Matrix Decomposition MFS Algorithms

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Abstract: *The Method of Fundamental Solutions (MFS) is a boundary–type meshless method for the solution of certain elliptic boundary value problems. We exploit the symmetries of the matrices appearing when this method is applied to certain three-dimensional elliptic problems and develop an efficient algorithm for their solution.*

1 MFS formulation for potential problems

We consider the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Δ denotes the Laplace operator and f is a given function. The region $\Omega \subseteq \mathbb{R}^3$ is axisymmetric, which means that it is formed by rotating a region $\Omega' \in \mathbb{R}^2$ about the z-axis. The boundaries of Ω and Ω' are denoted by $\partial\Omega$ and $\partial\Omega'$, respectively. The solution u is approximated by

$$u_{MN}(c,Q;P) = \sum_{m=1}^{M}\sum_{n=1}^{N} c_{m,n}\, k(P,Q_{m,n}), \qquad P \in \overline{\Omega}, \tag{1.2}$$

where $c = (c_{11}, c_{12}, \ldots, c_{1N}, \ldots, c_{M1}, \ldots, c_{MN})^T$ and Q is a $3MN$-vector containing the coordinates of the singularities $Q_{m,n}$, $m = 1,\ldots,M$, $n = 1,\ldots,N$, which lie outside $\overline{\Omega}$. The function $k_1(P,Q)$ is a fundamental solution of Laplace's equation in $\mathbb{R}^3$ given by

$$k_1(P,Q) = \frac{1}{4\pi|P-Q|}, \tag{1.3}$$

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where $|P-Q|$ is the distance between the points P and Q. The singularities $Q_{m,n}$ are fixed on the boundary $\partial\tilde{\Omega}$ of a solid $\tilde{\Omega}$ surrounding Ω. The solid $\tilde{\Omega}$ is generated by the rotation of the planar domain $\tilde{\Omega}'$ which is similar to Ω'. A set of MN collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$ is chosen on $\partial\Omega$ in the following way: We first choose N points on the boundary $\partial\Omega'$ of Ω'. These can be described by their polar coordinates (r_{P_j}, z_{P_j}), $j=1,\cdots,N$, where r_{P_j} denotes the vertical distance of the point P_j from the z-axis and z_{P_j} denotes the z- coordinate of the point P_j. The points on $\partial\Omega$ are taken to be $x_{P_{i,j}} = r_{P_j}\cos\varphi_i$, $y_{P_{i,j}} = r_{P_j}\sin\varphi_i$, $z_{P_{i,j}} = z_{P_j}$, where $\varphi_i = \dfrac{2(i-1)\pi}{M}$, $i=1,\ldots,M$. Similarly, we choose a set of MN singularities $\{Q_{m,n}\}_{i=m,n=1}^{M,N}$ on $\partial\tilde{\Omega}$ by taking $Q_{m,n}=(x_{Q_{m,n}}, y_{Q_{m,n}}, z_{Q_{m,n}})$, and $x_{Q_{i,j}} = r_{Q_j}\cos\varphi_i$, $y_{Q_{i,j}} = r_{Q_j}\sin\varphi_i$, $z_{Q_{i,j}} = z_{Q_j}$, where the N points Q_j are chosen on the boundary $\partial\tilde{\Omega}'$ of $\tilde{\Omega}'$.

The coefficients c are determined so that the boundary condition is satisfied at the boundary points:

$$u_N(c,Q;P_{i,j}) = f(P_{i,j}), \; i=1,\ldots,M, \; j=1,\ldots,N, \tag{1.4}$$

which yields an $MN\times MN$ linear system of the form

$$G^0 c \;=\; f, \tag{1.5}$$

for the coefficients c, where the elements of the matrix G^0 are given by

$$G^0_{(i-1)N+j,(m-1)N+n} = \frac{1}{4\pi|P_{i,j}-Q_{m,n}|}, \quad i,m=1,\ldots,M, \;\; j,n=1,\ldots,N. \tag{1.6}$$

The global matrix G^0 has the block circulant structure

$$G^0 \;=\; \begin{pmatrix} A_1 & A_2 & \cdots & A_M \\ A_M & A_1 & \cdots & A_{M-1} \\ \vdots & \vdots & & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{pmatrix}, \tag{1.7}$$

where the matrices A_ℓ, $\ell=1,\ldots,M$, are $N\times N$ matrices defined by

$$(A_\ell)_{j,n} = \frac{1}{4\pi|P_{1,j}-Q_{\ell,n}|}, \quad \ell=1,\ldots,M \;\; j,n=1,\ldots,N.. \tag{1.8}$$

The system (1.5) can therefore be written as

$$G^0 c \;=\; \left(I_M\otimes A_1 + P\otimes A_2 + P^2\otimes A_3 + \cdots + P^{M-1}\otimes A_M\right) c \;=\; f, \tag{1.9}$$

where the matrix P is the $M\times M$ permutation matrix $P = circ\,(0,1,0,\cdots,0)$ and $\otimes$ denotes the matrix tensor product.

1.1 Matrix decomposition algorithm

If $C = \mathrm{circ}(c_1, \ldots, c_M)$, then

$$C = U^* D U$$

where $D = \mathrm{diag}(d_1, \ldots, d_M)$, and

$$d_j = \sum_{k=1}^{M} c_k \omega^{(k-1)(j-1)},$$

where $\omega = e^{2\pi i/M}$ and U is the conjugate transpose of the Fourier matrix. In particular, the permutation matrix $P = \mathrm{circ}(0,1,0,\ldots,0)$ is diagonalized as $P = U^* D U$ where $D = \mathrm{diag}(d_1, \ldots, d_M)$, $d_j = \omega^{j-1}$.

Premultiplication of system (1.9) by $U \otimes I_N$ yields

$$(U \otimes I_N) \left(\sum_{k=1}^{M} P^{k-1} \otimes A_k \right) (U^* \otimes I_N)(U \otimes I_N)\, c \;=\; (U \otimes I_N)\, f, \qquad (1.10)$$

and

$$\left(I_M \otimes A_1 + D \otimes A_2 + D^2 \otimes A_3 + \cdots + D^{M-1} \otimes A_M \right) \tilde{c} \;=\; \tilde{f}, \qquad (1.11)$$

where $\tilde{c} = (U \otimes I_N)\, c$, $\tilde{f} = (U \otimes I_N)\, f$. The solution of system (1.11) can thus be decomposed into the solution of the M independent $N \times N$ systems

$$B_m \tilde{c}_m \;=\; \left(A_1 + d_m A_2 + d_m^2 A_3 + \cdots + d_m^{M-1} A_M \right) \tilde{c}_m \;=\; \tilde{f}_m \quad m = 1, 2, \ldots M. \quad (1.12)$$

The (r,s) entry of the matrix B_m is

$$(B_m)_{rs} = (A_1)_{rs} + \omega^{m-1}\, (A_2)_{rs} + \cdots + \omega^{(m-1)(M-1)}\, (A_M)_{rs},$$

where $r, s = 1, \ldots, N$ and $m = 1, \ldots, M$. Thus we have

$$\left((B_1)_{rs}, \ldots, (B_M)_{rs} \right)^T \;=\; M^{1/2} U^* \left((A_1)_{rs}, \ldots, (A_M)_{rs} \right)^T.$$

This observation enables us to reduce the cost of constructing the matrices $B_1, \ldots, B_M$ from $O(M^2 N^2)$ operations to $O(N^2 M \log M)$ operations, using Fast Fourier Transforms (FFTs). Further details can be found in [1].

2 MFS formulation for biharmonic problems

We also consider the boundary value problem

$$\begin{cases} \Delta^2 u = 0 & \text{in} \quad \Omega, \\ u = f, \; \dfrac{\partial u}{\partial n} = g & \text{on} \quad \partial\Omega, \end{cases} \qquad (2.1)$$

where $\partial/\partial n$ is the outward normal derivative at P, and f and g are given functions. The region Ω is axisymmetric, and as before it is formed by rotating a region $\Omega' \in \mathbb{R}^2$ about the z-axis. We now approximate the solution u of (2.1) by

$$u_{MN}(c,d,Q;P) = \sum_{m=1}^{M}\sum_{n=1}^{N}\left[c_{m,n}k_1(P,Q_{m,n}) + d_{m,n}k_2(P,Q_{m,n})\right], \quad P\in\overline{\Omega}, \tag{2.2}$$

where $c = (c_{11}, c_{12}, \ldots, c_{1N}, \ldots, c_{M1}, \ldots, c_{MN})^T$, $d = (d_{11}, d_{12}, \ldots, d_{1N}, \ldots, d_{M1}, \ldots, d_{MN})^T$, and Q is a $3MN$–vector containing the coordinates of the singularities $\{Q_{m,n}\}_{m=1,n=1}^{M,N}$, which lie outside $\overline{\Omega}$. The function $k_1(P,Q)$ is the fundamental solution of Laplace's equation in $\mathbb{R}^3$ and

$$k_2(P,Q) = \frac{1}{8\pi}|P-Q|, \tag{2.3}$$

is the fundamental solution of the biharmonic equation in $\mathbb{R}^3$. With the singularities and boundary points defined as in the potential problem, the coefficient vectors c and d are determined so that the boundary conditions are satisfied at the collocation points $\{P_{i,j}\}_{i=1,j=1}^{M,N}$:

$$u_N(c,d,Q;P_{i,j}) = f(P_{i,j}), \quad \frac{\partial u_N}{\partial n_P}(c,d,Q;P_{i,j}) = g(P_{i,j}), \quad i=1,\ldots,M, \quad j=1,\ldots,N. \tag{2.4}$$

These equations yield a $2MN \times 2MN$ linear system of the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c} c \\ \hline d \end{array}\right) = \left(\begin{array}{c} f \\ \hline g \end{array}\right), \tag{2.5}$$

where the $MN \times MN$ matrices A,B,C,D have a block circulant structure. Specifically,

$$A = \begin{pmatrix} A_1 & A_2 & \cdots & A_M \\ A_M & A_1 & \cdots & A_{M-1} \\ \vdots & \vdots & & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{pmatrix}, \tag{2.6}$$

where the $N \times N$ submatrices $A_\ell = \left((A_\ell)_{j,n}\right)$, $\ell = 1,\cdots,M$, are defined by

$$(A_\ell)_{j,n} = \frac{1}{4\pi}\frac{1}{|P_{1,j} - Q_{\ell,n}|}, \quad j,n = 1,\ldots,N, \tag{2.7}$$

and

$$B = \text{circ}(B_1\, B_2\, \cdots\, B_M), \quad C = \text{circ}(C_1\, C_2\, \cdots\, C_M), \quad D = \text{circ}(D_1\, D_2\, \cdots\, D_M),$$

where, for $\ell = 1,\ldots,M$, $\quad j,n = 1,\ldots,N$,

$$(B_\ell)_{j,n} = \frac{1}{8\pi}|P_{1,j} - Q_{\ell,n}|, \quad (C_\ell)_{j,n} = \frac{1}{4\pi}\frac{\partial}{\partial n}\left[\frac{1}{|P_{1,j} - Q_{\ell,n}|}\right], \quad (D_\ell)_{j,n} = \frac{1}{8\pi}\left[\frac{\partial}{\partial n}|P_{1,j} - Q_{\ell,n}|\right].$$

2.1 Matrix decomposition algorithm

With $P^0 = I_M$,

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \sum_{k=1}^{M} \left(\begin{array}{c|c} P^{k-1}\otimes A_k & P^{k-1}\otimes B_k \\ \hline P^{k-1}\otimes C_k & P^{k-1}\otimes D_k \end{array}\right)$$

$$= \sum_{k=1}^{M} \left[\left(\begin{array}{c|c} P^{k-1} & 0 \\ \hline 0 & 0 \end{array}\right) \otimes A_k + \left(\begin{array}{c|c} 0 & P^{k-1} \\ \hline 0 & 0 \end{array}\right) \otimes B_k + \left(\begin{array}{c|c} 0 & 0 \\ \hline P^{k-1} & 0 \end{array}\right) \otimes C_k + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & P^{k-1} \end{array}\right) \otimes D_k \right]. \tag{2.1}$$

Clearly,

$$P^{k-1}U^* = U^*E^{k-1}, \quad k = 1,\dots,M, \tag{2.2}$$

where $E = \text{diag}(e_1\ e_2\ \cdots\ e_M)$ with $e_j = \omega^{j-1}$. Premultiplying system (2.5) by

$$\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array}\right) \otimes I_N$$

and introducing the $MN \times MN$ identity matrix yields

$$\left[\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array}\right) \otimes I_N\right] \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \left[\left(\begin{array}{c|c} U^* & 0 \\ \hline 0 & U^* \end{array}\right) \otimes I_N\right] \left[\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array}\right) \otimes I_N\right] \left(\begin{array}{c} c \\ \hline d \end{array}\right)$$

$$= \left[\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array}\right) \otimes I_N\right] \left(\begin{array}{c} f \\ \hline g \end{array}\right). \tag{2.3}$$

On using (2.1), (2.2) and the properties of $\otimes$, (2.3) becomes

$$\sum_{k=1}^{M} \left[\left(\begin{array}{c|c} E^{k-1} & 0 \\ \hline 0 & 0 \end{array}\right) \otimes A_k + \left(\begin{array}{c|c} 0 & E^{k-1} \\ \hline 0 & 0 \end{array}\right) \otimes B_k + \left(\begin{array}{c|c} 0 & 0 \\ \hline E^{k-1} & 0 \end{array}\right) \otimes C_k + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & E^{k-1} \end{array}\right) \otimes D_k \right] \left(\begin{array}{c} \tilde{c} \\ \hline \tilde{d} \end{array}\right) = \left(\begin{array}{c} \tilde{f} \\ \hline \tilde{g} \end{array}\right), \tag{2.4}$$

where

$$\left(\begin{array}{c} \tilde{c} \\ \hline \tilde{d} \end{array}\right) = \left[\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array}\right) \otimes I_N\right] \left(\begin{array}{c} c \\ \hline d \end{array}\right), \quad \left(\begin{array}{c} \tilde{f} \\ \hline \tilde{g} \end{array}\right) = \left[\left(\begin{array}{c|c} U & 0 \\ \hline 0 & U \end{array}\right) \otimes I_N\right] \left(\begin{array}{c} f \\ \hline g \end{array}\right).$$

System (2.4) can then be written as

$$\sum_{k=1}^{M} \left(\begin{array}{c|c} E^{k-1}\otimes A_k & E^{k-1}\otimes B_k \\ \hline E^{k-1}\otimes C_k & E^{k-1}\otimes D_k \end{array}\right) \left(\begin{array}{c} \tilde{c} \\ \hline \tilde{d} \end{array}\right) = \left(\begin{array}{c} \tilde{f} \\ \hline \tilde{g} \end{array}\right),$$

or

$$\left(\begin{array}{c|c} \sum_{k=1}^{M} E^{k-1}\otimes A_k & \sum_{k=1}^{M} E^{k-1}\otimes B_k \\ \hline \sum_{k=1}^{M} E^{k-1}\otimes C_k & \sum_{k=1}^{M} E^{k-1}\otimes D_k \end{array}\right) \left(\begin{array}{c} \tilde{c} \\ \hline \tilde{d} \end{array}\right) = \left(\begin{array}{c} \tilde{f} \\ \hline \tilde{g} \end{array}\right).$$

This system reduces to the M independent $2N \times 2N$ linear systems

$$\left(\begin{array}{c|c} \tilde{A}_m & \tilde{B}_m \\ \hline \tilde{C}_m & \tilde{D}_m \end{array}\right) \left(\begin{array}{c} \tilde{c}_m \\ \hline \tilde{d}_m \end{array}\right) = \left(\begin{array}{c} \tilde{f}_m \\ \hline \tilde{g}_m \end{array}\right), \quad m = 1, 2, \cdots, M, \tag{2.5}$$

where

$$\tilde{A}_m = \sum_{i=1}^{M} e_m^{i-1} A_i, \tag{2.6}$$

with $\tilde{B}_m$, $\tilde{C}_m$, and $\tilde{D}_m$ defined similarly, and

$$\tilde{c}_m = [\tilde{c}_{m1}, \tilde{c}_{m2}, \cdots, \tilde{c}_{mN}]^T,$$

with $\tilde{d}_m$, $\tilde{f}_m$ and $\tilde{g}_m$ defined similarly. Further details can be found in [2].

References

[1] Y. S. Smyrlis and A. Karageorghis, *A matrix decomposition MFS algorithm for axisymmetric potential problems*, Technical Report TR/10/2002, Department of Mathematics and Statistics, University of Cyprus, 2002.

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