The Factorization Method for a Class of Inverse Elliptic Problems

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Extended Abstract

The factorization method is a fairly new approach for solving classes of inverse problems where an internal interface Γ has to be determined from measurements of the Cauchy data on the surface of a body B containing Γ in its interior.
Historically, this method was first developped for inverse scattering problems. Here, the shape $\Gamma = \partial\Omega$ of the scattering object Ω has to be determined by the far field pattern $u^\infty(\hat{x}, \hat{\theta})$ of the scattering field in direction $\hat{x}$ for all directions $\hat{\theta}$ of incident plane waves. Later, the method has been extended to crack problems and, quite recently, to scattering problems for periodic surfaces and for problems in impedance tomography. The original version of this method required that the scattering matrix had to be unitary – or, in the case of impedance tomography, – that the Neumann-Dirichlet operator had to be selfadjoint and positive (or negative) definite. This prevented applications with absorption. Recently, new versions of the factorization method have been developed which can handle problems with absorption as well. It is the aim of this talk to extend this approach to elliptic equations of the form

$$\operatorname{div}(\gamma \nabla u) + q\, u = 0 \quad \text{in some region } B\,, \tag{1}$$

$$\partial_\gamma u = f \quad \text{on } \partial B\,, \tag{2}$$

and wish to obtain information about γ from the knowledge of $\big(f, u|_{\partial B}\big)$ for all f. Here, for $x \in \partial B$ and smooth functions u the co-normal derivative $\partial_\gamma u|_{\partial B}$ of u at x is defined by $\partial_\gamma u(x) = n(x) \cdot (\gamma(x) \nabla u(x)$, where $n = n(x)$, $x \in \partial B$, denotes the exterior unit normal vector at $x \in \partial B$. The coefficient γ is space dependent and matrix valued. The problem (1), (2) defines the Neumann-Dirichlet operator Λ which assigns to each function f on ∂B the Dirichlet trace $u|_{\partial B}$ and, of course depends on γ and q. Therefore, we can rephrase the inverse problem to determine information about γ and q from the knowledge of Λ. We describe three possible applications of this problem.

(A) Impedance Tomography:
The first problem we have in mind is the problem of impedance tomography. On the surface of a body B one applies a set of known currents and measeures the corresponding voltages on the same surface. From this information one tries to recover the internal conductivity – or at least the shape of the region where the conductivity differs from

a known background situation. Applications of impedance tomography include medical imaging and non destructive testing. The mathematical model for the coupling of currents and voltages is given by the quasi-static Maxwell's equations

$$\operatorname{curl} H \;=\; (-i\omega\varepsilon + \sigma)E\,, \quad \operatorname{curl} E \;=\; 0\,, \tag{3}$$

which leads to the potential u through $E = \nabla u$ and, taking the divergence of the first equation, to

$$\operatorname{div}(\gamma\nabla u) \;=\; 0 \quad \text{in } B \text{ where } \gamma \;=\; \sigma - i\omega\varepsilon\,. \tag{4}$$

On the boundary ∂B the current and voltage distribution are given by $n \cdot (\gamma E) = \partial_\gamma u$ and $u|_{\partial B}$, respectively. Therefore, we are led to problem (1), (2) for $q \equiv 0$.

(B) Inverse Scattering Theory in $\mathbb{R}^3$:
Time harmonic (acoustic) waves of the form

$$v^{inc}(x,t) \;=\; \Re\left[a\frac{\exp\bigl(i\omega|x-y|/c_0 - i\omega t\bigr)}{4\pi|x-y|}\right], \quad x \neq y\,,$$

with frequency $\omega > 0$ and speed c_0 are scattered by a medium which is characterized by a density distribution $\rho = \rho(x)$ and speed of sound $c = c(x)$. The linearized wave equation is

$$\frac{\partial^2 v(x,t)}{\partial t^2} \;=\; c(x)^2\,\rho(x)\operatorname{div}\left[\frac{1}{\rho(x)}\nabla v(x,t)\right].$$

Outside of some region Ω the parameters ρ and c are constant with $c \equiv c_0$ and $\rho \equiv \rho_0$. The total wave is of the form $v(x,t) = \Re\bigl[a\,v(x)\,e^{-i\omega t}\bigr]$ and satisfies the reduced wave equation

$$\operatorname{div}\left[\frac{\rho_0}{\rho(x)}\nabla v(x)\right] \;+\; \frac{\omega^2\,\rho_0}{c(x)^2\,\rho(x)}\,v(x) \;=\; 0\,. \tag{5}$$

Therefore, in this situation $\gamma = \rho_0/\rho$ and $q = \bigl(\omega^2\rho_0\bigr)/\bigl(c^2\rho\bigr)$. Outside of Ω equation (5) reduces to the Helmholtz equation $\Delta v + k^2 v = 0$ with wave number $k = \omega/c_0$. The Sommerfeld radiation has to be imposed on $v^s = v - \Phi(\cdot,y)$. Here Φ denotes the fundamental solution in $\mathbb{R}^3$, given by

$$\Phi(x,y) \;:=\; \frac{\exp\bigl(ik|x-y|\bigr)}{4\pi\,|x-y|}\,, \quad x \neq y\,. \tag{6}$$

We write $v = v(x,y)$ to indicate the dependence on the source point y. The inverse problem is to determine ρ and/or c from the knowledge of $v(x,y)$ for all $x,y \in \partial B$ where B is some domain which contains $\overline{\Omega}$ in its interior. We note that $v = v(x,y)$ is just the Green's function of the differential equation. From the knowledge of $v(x,y)$ for $x,y \in \partial B$ we can compute the Neumann-Dirichlet $\Lambda : H^{-1/2}(\partial B) \to H^{1/2}(\partial B)$. Indeed, let $h \in H^{-1/2}(\partial B)$. We make an ansatz for u in the form of the single layer potential

$$u(x) \;=\; \int_{\partial B} \psi(y)\,v(x,y)\,ds(y)\,, \quad x \in B\,.$$

Then u solves the differential equation (5) in B, and $\partial u/\partial n = f$ on ∂B holds if and only if ψ solves the integral equation

$$\frac{1}{2}\psi(x) \; + \int\limits_{\partial B} \psi(y)\, \frac{\partial v(x,y)}{\partial n(x)}\, ds(y) \;=\; f(x)\,, \quad x \in \partial B\,. \tag{7}$$

This follows from the jump conditions for the traces of the double layer potential.[1] Equation (7) is a Fredholm equation of the second kind. Uniqueness holds provided the Neumann problem in B has a unique solution for all $f \in H^{-1/2}(\partial B)$. This is seen by standard arguments. Therefore, (7) is uniquely solvable for any $f \in H^{-1/2}(\partial B)$, and Λ can be expressed by

$$\Lambda \;=\; S\left(\frac{1}{2}I + K\right)^{-1}$$

where K and S are the extensions of the operators

$$K\psi(x) \;=\; \int\limits_{\partial B} \psi(y)\, \frac{\partial v(x,y)}{\partial n(x)}\, ds(y)\,, \quad x \in \partial B\,, \qquad \text{and}$$

$$S\psi(x) \;=\; \int\limits_{\partial B} \psi(y)\, v(x,y)\, ds(y)\,, \quad x \in \partial B\,,$$

respectively. Therefore, at least from a mathematical point of view inverse scattering problems are also subsumed under our problem.

(C) Thermal Imaging: This is also a technique in nondestructive testing and evaluation. A heat source is used to produce a heat flux at the boundary of an object and the resulting temperature distribution on the surface is measured. Assuming that the heat flux is time harmonic, i.e. of the form $\mathrm{Re}\left[f(x)\, e^{-i\omega t}\right]$ with frequency $\omega > 0$, the model is formulated by the boundary value problem

$$\operatorname{div}\bigl(\kappa\, \nabla u\bigr) \;+\; i\omega c\rho\, u \;=\; 0 \quad \text{in } B\,,$$

$$\kappa\, \frac{\partial u}{\partial n} \;=\; f \quad \text{on } \partial B\,,$$

where κ, c, ρ denote the theormal conductivity, specific heat, and density, respectively. The inverse problem is again, to determine internal interfaces from additional measurements of the temperature on ∂B. Therefore, we are again led to problem (1), (2) with $\gamma = \kappa$ and $q = i\omega c\rho$.

In this talk we will study in detail the case where q vanishes. First, the factorization of the difference of the Neumann-to-Dirichlet operators for the perturbed and unperturbed cases is factorized in the form $\Lambda - \Lambda_0 = G(T - T_0)G^*$ where G is "more explicit" than $\Lambda - \Lambda_0$, and the real part of $T - T_0$ is coercive. Then it is shown using arguments from spectral

[1]Note that v is as singular as the fundamental solution Φ – at least in a neighborhood of ∂B.

theory that the ranges of G and $|\Re\Lambda-\Lambda_0|^{1/2}$ coincide. Therefore, using an eigensystem of $\Re\Lambda-\Lambda_0$ the range of G can explicitely be characterized. Finally, functions φ_y depending on $y\in\mathbb{R}^n$ are constructed with the property that y belongs to the support Ω of the contrast $\gamma-\gamma_0$ if, and only if, φ_y belongs to the range of G. Combining the previous two results yields an explicit characterization of Ω by the eigensystem of $\Re\Lambda-\Lambda_0$.

At the end of the talk we will sketch the approach of the more general case where q is not vanishing. Here we will use compact perturbation arguments to characterize the range of G by $\Lambda-\Lambda_0$.