A) = 0$, $f(B) = 1$ for disjoint, closed $A, B \subset X$ can be extended to X, in particular, assuming X is T1, then it is normal

Proof. The main idea is to construct a sequence s_n of continuous functions, which converges uniformly to some function g with $g = f$ on A. Uniform convergence guarantees that g is continuous.

Let us consider the case (i), can assume that $[a, b] = [-1, 1]$. (Then (ii) will follow, identifying $\mathbb{R} \cong]-1, 1[$ and using (i).)

Let $B_1 = f^{-1}([-1, -\frac{1}{3}])$ and $C_1 = f^{-1}\left(\left[\frac{1}{3}, 1\right]\right)$. Then B_1, C_1 are closed in A, hence in X, as A is closed, and disjoint. By Urysohn's lemma, there exists

$$g_1 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right], \quad g_1(B_1) = -\frac{1}{3}, \quad g_1(C_1) = \frac{1}{3}.$$

In particular:

(i) $|g_1(x)| \leq \frac{1}{3}$, for $x \in X$,

(ii) $|f(a) - g_1(a)| \leq \frac{2}{3}$, for $a \in A$: note that $[-1, 1] = \left[-1, -\frac{1}{3}\right] \cup \left[-\frac{1}{3}, \frac{1}{3}\right] \cup \left[\frac{1}{3}, 1\right]$ and each of these sub-intervals has length 2/3, so it suffices to show that $f(a)$ and $g_1(a)$ are in the same sub-interval. Now have 3 cases:
   – $a \in B_1$: then $f(a) \in [-1, -\frac{1}{3}]$ and $g_1(a) = -\frac{1}{3}$;
   – $a \in C_1$: then $f(a) \in \left[\frac{1}{3}, 1\right]$ and $g_1(a) = \frac{1}{3}$;
   – $a \in A \setminus (B_1 \cup C_1)$: then $f(a), g_1(a) \in \left[-\frac{1}{3}, \frac{1}{3}\right]$.

Let $s_1 = g_1$.

Now apply the step above to the function $f - g_1 : A \to \left[-\frac{2}{3}, \frac{2}{3}\right]$ to obtain a function $g_2 : X \to \left[-\frac{2}{9}, \frac{2}{9}\right]$ such that

$$|f(a) - g_1(a) - g_2(a)| \leq \left(\frac{2}{3}\right)^2, \; a \in A.$$

Let $s_2 = g_1 + g_2$.

We can proceed by induction to define functions $g_n : X \to \left[-\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}, \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}\right]$ such that, writing $s_n = \sum_{i=1}^{n} g_i$, we get

$$|f(a) - s_n(a)| \leq \left(\frac{2}{3}\right)^n, \; a \in A.$$

In particular, for $a \in A$, $s_n(a) \to f(a)$, that is, s_n converges pointwise to f in A.

Moreover, since $|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$, by the comparison test for infinite series, for each $x \in X$, $s_n(x) = \sum_{i=1}^n g_i(x)$ converges and we define

$$g(x) = \sum_{n=1}^{\infty} g_n(x) = \lim_{n\to\infty} s_n(x).$$

We have that $g = f$ on A. Need only show that g is continuous, which will follow once we show that the convergence is uniform (exercise). □

In fact, the fact that the sequence s_n converges uniformly (can be proved directly but) follows from the fact that it is a *Cauchy sequence* and that $\mathbb{R}$ is *complete*. We will see these notions next.

6 Compactness

6.1 Compact spaces

Definition 6.1. Let X be a topological space. A collection of subsets $\{U_\alpha\}_{\alpha\in J}$ is a *cover* for X if $X = \cup_{\alpha\in J} U_\alpha$.

The space X is *compact* if for any open cover there is a finite subcover, that is, if $X = \cup_{\alpha\in J} U_\alpha$ with U_α open, there exist $\alpha_1, ...\alpha_p$ such that

$$X = \bigcup_{i=1}^{p} U_{\alpha_i}.$$

When considering subspaces $A \subset X$, it suffices to consider a cover by subsets of X, since open sets in A are of the form $A \cap U$, with U open in X. We refer to a cover of A as $\{U_\alpha\}$ such that $A \subset \cup U_\alpha$, $U_\alpha \subset X$.

Examples 6.2. 1. Finite sets are compact in any topology.

2. In the discrete topology: X is compact iff it is finite, as the sets $\{x\}$, $x \in X$ are an open cover.

3. In the cofinite topology: any set is compact. In the cocountable topology compacts are finite.

4. $\mathbb{R}$ not compact: $]-n, n[$, $n \in \mathbb{N}$ gives an open cover with no finite subcover.

   We see also that if a subset $A \subset \mathbb{R}$ is compact then $A \subset]-N, N[$ for some N, hence it is bounded.

5. $]0,1[$, $]0,1]$, $[0,1[$ not compact in $\mathbb{R}$ with the standard topology: e.g., $]-1/n, 1[$ is an open cover of $]0,1[$ with no finite subcover.

6. $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is compact in $\mathbb{R}$ with the standard topology: take an open cover $\{U_\alpha\}$, then $0 \in U_{\alpha_0}$, hence for $n > N$, $1/n \in \mathcal{U}_{\alpha_0}$. If we take U_{α_i} covering $\{\frac{1}{i}\}$ for $i = 1, ..., N$ we obtain a finite subcover.

Compactness is a topological property:

Proposition 6.3. *Let $f : X \to Y$ be continuous and X compact. Then $f(X)$ is compact.*

Proof. Let $\{U_\alpha\}$ be an open cover for $f(X)$. Then $f^{-1}(U_\alpha)$ is an open cover for X hence it has a finite subcover: $X = \cup_{i=1}^p f^{-1}(U_{\alpha_i})$. Then $f(X) \subset \cup_{i=1}^p U_{\alpha_i}$ (if $y \in f(X)$, then $y = f(x)$, $x \in U_{\alpha_i}$ for some $i = 1, ..., p$.) □

Corollary 6.4. *If $f : X \to Y$ is a homeomorphism, then X compact if, and only if, Y is compact.*

Examples 6.5. 1. If $X = \Pi X_\alpha$ is compact, then X_α is compact, for all α.

2. If Y is a quotient of a compact space X then Y is compact.

Proposition 6.6. *Let X be compact, and $A \subset X$ be closed. Then A is compact.*

Proof. (i) Let $\{U_\alpha\}$ be an open covering of A. Then $\{U_\alpha\} \cup X \setminus A$ is an open covering for X, hence has a finite subcover: $X = X \setminus A \cup U_{\alpha_1} \cup ... \cup U_{\alpha_p}$ and $A \subset U_{\alpha_1} \cup ... \cup U_{\alpha_p}$. □

The following result shows that in Hausdorff spaces, compact subsets can separated by open sets (ie, behave like points).

Lemma 6.7. *Let X be Hausdorff, A be compact,*

(i) if $x \notin A$ then there exist U, V open, disjoint, such that $U \supset A$ and $x \in V$.

(ii) if $B \subset X$ is compact, $A \cap B = \emptyset$, then there exist U, V open, disjoint, such that $U \supset A$ and $V \supset B$.

Proof. (i) For each $y \in A$, since $x \neq y$ and X is Hausdorff, there exist neighborhoods V_y of x and U_y of y such that$U_y \cap V_y = \emptyset$. Since $\{U_y\}_{y \in Y}$ is an open cover for A and A is compact, can take a finite subcover $A \subset U_{y_1} \cup ... \cup U_{y_p}$. Let $U = U_{y_1} \cup ... \cup U_{y_p}$ and $V = V_{y_1} \cap ... \cap V_{y_p}$. Then U, V are open, $A \subset U$, $x \in V$ and $U \cap V = \emptyset$ (since $V_{y_i} \cap U_{y_i} = \emptyset$ for all $i = 1, ..., p$).

(ii) Exercise.

□

Proposition 6.8. *Let X be Hausdorff, $A \subset X$ be compact. Then A is closed.*

Proof. We make use of the previous lemmma (i): let $x_0 \notin A$, then there exists a neighborhood U of x_0 that does not intersect A, hence $x_0 \notin \overline{A}$. Hence $\overline{A} \subset A$, that is, $\overline{A} = A$ and A is closed. □

In particular, if X is compact Hausdorff, the closed subsets coincide with the compacts. If the space is not Hausdorff, then compact sets need not be closed: e.g, in $\mathbb{R}$ with cofinite topology closed sets are finite but any set is compact (in particular, $\mathbb{R}$ is compact).

Lemma 6.9. *Let $f : X \to Y$ be continuous, where X is compact and Y is Hausdorff. Then f is a closed map.*

Proof. Let $A \subset X$ be closed. Then since X is compact, A is compact in X and $f(A)$ is compact in Y, since f is continuous. As Y is Hausdorff, any compact set is closed. □

Theorem 6.10. *Let $f : X \to Y$ be continuous, where X is compact and Y is Hausdorff. Then:*

(i) if f is bijective, then it is a homeomorphism.

(ii) if f is injective, then it is an embedding;

(iii) if f surjective, then it is a quotient map;

Proof. (i) A closed bijection has continuous inverse.
(ii) Follows from (i), as $f(X)$ is Hausdorff.
(iii) Any surjective, continuous, closed map is a quotient map. □

We have noted already that any compact in $\mathbb{R}$ must be bounded, and since $\mathbb{R}$ is Hausdorff, also closed. We now see:

Theorem 6.11. *Any closed, bounded interval in $\mathbb{R}$ is compact.*

Proof. Let $\{U_\alpha\}$ be an open cover for $[a, b]$ and let

$$A = \{x \in [a, b] : [a, x] \text{ has a finite subcover } \}.$$

(i) $A \neq \emptyset$ as $a \in A$ and is bounded above, as $A \subset [a, b]$. Hence there exists $c = \sup A \in [a, b]$.

(ii) $c > a$: since $a \in U_\alpha$ for some α, hence, as U_α is open, $[a, x[\subset U_\alpha$, for some $x > 0$, so $[a, x']$ has a finite subcover, $a < x' < x$.

(iii) $A = [a, c]$: since for some $\epsilon > 0$, $]c - \epsilon, c] \subset U_\beta$, as $c \in [a, b]$ and U_β is open. On the other hand, there is $x \in A\cap]c-\epsilon, c]$ (by definition of sup) so $[a, c] = [a, x]\cup]c-\epsilon, c] \subset U_1 \cup ... \cup U_p \cup U_\beta$ has a finite cover.

(iv) $c = b$: if not, $]c - \epsilon, c + \epsilon[\subset U_\beta$ and by the reasoning in (iii), there would be $y \in A$ with $y > c$.

□

Note that the proof above holds in an ordered space satisfying the sup axiom, endowed with the order topology.

Examples 6.12. 1. In $\mathbb{R}_l$, $[0,1]$ not compact: $\{[0,1-1/n[\}_{n\in\mathbb{N}} \cup [1,2[$ is an open cover with no finite subcover.

2. In $\mathbb{R}_K$ where basic open sets are of the form $]a,b[\backslash\{\frac{1}{n} : n \in \mathbb{N}\}$, $[0,1]$ is also not compact.

Recall that in a metric space, $A \subset A$ is bounded if $\mathrm{diam}(A) = \sup_{x,y\in A} d(x,y) < M$, for some $M > 0$.

Theorem 6.13. *Let X be a metric space. Then if $A \subset X$ is compact, then it is closed and bounded.*

Proof. A metric space is Hausdorff, hence if A is compact, it is closed. On the other hand, taking a covering of X by balls $B(x,\epsilon)$, $x \in X$, for some ϵ, since A is compact, $A \subset B(x_1,\epsilon) \cup ... \cup B_(x_p,\epsilon)$, hence A is bounded as for all $x, y \in A$,

$$d(x,y) \leq 2\epsilon + \max_{i,j=1,\ldots,p} d(x_i,x_j).$$

□

The converse is not true in general, for instance, any discrete topology is metric, and any set is closed and bounded, but the compact sets coincide with the finite sets.

Theorem 6.14 (Heine-Borel)**.** *In $\mathbb{R}^n$, a set is compact if, and only if, it is closed and bounded (with the standard or square metric).*

(Note that both metrics induce the same topology, and that bounded sets are also the same.)

Proof. Since $\mathbb{R}^n$ is a metric space, if $A \subset \mathbb{R}^n$ is compact, then it is closed and bounded.

If A is bounded, then it is contained in some r-ball is the square metric, that is a cube $[-r,r]^n$. We have seen that $[-r,r]$ is compact in $\mathbb{R}$ and will see that a finite product of compact sets is compact. Then if $A \subset [-r,r]^n$ is closed, with $[-r,r]^n$ compact, then A is compact. □

Theorem 6.15 (Weierstrass)**.** *If $f : X \to \mathbb{R}$ is continuous, and X is compact, then f has a maximum and minimum value on X.*

6.2 Products

The aim of this section is to show that an arbitrary product of compact spaces is compact. This is a deep result, called Tychonoff's theorem.

For finite (even countable products) we can give a direct geometric proof. It relies on what is known as the 'Tube lemma':

Lemma 6.16 (Tube Lemma)**.** *Let X, Y be topological spaces, with Y compact. If N is an open subset of $X \times Y$ such that $\{x_0\} \times Y \subset N$, then there exists $W \subset X$ open, $x_0 \in W$, such that $W \times Y \subset N$.*

(That is, if N contains the 'line' $\{x_0\} \times Y$, then it contains a 'tube' around $\{x_0\} \times Y$.)

Exercise: Use the Tube lemma to show that if Y is compact, then the projection $\pi : X \times Y \to X$ is a closed map.

Theorem 6.17. *If X and Y are compact, then $X \times Y$ is compact.*

(In fact, it is if and only if, as projections are continuous maps, hence if $X \times Y$ is compact, also X and Y are compact.)

For infinite products:

Theorem 6.18 (Tychonoff's theorem). *Let $X = \Pi_{j \in J} X_j$. Then X is compact if, and only if, X_j is compact for all $j \in J$.*

Finite initersection property

6.3 Locally compact spaces

Even if the space is not compact, it is useful to have some notion of compactness, even locally.

Definition 6.19.

X is locally compact

Definition 6.20. A *compactification* of a space X is a compact Hausdorff space Y such that $X \subset Y$ and $\overline{X} = Y$.

6.4 Compactness in metric spaces

6.5 Limit point and sequential compactness

We see now two other important versions of compacity, that turn out to be the same in metric spaces.

Definition 6.21. A space X is *limit point compact* (or has the *Bolzano-Weiertrass property*) if any infinite set has a limit point. A space X is *sequentially compact* if any sequence in X has a convergent subsequence.

Limit point compactness is always weaker than compactness and sequential, as we see now.

Proposition 6.22. *(i) If X is compact, then it is limit point compact.*

(ii) If X is sequentially compact, then it is limit point compact.

Proof. (i) Let $A \subset X$ be a set with no limit points. Then, for any $a \in A$, there is an open U such that $U \cap A = \{a\}$, since a is not a limit point of A, that is, A is discrete. On the other hand, since $\overline{A} = A \cup A'$, we have A is closed, hence compact, as X is compact. A discrete compact subset is finite.

(ii) Let A be an inifite set, take a sequence $a_n \in A$ such that $\{a_n : n \in \mathbb{N}\}$ is infinite. Then a_n has a convergent subsequence $a_{n_k} \to a \in A$ and a is a limit point of $\{a_n : n \in \mathbb{N}\}$, hence of A. □

The converse is not true.

Examples 6.23.

1. the space of sequence $[0, 1]^{\mathbb{N}}$ with the uniform topology is not limit point compact, hence not compact: $\{0, 1\}^{\mathbb{N}}$ is an infinite set with no limit points (it is discrete).

2. $\mathbb{R}^J$, J uncountable with the product topology is compact (from Tychonoff's theorem below) but not sequentially compact.

3. $X = \mathbb{N} \times \{0,1\}$ where $\{0,1\}$ has the indiscrete topology. Then any set has a limit point, but for instance the sequence $\{(n,0)\}$ has no convergent subsequence.

A limit point is not necessarily the limit of some sequence, though we shall see in shall see in Proposition 3.2 that this is true if each point of X has a *countable* basis of neighborhoods, that is if X is what is called *first countable*.

Proposition 6.24. *If X is first countable, then X limit point compact $\Leftrightarrow$ sequentially compact.*

There is, in general, no relation between compactness and sequential compactness (Exercise: if X T1 and second countable then compact $\Leftrightarrow$ sequentially compact.)

However, if we have a metric these notions coincide.

Theorem 6.25. *Let (X,d) be a metric space. Then*

$$\textit{compact} \Leftrightarrow \textit{sequentially compact} \Leftrightarrow \textit{limit point compact}$$

Metric spaces are first countable, hence sequentially compact $\Leftrightarrow$ limit point compact. We has already seen compact $\Rightarrow$ limit point compact. So we are left with showing that sequentially compact and metric yields compact. This can be proved using the *Lebesgue number*.

We have seen already that in a metric space (X,d)

$$\text{compact} \Leftrightarrow \text{sequentially compact} \Leftrightarrow \text{limit point compact}$$

Moreover, in a metric space, any compact set is closed and bounded, but the converse is not true in general (it is true in $\mathbb{R}^n$, according to the Heine-Borel theorem).

We have seen already that any compact metric space is complete. Our aim now is to give a generalization of Heine-Borel's theorem to metric spaces, using the notion of completeness. To do this, we need a richer notion of 'boundedness'.

Definition 6.26. A metric space (X,d) is *totally bounded* if for any $\epsilon > 0$ there is a finite covering of X by ϵ-balls:

$$X = B(x_1,\epsilon) \cup ... \cup B(x_p,\epsilon).$$

It is clear that X totally bounded yields X bounded: take $\epsilon = 1$, $x_1, ..., x_p$ as above and $M = \max_{i,j=1,...,p} d(x_i,x_j)$, then for $x,y \in X$, $x \in B(x_i,1)$ and $y \in B(x_j,1)$ for some x_i, x_j and

$$d(x,y) \leq d(x,x_i) + d(y,x_j) + d(x_j,x_i) < M + 2, \ \forall x,y \in X.$$

The converse is not true, for instance, with the standard bounded metric any set is bounded, but not necessarily totally bounded ($\mathbb{R}$ not totally bounded with $\overline{d}(x,y) = \min\{1, |x-y|\}$).

Note that for subspaces, $A \subset X$ is totally bounded if, and only if, given $\epsilon > 0$, there is a covering of A by ϵ-balls in X, that we can assume, centered in $a_i \in A$:

$$A \subset B(a_1,\epsilon) \cup ... \cup B(a_p,\epsilon), \ a_i \in A.$$

In particular, if A totally bounded and $B \subset A$ then B is totally bounded. Also, A totally bounded if, and only if, $\overline{A}$ is totally bounded (if $\epsilon' < \epsilon$ then $\overline{B(a,\epsilon')} \subset B(a,\epsilon)$[7]).

Examples 6.27.
1. In $\mathbb{R}^n$ with the usual Euclidean metric (or square), bounded $\Leftrightarrow$ totally bounded.
2. X infinite with the discrete metric, then any set is bounded but no infinite set is totally bounded: if $\epsilon < 1$ then an ϵ-ball only covers a one point set.

If X is a compact metric space, then X is always totally bounded: just take a covering of X by ϵ-balls $B(x,\epsilon)$, $x \in X$, then there is a finite subcover. Now we have:

Theorem 6.28. *A metric space (X,d) is compact if, and only if, it is complete and totally bounded.*

Proof. We need only show that if X is complete and totally bounded then it is compact. We show it is sequentially compact.

Let (a_n) be a sequence in X. We use total boundedness to construct a Cauchy subsequence of (a_n), which will then converge, as X is complete.

Since X can be covered by finite number of 1-balls, there is a 1-ball B_1 that contains an infinite number of a_n's, that is, such that $B_1 \cap \{a_n : n \in \mathbb{N}\}$ is infinite. Let $J_1 = \{n \in \mathbb{N} : x_n \in B_1\}$ is infinite.

Now, X is a finite union of $1/2$-balls, hence there is a $1/2$-ball B_2 such that $B_2 \cap \{x_n : n \in J_1\}$ is infinite. Let $J_2 = \{n \in J_1 : x_n \in B_2\}$.

By induction, we construct a sequence of infinite index sets

$$J_1 \supset J_2 \supset \ldots \supset J_n \supset \ldots$$

such that $J_{k+1} = \{n \in J_k : x_n \in B_{k+1}$, with B_{k+1} a $\frac{1}{k+1}$-ball. Now we define our Cauchy subsequence: just pick $n_1 \in J_1$, and $n_2 \in J_2$ with $n_2 > n_1$ (possible since J_2 infinite) and in general, given $n_k \in J_k$, pick $n_{k+1} > n_k$, with $n_{k+1} \in J_{k+1}$. Then, for any $k \in \mathbb{N}$:

$$\begin{aligned} i, j, > k &\Rightarrow n_i, n_j \in J_k \\ &\Rightarrow a_{n_i}, a_{n_j} \in \frac{1}{k} - \text{ball} \\ &\Rightarrow d(a_{n_i}, a_{n_j}) < \frac{2}{k}. \end{aligned}$$

Hence (a_{n_k}) is a Cauchy subsequence of (a_n), hence it converges, as X is complete. □

In fact, we saw that if X is totally bounded, then any sequence in X has a Cauchy subsequence. (The converse is also true.)

[7] It is not always true that $\overline{B(a,\epsilon')} = \{x \in X : d(x,a) \leq \epsilon'\}$ - discrete topology.

Corollary 6.29. *Let (X, d) be complete. Then $A \subset X$ is compact if, and only if, it is closed and totally bounded.*

A subset $A \subset X$, X a topological space, is said to be *relatively compact* if $\overline{A}$ is compact.

Corollary 6.30. *Let (X, d) be complete. Then $A \subset X$ is totally bounded if, and only if, A is relatively compact.*

Example 6.31. In $\mathbb{R}^{\mathbb{N}}$ with the uniform topology: it is a complete space, and $B = \{x \in \mathbb{R}^{\mathbb{N}} : \overline{\rho}(x) \leq 1\}$ is closed. However, it is not totally bounded as it contains an infinite dense subset. Hence it is not compact.

7 Complete metric spaces and function spaces

7.1 Completeness

Let (X, d) be a metric space.

Definition 7.1. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is a *Cauchy sequence* if for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that

$$n, m > N \Rightarrow d(x_n, x_m) < \epsilon.$$

It is clear that any convergent sequence is a Cauchy sequence: if $x_n \to x$ then

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x).$$

It is also easy to check that any Cauchy sequence is bounded: take N such that $d(x_n, x_m) < 1$ for $n, m > N$ and $M = \max_{n,m=1,\dots,N}\{d(x_n, x_m)\}$, then

$$d(x_n, x_m) \leq \max\{M, 1\}, \forall n, m \in \mathbb{N} \;\Rightarrow\; \operatorname{diam}(\{x_n : n \in \mathbb{N}\}) \leq \max\{M, 1\}.$$

Definition 7.2. A metric space is *complete* if any Cauchy sequence is convergent.

A subspace A of a metric space (X, d) is complete if, and only if, it is complete for the induced metric $d_{|A}$.

Examples 7.3.

1. $]0, 1[$ not complete: $x_n = 1/n$ not convergent.
2. $\mathbb{Q}$ not complete.
3. Any discrete topological space X with the metric

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y, \end{cases}$$

is complete, as any Cauchy sequence is eventually constant.

Proposition 7.4. *Let (X, d) be a metric space*

1. *If $A \subset X$ is closed and X is complete then A is complete.*
2. *If $A \subset X$ is complete, then A is closed.*

Proof. (i) If (a_n) is a Cauchy sequence in A, then it is a Cauchy sequence in X, hence it is convergent in X, as X complete. Since A closed: $\lim a_n \in \overline{A} = A$.

(ii) In a metric space[8] A is closed if, and only if, for any sequence $a_n \to x$, $a_n \in A \Rightarrow x \in A$. Let then $a_n \in A$, with $a_n \to x$. Then (a_n) is a Cauchy sequence in X, hence in A, hence it converges in A. Since there is unicity of limits (X Hausdorff), $\lim a_n = x \in A$.

(Or: if $x \in \overline{A}$, then there is $a_n \in A$ with $a_n \to x$ - as X is metric / first countable. If $x \notin A$, then a_n is a Cauchy sequence in A that does not converge in A, so A not complete.) □

Example 7.5. $\mathbb{Q} \cap [0, 2]$ is closed in $\mathbb{Q}$ but not complete.

Lemma 7.6. *X is complete if, and only if, any Cauchy sequence has a convergent subsequence.*

Proof. Let (x_n) be a Cauchy sequence and $x_{n_k} \to x$, $k \in \mathbb{N}$, ($n_k \in \mathbb{N}$ an increasing sequence) be a convergent subsequence. We prove that $x_n \to x$:

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x).$$

Given $\epsilon > 0$, choose N such that $d(x_n, x_m) < \epsilon/2$ for $n, m > N$ and $k \in \mathbb{N}$ such that $n_k > N$ and $d(x_{n_k}, x) < \epsilon/2$. Then $n > N \Rightarrow d(x_n, x) < \epsilon$. □

Corollary 7.7. *If (X, d) is compact then X is complete.*

Proof. In metric spaces, compact $\Leftrightarrow$ sequentially compact, hence any sequence has a convergent subsequence. □

Now we check completeness of well-known spaces.

Theorem 7.8. *$\mathbb{R}^n$ complete with usual or square metric $\rho(x, y) = \max_{i=1,\dots,n} |x_i - y_i|$.*

Proof. First note that $\mathbb{R}^n$ with the usual Euclidean metric is complete if, and only if, $(\mathbb{R}^n, \rho)$ is complete: the induced topologies are the same, hence the convergent sequences are the same, and check that (x_n) is Cauchy sequence in the Euclidean metric, if, and only if, it is a Cauchy sequence in $(\mathbb{R}^n, \rho)$.

Let then (x_n) be a Cauchy sequence in $(\mathbb{R}^n, \rho)$, then (x_n) is bounded in $\mathbb{R}^n$, hence it is contained in some ρ-ball:

$$\{x_n : n \in \mathbb{N}\} \subset B_\rho(0, r) = [-r, r]^n.$$

Since $[-r, r]^n$. is compact, hence sequentially compact, (x_n) has a convergent subsequence, and completeness follows from the previous lemma. □

[8] In fact, in any first countable space.

Remark 7.9. Completeness is not a topological property: $\mathbb{R} \simeq]0,1[$, and $\mathbb{R}$ complete, $]0,1[$ not complete.

In fact, being a Cauchy sequence is not a topological property: $f :]0,1] \to [1,+\infty[, x \mapsto 1/x$ is a homeomorphism but $x_n = 1/n$ is Cauchy in $]0,1]$, $f(x_n) = n$ not Cauchy in $[1,+\infty[$. It is true however that if $f : X \to Y$ is *uniformly continuous*[9] and (x_n) is a Cauchy sequence in X then $(f(x_n))$ is a Cauchy sequence in Y.

The finite product of complete metric spaces is always complete in the square metric, which induces the product topology. The countable product of complete metric spaces (Y_i, d_i) is also complete, with the metric

$$D(x,y) = \sup_{i\in\mathbb{N}}\{\overline{d_i}(x_i,y_i)/i\}$$

where $\overline{d_i}$ is the standard bounded metric with respect to d_i. In particular, if (Y,d) is complete, the space of sequences $(Y^{\mathbb{N}}, D)$ is complete, where D induces the product topology.

Example 7.10. l^p is complete

When we take uncountable products, then the product topology on Y^X, X uncountable, is not even metrizable. But if we take the uniform metric

$$\overline{\rho}(f,g) := \sup\{\overline{d}(f(x), g(y)) : x \in X\}$$

where $\overline{d}$ is the standard bounded metric with respect to d on Y, i.e., $\overline{d}(f(x), g(x)) = \min\{d(f(x), g(x)), 1\}$ then we do obtain a complete metric space.

Theorem 7.11. *If (Y,d) is complete then $(Y^X, \overline{\rho})$ is complete.*

Proof. First note that if (Y,d) is complete, then $(Y, \overline{d})$ is also complete. Let (f_n) be a Cauchy sequence in $(Y^X, \overline{\rho})$. Then, for each $x \in X$,

$$\overline{d}(f_n(x), f_m(x)) \leq \overline{\rho}(f_n(x), f_m(x)),$$

hence $(f_n(x)$ is a Cauchy sequence in Y, and therefore it is convergent, as Y is complete.

Define $f : X \to Y$ such that $f(x) := \lim f_n(x)$. Then, by definition, $f_n \to f$ pointwise (ie in the product topology). We now prove that $f_n \to f$ in the $\overline{\rho}$ metric.

Given $\epsilon > 0$, let N be such that $n, m > N$ then

$$\overline{\rho}(f_n(x), f_m(x)) < \epsilon \;\Rightarrow\; \overline{d}(f_n(x), f_m(x)) < \epsilon, \forall x \in X.$$

Fix $n > N$, $x \in X$ and let $m \to \infty$, then it follows that also $\overline{d}(f_n(x), f(x)) < \epsilon$, hence

$$\overline{\rho}(f_n(x), f(x)) = \sup_{x\in X} \overline{d}(f_n(x), f(x)) < \epsilon,\; n > N \;\Rightarrow\; f_n \to f \text{ in } (Y^X, \overline{\rho}).$$

□

[9] f is uniformly continuous in A if given $\epsilon > 0$, there is $\delta > 0$ such that for all $x, y \in A$: $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$

(Note that we proved in particular that if (f_n) is a Cauchy sequence in the uniform metric and $f_n \to f$ pointwise, then $f_n \to f$ uniformly.)

Now assume that X is also a topological space and consider the subspace of Y^X given by *continuous* functions:

$$\mathbb{C}(X, Y) := \{f : X \to Y : f \text{ is continuous}\}.$$

Recall the uniform limit theorem: if (Y, d) is a metric space and $(f_n) \in \mathbb{C}(X, Y)$ is such that, for some $f \in Y^X$, given $\epsilon > 0$ one can find N such that $d(f_n(x), f(x)) < \epsilon$, for all $x \in X$, then $f \in \mathbb{C}(X, Y)$.

It is easy to check that the condition above is equivalent to $f_n \to f$ in $(Y^X, \overline{\rho})$, hence the uniform limit theorem basically says that $\mathbb{C}(X, Y)$ is closed in Y^X.

Theorem 7.12. *Let (Y, d) be complete and X be a topological space. Then $\mathbb{C}(X, Y)$ is complete in the uniform metric.*

Proof. We have just seen that $\mathbb{C}(X, Y)$ is closed in the complete metric space $(Y^X, \overline{\rho})$. □

In particular, $\mathbb{C}(X, \mathbb{R}^n)$ is complete.

Another closed subspace, as we shall see next, is the space of *bounded* functions:

$$\mathcal{B}(X, Y) := \{f : X \to Y : f \text{ is bounded}\}.$$

Here, we can take the equivalent metric, usually called the *sup metric* given by

$$\rho(f, g) = \sup_{x \in X}\{d(f(x), g(x)\},$$

that is, $\overline{\rho} = \min\{\rho, 1\}$ is the standard bounded metric with respect to ρ. Hence it is clear that a sequence is Cauchy with respect to ρ if, and only if, it is Cauchy with respect to $\overline{\rho}$. Note that when X is compact,

$$\mathbb{C}(X, Y) \subset \mathcal{B}(X, Y)$$

as we have seen in Theorem 6.13 that any compact set in a metric space is bounded (and a continuous function maps compacts to compacts).

Theorem 7.13. *Let (Y, d) be complete and X be a topological space. Then $\mathcal{B}(X, Y)$ is complete in the uniform / sup metric.*

Proof. We show that $\mathcal{B}(X, Y)$ is closed. Let $f_n \in \mathcal{B}(X, Y)$ be such that $f_n \to f$ uniformly, or equivalently, in the sup metric. Then, given $\epsilon > 0$,

$$\sup_{x \in X} d(f_n(x), f(x)) < \epsilon, n > N.$$

Take N such that $d(f_N(x), f(x)) < 1/2$, for all $x \in X$ and let $M > 0$ be such that $d(f_N(x), f_N(y)) < M$, for all $x, y \in X$, which exists since $f_N(X)$ is bounded. Then

$$d(f(x), f(y)) \leq d(f_N(x), f(x)) + d(f_N(y), f(y)) + d(f_N(x), f_N(y)) < M + 1, \ \forall x, y \in X$$

hence $f \in \mathcal{B}(X, Y)$ and completeness follows. □

We now see that any metric space can be embedded in a complete metric space.

Theorem 7.14. *(X, d) be a metric space. Then there is an isometric embedding of X in a complete metric space.*

Proof. To prove this result, we embed X in $\mathcal{B}(X, \mathbb{R})$.

Let $x_0 \in X$ be fixed. Given $a \in X$, let

$$\phi_a(x) := d(x, a) - d(x, x_0), \quad \phi_a : X \to \mathbb{R}.$$

Always have $|d(x, a) - d(x, b)| \leq d(a, b)$, hence $|\phi_a(x)| \leq d(a, x_0)$ and $\phi_a \in \mathcal{B}(X, \mathbb{R})$.

Show now that the map $\Phi : X \to \mathcal{B}(X, \mathbb{R})$, $a \mapsto \phi_a$ is an isometric embedding:

$$\rho(\phi_a, \phi_b) = \sup_{x\in X} |\phi_a(x) - \phi_b(x)| = \sup_{x\in X} |d(x, a) - d(x, b)| = d(a, b)$$

since $|d(x, a) - d(x, b)| \leq d(a, b)$, for all $x \in X$ and for $x = a$, $|d(x, a) - d(x, b)| = d(a, b)$. □

Definition 7.15. Let (X, d) be a metric space and $h : (X, d) \to (Y, D)$ be an isometric embedding, with (Y, D) complete. The *completion* of X is defined as $h(\overline{X})$,

The completion is unique, up to an isometry.

Remark 7.16. There is an alternative way of proving the existence of a completion: take equivalence classes of Cauchy sequences (so proceed in a similar way as to define $\mathbb{R}$ as the completion of $\mathbb{Q}$): let

$$\tilde{X} := \{x = (x_n) \in X^{\mathbb{N}} : (x_n) \text{ is Cauchy } \}$$

and define an equivalence relation on $\tilde{X}$ such that $x \sim y$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Let $Y = X/\sim$ and $D : Y^2 \to \mathbb{R}$, $D([x], [y]) := \lim_{n\to\infty} d(x_n, y_n)$. Show that D is well-defined (let $z_n = d(x_n, y_n)$, then show (z_n) is Cauchy sequence in $\mathbb{R}$ hence it converges) and it is a metric such that (Y, D) is complete.

Now let $h : X \to Y$ given by $h(x) = [(x, x, ..., x, ...)]$: it is injective, and $D(h(x), h(y)) = d(x, y)$ hence it is an isometric embedding.

The space $h(X)$ is actually dense in $Y = X/\sim$, that is Y is the completion of X.

Examples 7.17. $\mathbb{R}$ is the completion of $\mathbb{Q}$.

We shall use the characterization above to study compact subsets of spaces of *continuous funtions*. We make use of the following notion, often easier to check than total boundedness:

Definition 7.18. Let X be a topologcal space, (Y, d) be a metric space. A subset $\mathcal{F} \subset \mathbb{C}(X, Y)$ is said to be *equicontinuous* if for $x_0 \in X$, there is an open set U, $x_0 \in U$, such that for *all* $f \in \mathcal{F}$,

$$x \in U \Rightarrow d(f(x), f(y)) < \epsilon.$$

The point is that the same neighborhood U can be chosen, independently of $f \in \mathcal{F}$. We now see what is the relation between equicontinuous families and total boundedness in $\mathbb{C}(X, Y)$.

Lemma 7.19. *Let X be a topological space, (Y, d) be a metric space, and $\mathcal{F} \subset \mathbb{C}(X, Y)$ endowed with the uniform metric*

(i) $\mathcal{F}$ is totally bounded then is equicontinuous

(ii) If X, Y are compact: $\mathcal{F}$ equicontinuous then totally bounded (with respect to uniform or sup metric).

Proof. (i) Let $x_0 \in X$ and $\epsilon > 0$ be given, can assume $\epsilon < 1$. Since $\mathcal{F}$ is totally bounded, cover $\mathcal{F}$ by finite union of δ-balls: $\mathcal{F} \subset B(f_1, \delta) \cup ... \cup B(f_p, \delta)$. Given $f \in \mathcal{F}$, take i such that

$$f \in B(f_i, \delta) \Leftrightarrow \bar{d}(f(x), f_i(x)) < \delta, \forall x \in X \Leftrightarrow d(f(x), f_i(x)) < \delta, \forall x \in X$$

(assuming $\delta < 1$). Then

$$d(f(x), f(x_0)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(x_0)) + d(f_i(x_0), f(x_0)).$$

Now let $\delta < \epsilon/3$ and note that since each f_k is continuous, we can choose U_k such that $d(f_k(x), f_k(x_0)) < \epsilon/3$, for $x \in U_k$. Setting $U = U_1 \cap ... \cap U_p$, we have $d(f_k(x), f_k(x_0)) < \epsilon/3$, for any $k = 1, ..., p$, $x \in U$, and

$$x \in U \Rightarrow d(f(x), f(x_0)) < \epsilon.$$

Since $f \in \mathcal{F}$ was arbitrary and U is independent of f, $\mathcal{F}$ is equicontinuous.

(ii) Let $\mathcal{F}$ be equicontinuous and $\epsilon > 0$. Show that $\mathcal{F}$ can be covered by finite ϵ-balls. For each $a \in X$, let $U_a \subset X$ open such that

$$x \in U_a \Rightarrow d(f(x), f(a)) < \epsilon/3, \forall f \in \mathcal{F}.$$

Then cover X by finitely many such U_a's: $X = U_{a_1} \cup ... \cup \mathcal{U}_{a_p}$.

Now cover Y by finitely many $\epsilon/6$-balls: $Y = B(b_1, \epsilon/6) \cup ... \cup B(b_k, \epsilon/6)$, $b_i \in Y$, $i = 1, ..., k$.

Let $A = \{a_1, ..., a_p\}$, $B = \{b_1, ..., b_k\}$. Then the set B^A of maps $\alpha : A \to B$ is finite. To each such map α, let $f_\alpha \in \mathcal{F}$ be such that $f_\alpha(a_i) \in B(\alpha(a_i), \epsilon/6)$, if it exists. Then $\mathcal{F}$ is covered by ϵ-balls centered in f_α: given $f \in \mathcal{F}$, for each $i = 1, ..., p$, $f(a_i) \in B(b_j, \epsilon/6)$, for some b_j and we let $\alpha \in B^A$ be such that $\alpha(a_i) = b_j$. In this case, for $x \in U_{a_i}$,

$$d(f(x), f_\alpha(x)) \leq d(f(x), f(a_i)) + d(f(a_i), f_\alpha(a_i)) + d(f_\alpha(a_i), f_\alpha(x)) < \epsilon,$$

by definition of U_{a_i} and since $f(a_i), f_\alpha(a_i) \in B(\alpha(a_i), \epsilon/6)$. Hence,

$$\rho(f, f_\alpha) = \sup_{x \in X} d(f(x), f_\alpha(x)) < \epsilon \;\Rightarrow f \in B_\rho(f_\alpha, \epsilon),$$

and therefore $\mathcal{F} \subset \cup_{\alpha \in B^A} B_\rho(f_\alpha, \epsilon)$. □

We now characterize compact subsets of $\mathbb{C}(X, \mathbb{R}^n)$, X compact, with the sup metric, which, as we have seen is a complete metric space (since $\mathbb{R}^n$ is complete).

A family $\mathcal{F} \subset \mathbb{C}(X, Y)$ is said to be *pointwise bounded* if for any $a \in X$ the set

$$\mathcal{F}_a := \{f(a) : f \in \mathcal{F}\} \text{ is bounded in } Y.$$

Theorem 7.20 (Ascoli)**.** *Let X be compact and $\mathbb{R}^n$ have the usual / square metric. A subspace $\mathcal{F} \subset \mathbb{C}(X, \mathbb{R}^n)$ is relatively compact if, and only if, it is equicontinuous and pointwise bounded.*

Proof. We have seen that, since $\mathbb{C}(X, \mathbb{R}^n)$ is complete

$$\text{relatively compact} \Leftrightarrow \text{totally bounded.}$$

Assume $\mathcal{F}$ is totally bounded, then it is equicontinuous by the previous lemma (i), and bounded with the sup norm, hence also pointwise bounded.

Conversely, we show that equicontinuous and pointwise bounded yields totally continuous. By the previous lemma (ii), it suffices to check that there is Y compact such that $f(X) \subset Y$ for all $f \in \mathcal{F}$.

For each $a \in X$, let $U_a \subset X$ open such that

$$x \in U_a \Rightarrow d(f(x), f(a)) < 1, \forall f \in \mathcal{F}.$$

Since X is compact, we have that

$$X = U_{a_1} \cup ... \cup U_{a_p}.$$

Moreover, since $\mathcal{F}_{a_i}$ is bounded in $\mathbb{R}^n$, the finite union $\mathcal{F}_{a_1} \cup ... \cup \mathcal{F}_{a_p}$ is also bounded, hence it is contained in some ball $B(0, R)$, for some $R > 0$. Then, for any $f \in \mathcal{F}$, if $x \in U_{a_i}$,

$$d(f(x), 0) \leq d(f(x), f(a_i)) + d(f(a_i, 0) < 1 + R.$$

We have then $f(X) \subset B(0, R+1)$, for all $f \in \mathcal{F}$. Let $Y = \overline{B(0, R+1)}$, closed and bounded in $\mathbb{R}^n$, hence compact. □

Corollary 7.21. *Let X be compact and $\mathbb{R}^n$ have the usual / square metric. A subspace $\mathcal{F} \subset \mathbb{C}(X, \mathbb{R}^n)$ is compact if, and only if, it is closed, equicontinuous and bounded (under sup norm).*

Proof. $\mathcal{F}$ compact, then always closed and bounded, and equicontinuous by previous result.

If $\mathcal{F}$ is closed then $\mathcal{F}$ compact iff relatively compact. □

Example 7.22. In $\mathbb{C}([0, 1])$ with the sup norm, $B = \{f \in C([0, 1]) : \|f\| \leq 1\}$ is closed and bounded but not compact: take $f_n(t) = t^n$. Then f_n does not have a convergent subsequence, since a convergent subsequence would converge pointwise and for each $t \in [0, 1]$, $f_n(t) \to f(t)$ with $f(t) = 0, 0 \leq t < 1$, $f(1) = 1$, f not continuous. Check that $\{f_n\}$ not equicontinuous at $t = 1$.

The following result is often used, e.g., to prove existence of solutions of differential equations:

Theorem 7.23 (Ascoli-Arzela)**.** *Let X be compact and $f_n \in \mathbb{C}(X, \mathbb{R}^n)$. If $\{f_n\}$ is pointwise bounded and equicontinuous then (f_n) has a convergent subsequence.*

Proof. Let $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$. By Ascoli's theorem, $\overline{\mathcal{F}}$ is compact, hence sequentally compact. Then any sequence in $\overline{\mathcal{F}}$, in particular (f_n), has a convergent subsequence. □

Theorem 7.24. *(X, d) is complete if and only if, any decreasing sequence of closed, non-empty sets $F_1 \supset ... \supset F_n \supset ...$ such that* $\mathrm{diam}(F_n) \to 0$ *has non-empty intersection $\cap_{n\in\mathbb{N}} F_n \neq \emptyset$.*

(Note that if (X, d) compact then the previous is clear as the family $\{F_n\}$ has the finite intersection property.)

Proof. Suppose (X, d) is complete and let F_n be a family of closed sets as above. Pick $x_n \in F_n$, for each $n \in \mathbb{N}$ and $X_n = \{x_p : p \geq n\}$. Then $X_n \subset F_n$ hence $\mathrm{diam}(X_n) \to 0$ and therefore (x_n) is Cauchy, so it converges to some $x = \lim x_n$. We have $x \in \cap_{n\in\mathbb{N}} \overline{X_n} \subset \cap_{n\in\mathbb{N}} F_n \neq \emptyset$.

Conversely: let (x_n) be a Cauchy sequence and again $X_n = \{x_p : p \geq n\}$. Define $F_n = \overline{X_n}$. Then, since (x_n) Cauchy, $\mathrm{diam}\, F_n = \mathrm{diam}\, X_n \to 0$ and clearly F_n closed such that $F_{n+1} \subset F_n$, hence $\cap_{n\in\mathbb{N}} F_n \neq \emptyset$. But if $x \in \cap_{n\in\mathbb{N}} F_n$ then any neighborhood of x intersects X_n so x is a sublimit of (x_n). We conclude that X is complete. □

Theorem 7.25 (Baire)**.** *Let (X, d) be a complete metric space. If X is given by a countable union of closed sets then at least one has non-empty interior.*

Hence $\mathbb{Q}$ not complete. In fact, a complete metric space with no isolated points is necessarily uncountable.