A MARKOVIAN STORAGE MODEL

ANTÓNIO PACHECO
Technical University of Lisbon
Departamento de Matemática - Instituto Superior Técnico
Av. Rovisco Pais, 1096 Lisboa Codex, PORTUGAL
e-mail: apacheco@math.ist.utl.pt

N.U. PRABHU
School of Operations Research and Industrial Engineering
Cornell University, Rhodes Hall
Ithaca, NY 14853-3801, USA
e-mail: questa@orie.cornell.edu


Abstract: We investigate a storage model where the input and the demand are additive functionals on a Markov chain $J$. The storage policy is to meet the largest possible portion of the demand. We first derive results for the net input process embedded at the epochs of transitions of $J$, which is a Markov random walk. Our analysis is based on a Wiener-Hopf factorization for this random walk; this also gives results for the busy period of the storage process. The properties of the storage level and the unsatisfied demand are then derived.

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1 Introduction

In this paper we investigate the storage model in which the storage level $Z(t)$ at time $t$ satisfies almost surely (a.s.) the integral equation

$$Z(t) = Z(0) + \int_0^t a(J(s)) \, ds - \int_0^t r(Z(s), J(s)) \, ds$$  \hspace{1cm} (1)

where

$$r(x,j) = \begin{cases} d(j) & x > 0 \\ \min(a(j), d(j)) & x \leq 0 \end{cases}$$  \hspace{1cm} (2)

with the condition $Z(0) \geq 0$. Here $J = \{J(t), t \geq 0\}$ is a non-explosive Markov chain on a countable state space $E$, and $a$ and $d$ are nonnegative functions on $E$. The equation (1) states that when the Markov chain $J$ is in state $j$ at time $t$, input into the storage (buffer) occurs at rate $a(j)$, while the demand occurs at rate $d(j)$ and the storage policy is to meet the largest possible portion of this demand. Let us denote by

$$A(t) = \int_0^t a(J(s)) \, ds, \quad D(t) = \int_0^t d(J(s)) \, ds$$  \hspace{1cm} (3)

the input and the (actual) demand during a time interval $(0, t]$. It can be proved that $(A, J) = \{(A, J)(t), t \geq 0\}$ and $(D, J) = \{(D, J)(t), t \geq 0\}$ are Markov additive processes (MAPs) on the state space $\mathbb{R}_+ \times E$. A storage model with a more general input process $(X, J)$ of the Markov additive type has been investigated by Prabhu and Pacheco [11]. Here $X(t)$ consists of jumps of positive size as well as a (cumulative) drift $A(t)$. However, the analysis of that paper cannot be applied in the present model as there the assumption is made that

$$a(j) < d(j) \quad \text{for } j \in E.$$  \hspace{1cm} (4)

This assumption will make (1) trivial since then the storage level is decreasing so that it will eventually reach zero after a random length of time and remain at zero after that. Therefore in this paper we do not assume (4) and use a completely different approach to analyze our
model. We denote

\[ E_1 = \{ j \in E : a(j) \leq d(j) \}, \quad E_0 = \{ j \in E : a(j) > d(j) \}. \] (5)

The model represented by (1) occurs in data communication systems. Virtamo and Norros [17] have investigated a model in which a buffer receives input of data from an M/M/1 queueing system at a constant rate \( c_0 \) so long as the system is busy, and transmits these data at a maximum rate \( c_1 \) (\(< c_0 \)). Denoting by \( J(t) \) the queue length, we can represent the buffer content \( Z(t) \) at time \( t \) by

\[ Z(t) = Z(0) + \int_0^t c_0 1\{J(s) > 0\} \, ds - \int_0^t c_1 1\{J(s) > 0 \lor Z(s) > 0\} \, ds. \] (6)

This equation is of type (1) with

\[ a(j) = \begin{cases} c_0 & j > 0 \\ 0 & j = 0 \end{cases}, \quad d(j) = c_1 \quad (j \geq 0). \]

Clearly \( a(0) < d(0) \), but \( a(j) > d(j) \) for \( j > 0 \). Thus \( E_0 = \{ 1, 2, \ldots \} \) and \( E_1 = \{ 0 \} \). The input during \((0, t]\) is given by \( c_0 B(t) \), where

\[ B(t) = \int_0^t 1\{J(s) > 0\} \, ds, \]

this being the part of the time interval \((0, t]\) during which the server is busy. For a survey of earlier storage models of data communication systems satisfying equation (1) see Prabhu and Pacheco [11]. A brief description of two of these models is the following.

Anick, Mitra and Sondhi [2] study a model for a data-handling system with \( N \) sources and a single transmission channel. The input rate is \( a(j) = j \), where \( j \) is the number of sources that are “on”, and the maximum output rate is a constant \( c \), so that \( d(j) = c \). We see that in this case \( E_0 = \{ [c] + 1, [c] + 2, \ldots \} \) and \( E_1 = \{ 0, 1, \ldots, [c] \} \).

Gaver and Lehoczky [5] investigate a model for an integrated circuit and packet switching multiplexer, with input of data and voice calls. There are \( s + u \) output channels, of which \( s \) are for data transmission, while the remaining \( u \) are shared by data and voice calls (with
calls having preemptive priority over data). Here calls arrive in a Poisson process and have exponentially distributed holding times. The model gives rise to equation (1) for the data buffer content $Z(t)$ with

$$a(j) = c_0, \quad d(j) = c_2(s + j), \quad j = 0, 1, \ldots, u,$$

where $j$ is the number of channels out of $u$ not occupied by calls, $c_0$ is the (constant) data arrival rate and $c_2$ is the output rate capacity per channel.

The net input of our model $X(t)$ is given by

$$X(t) = A(t) - D(t) = \int_0^t x(J(s)) \, ds.$$  

(7)

with $x = a - d$. The net input process is thus an MAP which is nonincreasing during periods in which the environment $J$ is in $E_1$ and is increasing when $J$ is in $E_0$. Its sample functions are continuous a.s., and differentiable everywhere except at the transition epochs $T_n$ ($n \geq 0$) of the Markov chain $J$. Since

$$\int_0^t r(Z(s), J(s)) \, ds = \int_0^t d(J(s)) \, ds + \int_0^t \min\{a(J(s)) - d(J(s)), 0\} 1\{Z(s) \leq 0\} \, ds,$$

we can rewrite equation (1) in the form

$$Z(t) = Z(0) + X(t) + \int_0^t x(J(s))^{-} \, ds,$$

(8)

where $y^- = \max\{-y, 0\}$. Here the integral

$$I(t) = \int_0^t x(J(s))^- \, ds$$

(9)

represents the amount of unsatisfied demand during $(0, t]$. If we let $J_n = J(T_n)$, then from equation (8) we obtain from $T_n \leq t \leq T_{n+1}$

$$Z(t) = Z(T_n) + x(J_n)(t - T_n) + x(J_n)^- \int_{T_n}^t 1\{Z(s) \leq 0\} \, ds$$

(10)

and from (9)

$$I(t) = I(T_n) + x(J_n)^- \int_{T_n}^t 1\{Z(s) \leq 0\} \, ds.$$  

(11)
This shows that in order to study the process \((Z, I, J)\) it may be of interest to first study
the properties of the embedded process \((Z(T_n), I(T_n), J(T_n))\).

The following is a brief summary of the results of this paper. In section 2 we give the
solution of the integral equation (8); a particular consequence of the solution is that \(Z(T_n)\)
and \(I(T_n)\) may be identified as functionals on the process \((T_n, X(T_n), J(T_n))\), which is a
Markov random walk (MRW – the discrete time analogue of an MAP). So the properties of
this MRW are investigated in section 3, the key result being a Wiener-Hopf factorization due
to Presman [14]; see Prabhu and Tang [12], and Prabhu, Tang and Zhu [13]. In section 4 the
results of section 3 are used to study the properties of the storage level and the unsatisfied
demand.

Rogers [15] investigates the model we consider, with the Markov chain having finite state
space. His analysis is based on the Wiener-Hopf factorization of finite Markov chains, from
which the invariant distribution of the storage level is derived. Methods for computing the
invariant law of the storage level are discussed by and Rogers and Shi [16]. Asmussen [3]
and Karandikar and Kulkarni [7] investigate a storage process identified as the reflected
Brownian Motion (BM) modulated by a finite state Markov chain. In the case where the
variance components of this BM are all zero their storage process reduces to that of our
paper.

Our analysis allows for the Markov chain to have infinite state space, but in [15], [3],
and [7] this space is finite. We believe that some results in the paper by Asmussen [3]
and specially in the paper by Rogers [15] could be generalized, without much effort, to the
infinite state space case. The same is not true for numerical methods to compute the storages
quantities of interest. In fact, the development of efficient numerical methods when the
Markov chain has infinite state space is likely to be the subject of future research in the area
of communication systems. In [15], [3], and [7] only the steady state behaviour of the storage
level is studied, whereas we derive the time dependent as well as the steady state behaviour
of both the storage level and the unsatisfied demand (see Examples 1 and 2). It should also
be noted that the specificity of our net input (namely, piecewise linearity) is not too relevant
for our analysis, so the techniques of the paper can be applied to other net inputs. This
makes our approach potentially more powerful.

We shall denote by $N = (\nu_{jk})$ the generator matrix of $J$ and assume that $J$ has a
stationary distribution $(\pi_j, j \in E)$. For analytical convenience we assume that $x(j) \neq 0$ for
$j \in E$.

2 Preliminary Results

We start by solving the integral equation (8). Proceeding as in the proof of Theorem 1 in
Prabhu and Pacheco [11], we have the following result.

Lemma 1 We are given a stochastic process $J$, as defined above, on a probability space
$(\Omega, \mathcal{F}, \mathcal{P})$, and additive functionals $A$ and $D$ on $J$ as given in (3). The integral equation
equation (8) with $Z(0) \geq 0$ a.s., has $\mathcal{P}$-a.s. the unique solution

$$Z(t) = Z(0) + X(t) + I(t)$$

where

$$I(t) = [Z(0) + m(t)]^- = \left[ Z(0) + \inf_{0 \leq \tau \leq t} X(\tau) \right]^-.$$  \hfill (13)

One of the consequences of the solution (12) is (10) since

$$Z(t) = \max\{Z(0) + X(t), X(t) - m(t)\} \geq [Z(0) + X(t)]^+ \geq 0.$$  \hfill (14)

We next prove some preliminary results concerning the embedded process $(Z(T_n), I(T_n), J(T_n))$.

We let

$$Z_n = Z(T_n), \quad I_n = I(T_n), \quad S_n = X(T_n), \quad X_{n+1} = S_{n+1} - S_n$$

so that $X_{n+1} = x(J_n)(T_{n+1} - T_n)$.  \hfill (15)
Lemma 2  For $T_n \leq t \leq T_{n+1}$ we have

\[ Z(t) = [Z_n + x(J_n)(t - T_n)]^+, \quad I(t) = I_n + [Z_n + x(J_n)(t - T_n)]^- \]

where

\[ Z_n = Z_0 + S_n + I_n, \quad I_n = [Z_0 + m_n]^\sim = \left[ Z_0 + \min_{0 \leq r \leq n} S_r \right]^\sim. \]

Proof: Using (10) we may conclude that

\[ Z(t) = \max\{0, Z_n + x(J_n)(t - T_n)\} \quad (n \geq 0, t \in [T_n, T_{n+1}]), \]

which proves (16). As a consequence $Z_{n+1} = \max\{0, Z_n + X_{n+1}\}$ for $n \geq 0$, which implies (17) in view of a result familiar in queueing systems (e.g. Theorem 8 in [10, Chap.2]).

Lemma 2 shows that in order to study the process $(Z_n, I_n, J_n)$ it suffices to investigate the MRW $(T_n, S_n, J_n)$, which we do in the next section. We note that if $T_n \leq t \leq T_{n+1}$, then $X(t) = S_n + x(J_n)(t - T_n)$. Thus $\min(S_n, S_{n+1}) \leq X(t) \leq \max(S_n, S_{n+1})$, which in turns implies that a.s. $\liminf X(t) = \liminf S_n$ and $\limsup X(t) = \limsup S_n$. Similarly, if we denote for $t \geq 0$ and $n = 0, 1, \ldots$,

\[ M(t) = \sup_{0 \leq \tau \leq t} X(\tau), \quad M_n = \max_{0 \leq r \leq n} S_r \]

we may conclude that,

\[ \lim_{t \to \infty} M(t) = \lim_{n \to \infty} M_n = M \leq +\infty \]

\[ \lim_{t \to \infty} m(t) = \lim_{n \to \infty} m_n = m \geq -\infty. \]

These statements show that some conclusions about the fluctuation behaviour of the net input process may be drawn from the associated MRW $(S_n, J_n)$. This in turn has implications for the storage level and unsatisfied demand since these processes depend on the net input.

We denote by $(\pi_j^*, j \in E)$ the stationary distribution of $(J_n)$, so that

\[ \pi_j^* = \frac{(-\nu_{jj}) \pi_j}{\sum_{k \in E} (-\nu_{kk}) \pi_k}. \]

Also, define the net input rate $\pi = \sum_{j \in E} \pi_j x(j)$, where we assume the sum exists, but may be infinite. We have then the following.
Theorem 1 (Fluctuation behaviour of $X(t)$) We have a.s.

(i). $X(t) \to x$.

(ii). If $x > 0$, then $\lim X(t) = +\infty$, $m > -\infty$, and $M = +\infty$.

(iii). If $x = 0$, then $\lim \inf X(t) = -\infty$, $\lim \sup X(t) = +\infty$, $m = -\infty$, and $M = +\infty$.

(iv). If $x < 0$, then $\lim X(t) = -\infty$, $m = -\infty$, and $M < +\infty$.

Proof: The proof of (i) is standard, but is given here for completeness. We have

$$\frac{X(t)}{t} = \frac{1}{t} \int_0^t x(J(s)) \, ds = \frac{1}{t} \int_0^t \left\{ [x(J(s))]^+ - [x(J(s))]^- \right\} \, ds. \quad (22)$$

Now since $J$ is ergodic

$$\frac{1}{t} \int_0^t [x(J(s))]^+ \, ds = \sum_{j \in E} \frac{[x(j)]^+}{t} \int_0^t 1_{J(s) = j} \, ds \to \sum_{j \in E} [x(j)]^+ \pi_j, \text{ as } t \to \infty, \text{ a.s.} \quad (23)$$

and similarly $\frac{1}{t} \int_0^t [x(J(s))]^- \, ds \to \sum_{j \in E} [x(j)]^- \pi_j \text{ a.s. as } t \to \infty$, so that using (22) we conclude that $\lim_{t \to \infty} X(t)/t = \bar{x}$ a.s. Define the mean increment in the MRW $(S_n, J_n)$

$$\mu^* = \sum_{j \in E} \pi_j^* E [X_1 | J_0 = j]. \quad (24)$$

Since $E [X_1 | J_0 = j] = x(j)/(-\nu_{jj})$ it follows that $\bar{x} = \mu^* \sum_{k \in E} (-\nu_{kk}) \pi_k$. The statements (ii) – (iv) follow from this, and (19)-(20), by using Proposition 2 of Prabhu and Tang [12] and Theorem 8 of Prabhu, Tang and Zhu [13]. These last two results describe the fluctuation behaviour of the MRW $(S_n, J_n)$. $\Box$

3 The MRW $(T_n, S_n, J_n)$

In this section we investigate the properties of the MRW $(T_n, S_n, J_n)$. We note that the conditional distribution of the increments $(T_n - T_{n-1}, S_n - S_{n-1})$, given $J_{n-1}$ is singular, since $X_n = S_n - S_{n-1} = x(J_n)(T_n - T_{n-1})$ a.s. The distribution of $(T_1, X_1, J_1)$ is best described by the transform matrix

$$\Phi(\theta, \omega) = (\phi_{jk}(\theta, \omega)) = \left( E \left[ e^{-\theta T_1 + i\omega X_1}; J_1 = k \mid J_0 = j \right] \right) \quad (25)$$
for $\theta > 0$, $\omega$ real and $i = \sqrt{-1}$. We find that

$$\Phi(\theta, \omega) = (\phi_{jk}(\theta, \omega)) = (\alpha_j(\theta, \omega) p_{jk}) = (\alpha_j(\theta, \omega) \delta_{jk})(p_{jk}) = \alpha(\theta, \omega) P$$

where

$$\alpha_j(\theta, \omega) = \frac{-\nu_{jj}}{-\nu_{jj} + \theta - i\omega x(j)}, \quad p_{jk} = \frac{\nu_{jk}}{(-\nu_{jj})}(k \neq j), \quad p_{jj} = 0.$$  \hfill (27)

For the time-reversed MRW $(\hat{T}_n, \hat{S}_n, \hat{J}_n)$ corresponding to the given MRW we have

$$\hat{\Phi}(\theta, \omega) = \left(\hat{\phi}_{jk}(\theta, \omega)\right) = \left(E \left[ e^{-\theta T_1 + i\omega X_1}; \hat{J}_1 = k \mid \hat{J}_0 = j \right]\right)$$

$$= \left(\frac{\pi_k^*}{\pi_j^*} E \left[ e^{-\theta T_1 + i\omega X_1}; J_1 = j \mid J_0 = k \right]\right) = \hat{P} \alpha(\theta, \omega)$$

where $\hat{P}$ is the transition probability matrix of the time-reversed chain $\hat{J}$, namely

$$\hat{P} = (\hat{p}_{jk}) = \left(\frac{\pi_k^*}{\pi_j^*} p_{kj}\right). \hfill (29)$$

Since the $T_n$ are non-decreasing a.s., the fluctuating theory of the MRW $(T_n, S_n, J_n)$ is adequately described by $(S_n)$. We now define the descending ladder epoch $\overline{N}$ of this MRW $(T_n, S_n, J_n)$, and the ascending ladder epoch $N$ of the time-reversed MRW $(\hat{T}_n, \hat{S}_n, \hat{J}_n)$

$$\overline{N} = \min \left\{n : S_n < 0\right\}, \quad N = \min \left\{n : \hat{S}_n > 0\right\}. \hfill (30)$$

(Here we adopt the convention that the minimum of an empty set is $+\infty$.) It should be noted that both $N$ and $\overline{N}$ are strong ladder epochs, which is reasonable since the increments of $S_n$ and $\hat{S}_n$ in each case have an absolutely continuous distribution. The random variables $S_{\overline{N}}$ and $\hat{S}_N$ are the ladder heights corresponding to $\overline{N}$ and $N$. We also denote the transforms (in the matrix form)

$$\overline{X} = (\overline{x}_{jk}(z, \theta, \omega)) = \left(E \left[ z^{\overline{N}} e^{-\theta T_{\overline{N}} + i\omega S_{\overline{N}}}; \overline{J}_{\overline{N}} = k \mid J_0 = j \right]\right) \hfill (31)$$

$$\chi = (\chi_{jk}(z, \theta, \omega)) = \left(\frac{\pi_k^*}{\pi_j^*} E \left[ z^N e^{-\theta T_N + i\omega S_N}; \hat{J}_N = j \mid \hat{J}_0 = k \right]\right) \hfill (32)$$
where $0 < z < 1$, $\theta > 0$, $i = \sqrt{-1}$ and $\omega$ is real. Connecting these two transforms is the Wiener-Hopf factorization, first established by Presman [14] analytically, and interpreted in terms of the ladder variables defined above by Prabhu, Tang and Zhu [13]. The result is the following.

**Lemma 3 (Wiener-Hopf factorization)** For the MRW $(T_n, S_n, J_n)$ with $0 < z < 1$, $\theta > 0$ and $\omega$ real

$$I - z\Phi(\theta, \omega) = [I - \chi(z, \theta, \omega)] [I - \overline{\chi}(z, \theta, \omega)]. \quad \square \quad (33)$$

We shall use this factorization and the special structure of our MRW to indicate how the transforms $\chi$ and $\overline{\chi}$ can be computed in the general case. It turns out that our results contain information concerning the descending ladder epoch $T$ of the net input process $(X, J)$ and the ascending ladder epoch $\hat{T}$ of the time-reversed process $(\hat{X}, \hat{J})$, which is defined as follows:

$$\hat{J}(t) = \hat{J}_{n-1} \quad (\hat{T}_{n-1} < t \leq \hat{T}_n), \quad \hat{X}(t) = \int_0^t x(\hat{J}(s)) \, ds. \quad (34)$$

Thus

$$\hat{T} = \inf\{t > 0 : X(t) \leq 0\}, \quad T = \inf\{t > 0 : \hat{X}(t) \geq 0\}. \quad (35)$$

We note that $X(\hat{T}) = 0$ and $\hat{X}(T) = 0$ a.s. For $0 < z < 1$, $\theta > 0$ we define the transforms

$$\zeta = (\zeta_{jk}(z, \theta)) = \left( E \left[ z_N e^{-\theta T}; J(T) = k \mid J(0) = j \right] \right) \quad (36)$$

$$\eta = (\eta_{jk}(z, \theta)) = \left( \frac{n^*}{\pi^*} E \left[ z_N e^{-\theta T}; \hat{J}(T) = j \mid \hat{J}(0) = k \right] \right). \quad (37)$$

**Theorem 2** For $0 < z < 1$, $\theta > 0$ and $\omega$ real we have

$$\overline{\chi}(z, \theta, \omega) = \zeta(z, \theta) \Phi(\theta, \omega), \quad \chi(z, \theta, \omega) = \alpha(\theta, \omega) \eta(z, \theta). \quad (38)$$

**Proof:** An inspection of the sample paths of $(X, J)$ will show that $J(\hat{T}) = J_{\overline{N} - 1}$ and

$$T_N - \hat{T} = \frac{S_N}{x(J(\hat{T}))} \quad \text{a.s.}$$

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Since $\overline{T}$ is a stopping time for $(X, J)$, we see that given $J(\overline{T}) = l$, $S_{\overline{N}}/x(J(\overline{T}))$ is independent of $\overline{T}$ and has the same distribution as $T_1$, given $J_0 = l$. Therefore

$$\overline{\chi}_{jk}(z, \theta, \omega) = \sum_{i \in E} E \left[ z^N \exp \left( -\theta \left( \overline{T} + \frac{S_{\overline{N}}}{x(J_{N-1})} \right) + i\omega S_{\overline{N}} \right) ; J_{N-1} = l, J_N = k \mid J_0 = j \right]$$

$$= \sum_{i \in E} E \left[ z^N e^{-\theta T} ; J(T) = l \mid J_0 = j \right] E \left[ \exp \left( -\theta \frac{S_{\overline{N}}}{x(J_{N-1})} + i\omega S_{\overline{N}} \right) ; J_{N-1} = l \right]$$

$$= \sum_{i \in E} \zeta_{ij}(z, \theta) E \left[ e^{-\theta T_1 + i\omega X_1} ; J_1 = k \mid J_0 = l \right] = \sum_{i \in E} \zeta_{ij}(z, \theta) \phi_{ik}(\theta, \omega).$$

Thus $\overline{\chi}(z, \theta, \omega) = \zeta(z, \theta) \Phi(\theta, \omega)$. The proof of $\chi(z, \theta, \omega) = \alpha(\theta, \omega) \eta(z, \theta)$ is similar. $\square$

In general, for an $(|E| \times |E|)$-matrix $A$ we block-partition $A$ in the form

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

with the rows and columns of $A_{00}$ corresponding to the states in $E_0$. We have now

$$\zeta = \begin{pmatrix} 0 & \zeta_{01} \\ 0 & \zeta_{11} \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_{00} & \eta_{01} \\ 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} I_{00} & 0 \\ 0 & I_{11} \end{pmatrix} \quad (39)$$

where $I$ is the identity matrix. From (33) and Theorem 2 we have the following.

**Theorem 3** We have for $0 < z < 1$, $\theta > 0$ and $\omega$ real

$$\chi = \begin{pmatrix} \alpha_{00}\eta_{00} & \alpha_{00}\eta_{01} \\ 0 & 0 \end{pmatrix}, \quad \overline{\chi} = \begin{pmatrix} \zeta_{01}\Phi_{10} & \zeta_{01}\Phi_{11} \\ z\Phi_{10} & z\Phi_{11} \end{pmatrix}, \quad (40)$$

and

$$I_{00} - \chi_{00} = (I_{00} - \chi_{00})^{-1} \left[ (I_{00} - z\Phi_{00}) - z\chi_{01}\Phi_{10} \right] \quad (41)$$

$$\chi_{01} = (I_{00} - \chi_{00})^{-1} \left[ z\Phi_{01} - \chi_{01}(I_{11} - z\Phi_{11}) \right] \quad (42)$$

where the inverse exists in the specified domain. $\square$

We note that if we let

$$\overline{T} = (\overline{\tau}_{jk}(z, \theta)) = \left( E \left[ z^N e^{-\theta T} ; J_{N-1} = k \mid J_0 = j \right] \right)$$

$$= \begin{pmatrix} \overline{T}_{00} & \overline{T}_{01} \\ \overline{T}_{10} & \overline{T}_{11} \end{pmatrix} \quad (43)$$
\[
\gamma = (\gamma_{jk}(z, \theta)) = \left(\frac{\pi_k^*}{\pi_j} \mathbb{E} \left[ z^N e^{-\theta T_N}; \hat{J}_N = j \mid \hat{J}_0 = k \right] \right)
\] (44)

then with \( I^\theta = (-\delta_{jk}\nu_{jj}/[-\nu_{jj} + \theta]) \)

\[
\bar{\gamma} = \zeta I^\theta, \quad \gamma = I^\theta \eta
\] (45)

with Due to (45) the results for \((\zeta, \eta)\) are equivalent to those for \((\bar{\gamma}, \gamma)\). As a matter of convenience we express some of the remaining results of this section in terms of \((\bar{\gamma}, \gamma)\).

**Corollary 1** For \(0 < z < 1, \theta > 0\), with \( R = (r_{jk}(\theta)) = (\delta_{jk} \mid x(j) \mid /[-\nu_{jj} + \theta]) \), we have

\[
\gamma_{00} + (I_{00} - \gamma_{00}) \bar{\gamma}_{01} P_{10} = z \left[I_{00}^\theta P_{00} + \gamma_{01} I_{11}^\theta P_{10}\right]
\] (46)

\[
\gamma_{01} + (I_{00} - \gamma_{00}) \bar{\gamma}_{01} P_{11} = z \left[I_{00}^\theta P_{01} + \gamma_{01} I_{11}^\theta P_{11}\right]
\] (47)

\[
R_{00} \bar{\gamma}_{01} P_{10} + (I_{00} - \gamma_{00}) \bar{\gamma}_{01} R_{11} P_{10} = z \gamma_{01} R_{11} I_{11}^\theta P_{10}
\] (48)

\[
R_{00} \bar{\gamma}_{01} P_{11} + (I_{00} - \gamma_{00}) \bar{\gamma}_{01} R_{11} P_{11} = z \gamma_{01} R_{11} I_{11}^\theta P_{11}.
\] (49)

**Proof:** We equate the real parts of the identity (41) and put \( \omega = 0 \). This yields (46). We also equate the imaginary parts of (41), divide by \( \omega \) and let \( \omega \to 0 \). This yields (48), in view of (46). The proof of (47) and (49) is similar, starting with the identity (42). \( \square \)

Theorem 3 shows that the submatrices \( \overline{\chi}_{00} \) and \( \overline{\chi}_{01} \) are determined by \( \chi_{00} \) and \( \chi_{01} \). Corollary 1 can be used in some important cases to reduce the computation to a single (matrix) equation for \( \overline{\gamma}_{01} \), as we will show in the following. Case (i) arises in models with \(|E_1| > 1\), while case (ii) covers the situation with \(|E_1| = 1\). Details of the computations are omitted.

**Case (i).** If the submatrix \( P_{11} \) has an inverse, then

\[
\gamma_{00} = z I_{00}^\theta P_{00} + R_{00} \bar{\gamma}_{01} R_{11}^{-1} P_{10}
\] (50)

\[
\gamma_{01} = z I_{00}^\theta P_{01} + R_{00} \bar{\gamma}_{01} R_{11}^{-1} P_{11}
\] (51)

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where $\gamma_{01}$ satisfies the equation

$$
\gamma_{01} \left[ R_{11}^{-1} P_{10} \right] \gamma_{01} - \left[ R_{01}^{-1} \left( I_{00} - z R_{00} P_{00} \right) \right] \gamma_{01} + \gamma_{01} R_{11}^{-1} \left( I_{11} - z P_{11} I_{11}^0 \right) + z^2 R_{00}^{-1} R_{00}^0 P_{01} I_{11}^0 = 0.
$$

(52)

**Case (ii).** If $P_{11} = 0$ and $r_{jj}(\theta) = r_{11}^j$ ($j \in E_1$), then

$$
\gamma_{00} = z R_{00}^0 P_{00} + \frac{1}{r_{11}^1} R_{00} \gamma_{01} P_{10}, \quad \gamma_{01} = z R_{00}^0 P_{01}
$$

(53)

$$
(\gamma_{01} P_{10})^2 - \left[ r_{11}^1 R_{00}^{-1} \left( I_{00} - z R_{00}^0 P_{00} \right) + I_{00} \right] (\gamma_{01} P_{10}) + z^2 R_{00}^{-1} P_{00} I_{11}^0 P_{10} = 0.
$$

(54)

**Example 1** Consider the Gaver-Lehoczky [5] model with a single output channel, in which the channel is shared by data and voice calls (with calls having preemptive priority over data). Here $J(t) = 0$ if a call is in progress at time $t$ (i.e. the channel is not available for data transmission), and $J(t) = 1$ otherwise. Thus $J$ has a two state space $\{0, 1\}$ and

$$
a(0) = a(1) = c_0, \quad d(0) = 0, \quad d(1) = c_2 \quad (c_0 < c_2),
$$

so that $E_0 = \{0\}$, $E_1 = \{1\}$, $x(0) = c_0$ and $x(1) = c_0 - c_2 = -c_1$. Let the arrival and service rate of calls be denoted by $\lambda$ and $\mu$ respectively, then

$$
P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad N = (\nu_{jk}) = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}.
$$

We may now use (45) and (53)-(54) to conclude that

$$
\eta_{01} = z, \quad \eta_{00} = \frac{\sigma_1}{\sigma_0} \zeta_{01}, \quad \sigma_1 \zeta_{01}^2 - \left[ (\sigma_0 + \sigma_1) + \left( \frac{1}{c_0} + \frac{1}{c_1} \right) \theta \right] \zeta_{01} + z^2 \sigma_0 = 0
$$

where $\sigma_0 = \frac{\lambda}{c_0}$ and $\sigma_1 = \frac{\lambda}{c_1}$. This implies that

$$
\zeta_{01}(z, \theta) = \frac{(\sigma_0 + \sigma_1) + \left( \frac{1}{c_0} + \frac{1}{c_1} \right) \theta - \sqrt{\left[ (\sigma_0 + \sigma_1) + \left( \frac{1}{c_0} + \frac{1}{c_1} \right) \theta \right]^2 - 4 z^2 \sigma_0 \sigma_1}}{2 \sigma_1}.
$$

(55)

For an $M/M/1$ queue with arrival and service rates $\sigma_1$ and $\sigma_0$ respectively, we denote the busy period by $T^*$ and the number of customers served during the busy period by $N^*$. From (55) we have the following (see Section 2.8 of Prabhu [9])

$$
\zeta_{01}(z, \theta) = E \left[ z N^* e^{-\theta T^*} ; J(T) = 1 \mid J(0) = 0 \right] = E \left[ z^2 N^* e^{-\theta \left( \zeta_{00}^1 + \zeta_{01}^1 \right) T^*} \right]. \quad \square
$$
Example 2 Consider the Virtamo and Norros [17] model, given by (6). Denote \( c = c_1/c_0 \) and \( \rho = \lambda/\mu \), in which case \( 0 < c < 1 \) and \( \rho > 0 \). Equation (54) holds for this model and may be proved to be equivalent to:

\[
\zeta_{n0} \left[ \zeta_{10} - \frac{c\mu + \lambda + \mu}{\lambda (1 - c)} \right] - \frac{z c}{\rho (1 - c)} \left[ \zeta_{n-1,0} + \rho \zeta_{n+1,0} \right] = 0, \quad n \geq 1.
\]  \hspace{1cm} (56)

If \( J(0) = 1 \), it is known from Aalto [1] that \( \mathbf{T} \) is equal to the busy period of an \( M/M/1 \) queue with arrival rate \( \lambda \) and service rate \( c\mu \). Thus

\[
\zeta_{10}(1, \theta) = \frac{c\mu + \lambda + \mu}{2\lambda} - \sqrt{\left( \frac{c\mu + \lambda + \mu}{2\lambda} \right)^2 - \frac{c\mu}{2\lambda}}.
\]  \hspace{1cm} (57)

Using (56), the transforms

\[
\zeta_{n0}(1, \theta) = \left( E \left[ e^{-\theta\mathbf{T}} \mid J(0) = n \right] \right), \quad n \geq 1,
\]

may be computed recursively, starting with \( \zeta_{10}(1, \theta) \) as given in (57). \( \square \)

4 The main results

With the properties for the MRW \( (T_n, S_n, J_n) \) established in Section 3, we are now in a position to derive the main results of the paper. We first state the following results for the embedded process \( (Z_n, I_n, J_n) \), which follow easily from Theorem 3 and 4 of Prabhu and Tang [12].

Lemma 4 If \( Z_0 = 0 \) a.s. then for \( \theta > 0 \) and \( \omega_1, \omega_2 \) real we have

\[
\left( \sum_{n=0}^{\infty} E \left[ e^{-\theta T_n + i\omega_1 Z_n + i\omega_2 I_n} \mid J_n = k, J_0 = j \right] \right)^{-1} = [I - \chi(z, \theta, \omega_1)] [I - \chi(z, \theta, -\omega_2)]. \quad \square \hspace{1cm} (58)
\]

Lemma 5 Suppose that \( \mu^* < 0 \). Then \( (Z_n, J_n) \xrightarrow{D} (Z^*_\infty, J^*_\infty) \) as \( n \to \infty \), for all initial distributions, where \( J^*_\infty \) is the stationary version of \( (J_n) \) and \( E \left[ e^{i\omega Z^*_\infty} \mid J^*_\infty = k \right] \) is the \( k \)th element of the row vector

\[
\pi^* \left[ I - \chi(1, 0, 0) \right] [I - \chi(1, 0, \omega)]
\]

where \( \pi^* = (\pi^*_j, j \in E) \). \( \square \)
For finite $t$, the distribution of $(Z(t), I(t), J(t))$ can be found from equation (16) using Lemma 4. We note that since $T = \inf\{t > 0 : X(t) \leq 0\}$ we have

$$T = \inf\{t > 0 : Z(t) = 0 \mid Z(0) = 0\}. \quad (59)$$

Thus $T$ is the busy period of the storage. The transform of $(T, J(T))$ is $\zeta(1, \theta)$ given by

$$\zeta(1, \theta) = (\zeta_{jk}(1, \theta)) = \left(E\left[e^{-\theta T} ; J(T) = k \mid J(0) = j\right]\right) \quad (60)$$

In Section 3 it was shown how this transform can be computed. We recall that $\varpi$ is the net input rate.

**Theorem 4** The busy period $T$ defined by (59) has the following properties:

(i). Given $J(0) \in E_1, T = 0 \text{ a.s.}$

(ii). If $\varpi < 0$ then $T < \infty \text{ a.s.}$

**Proof:** (i). The statement follows immediately from the fact that $x(j) < 0$ for $j \in E_1$.

(ii). Since $\varpi < 0$ we have $\mu^* < 0$ (see the proof of Theorem 1 (ii)). This implies that the descending ladder epoch of the MRW $(T_n, S_n, J_n)$ is finite ($N < \infty \text{ a.s.}$) in virtue of Proposition 2 of Prabhu and Tang [12]. This in turn implies that $T_N < \infty \text{ a.s.}$ The statement now follows since $T \leq T_N$. $\square$

The limit behaviour of the process $(Z(t), I(t), J(t))$ as $t \to \infty$ can also be obtained from that of the embedded process $(Z_n, I_n, J_n)$ as $n \to \infty$, by using Lemma 5. The following theorems characterize this limit behaviour.

**Theorem 5** The process $(Z(t), I(t), J(t))$ has the following properties:

(i). If $\varpi > 0$, then $I(t) \to (Z(0) + m)^- < +\infty$ and $\frac{Z(t)}{t} \to \varpi \text{ a.s.}$;

in particular $Z(t) \to +\infty$ a.s.

(ii). If $\varpi = 0$, then $I(t) \to +\infty$ and $\limsup Z(t) = +\infty$ a.s.

(iii). If $\varpi < 0$, then $\frac{I(0)}{t} \to -\varpi$ and $\frac{Z(t)}{t} \to 0$ a.s.; in particular $I(t) \to +\infty$ a.s.
**Proof:** (i). We first note that $I(t)$ converges as indicated by Theorem 1 (ii) and (13). The rest of the statement follows directly from Theorem 1 (i) and (1).

(ii). From Theorem 1 (iii) and (13) $I(t) \to (Z(0) + m)^- = +\infty$. Also, from (12), since $I(t)$ is nonnegative, $Z(t) \geq Z(0) + X(t)$. Using Theorem 1 (iii) we obtain

\[ \limsup Z(t) \geq \limsup[Z(0) + X(t)] = +\infty. \]

(iii). Since $X(t)$ has continuous sample functions and $\frac{X(t)}{t} \to \varpi < 0$, by Theorem 1 (i), standard analytical arguments show that

\[ \lim \frac{m(t)}{t} = \lim \frac{X(t)}{t} = \varpi < 0. \]

The desired results now follow from (13)-(14). □

**Theorem 6** If $\varpi < 0$ then for $z_0, z \geq 0$ and $j, k \in E$, and with $(Z_\infty^*, J_\infty^*)$ being the limit distribution of $(Z_n, J_n)$ as given in Lemma 5,

\[ \lim_{t \to \infty} P \{ Z(t) \leq z; J(t) = k \mid Z(0) = z_0, J(0) = j \} = \pi_k \int_{0-}^{\infty} P \{ Z_\infty^* \in dv \mid J_\infty^* = k \} P \{ X_1 \leq z - v \mid J(0) = k \}. \]

(61)

**Proof:** Let $N(t) = \sup \{ n : T_n \leq t \}$. We have

\[ P \{ Z(t) \leq z; J(t) = k \mid Z(0) = z_0, J(0) = j \} = \int_{0-}^{\infty} P \{ Z_N(t) \in dv; J_N(t) = k \mid Z_0 = z_0, J_0 = j \} \]

\[ . P \{ Z(t) \leq z \mid Z(T_N(t)) = v, J(T_N(t)) = k, Z(0) = z_0, J(0) = j \} \]

\[ = P \{ J_N(t) = k \mid J_0 = j \} \int_{0-}^{\infty} P \{ Z_N(t) \in dv \mid Z_0 = z_0, J_N(t) = k, J_0 = j \} \]

\[ . P \{ Z(t) \leq z \mid Z(T_N(t)) = v, J(T_N(t)) = k \} \]

\[ = P \{ J(t) = k \mid J(0) = j \} \int_{0-}^{\infty} P \{ Z_N(t) \in dv \mid Z_0 = z_0, J_N(t) = k, J_0 = j \} \]

\[ . P \{ [v + x(k)(t - T_N(t))]^+ \leq z \mid J(T_N(t)) = k \} \]

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Since \( P \{ J(t) = k \mid J(0) = j \} \rightarrow \pi_k \) a.s. as \( t \rightarrow \infty \), the statement follows from the fact that as \( t \rightarrow \infty \) the following two results hold. Given \( J(T_{N(t)}) = k \), \( (t - T_{N(t)}) \) has by limit distribution \( T_1 \), given \( J(0) = k \), so that

\[
P \left\{ [v + X(t - T_{N(t)})]^+ \leq z \mid J(T_{N(t)}) = k \right\} \rightarrow P \left\{ [v + X_1]^+ \leq z \mid J(0) = k \right\} = P \{ X_1 \leq z - v \mid J(0) = k \}.
\]

Since \( \pi < 0 \) we have \( \mu^* < 0 \), and \( N(t) \rightarrow \infty \); thus using Lemma 5 we conclude that

\[
P \left\{ Z_{N(t)} \in dv \mid Z_0 = z_0, J_{N(t)} = k, J_0 = j \right\} \rightarrow P \{ Z^*_\infty \in dv \mid J^*_\infty = k \}. \quad \square
\]

In case \( \pi < 0 \), we denote by \( (Z^*_\infty, J^*_\infty) \) the limit random variable of \( (Z, J)(t) \), which in view of Theorem 6 is independent of the initial distribution.

**Corollary 2** If \( \pi < 0 \) we have the following:

(i). For \( z \geq 0 \) we have

\[
P \{ Z^*_\infty \leq z \mid J^*_\infty = k \} = P \{ Z^*_\infty \leq z \mid J^*_\infty = k \}
\]

\[
\cdot \left( 1 - E \left[ e^{-\frac{\mu k}{1-k}(Z^*_\infty - z)} \mid Z^*_\infty \leq z, J^*_\infty = k \right] \right) \quad (k \in E_0)
\]

(ii). We have

\[
P \{ Z^*_\infty = 0 \mid J^*_\infty = k \} = \begin{cases} 0 & k \in E_0 \\ E \left[ e^{-\frac{\mu k}{1-k}Z^*_\infty} \mid J^*_\infty = k \right] & k \in E_1 \end{cases}
\]

**Proof:** (i). Let \( z \geq 0 \) and \( k \in E_0 \). From (61) we have

\[
P \{ Z^*_\infty \leq z \mid J^*_\infty = k \} = \int_0^z P \{ Z^*_\infty \in dv \mid J^*_\infty = k \} P \{ X_1 \leq z - v \mid J(0) = k \}
\]

\[
= \int_0^z P \{ Z^*_\infty \in dv \mid J^*_\infty = k \} \left( 1 - e^{-\frac{\mu k}{1-k}(z-v)} \right)
\]

\[
= P \{ Z^*_\infty \leq z \mid J^*_\infty = k \} \left( 1 - \int_0^z P \{ Z^*_\infty \in dv \mid Z^*_\infty \leq z, J^*_\infty = k \} e^{-\frac{\mu k}{1-k}(v-z)} \right)
\]

\[
= P \{ Z^*_\infty \leq z \mid J^*_\infty = k \} \left( 1 - E \left[ e^{-\frac{\mu k}{1-k}(Z^*_\infty - z)} \mid Z^*_\infty \leq z, J^*_\infty = k \right] \right).
\]
This gives (62). The proof of (63) is similar.

(ii). The statement follows from the fact that using (61) we have

\[ P\{Z_\infty = 0 \mid J_\infty = k\} = \int_{0}^{\infty} P\{Z_\infty^* < dv \mid J_\infty^* = k\} P\{X_1 \leq -v \mid J(0) = k\}. \]

\[ \square \]

We denote by \( I_k(t) \) the unsatisfied demand in state \( k \) during \((0, t] \), so that

\[ I_k(t) = \int_{0}^{t} x(J(s))^{-1}1_{\{Z(s) = 0, J(s) = k\}} ds = x(k)^{-1} \int_{0}^{t} 1_{\{Z(s) = 0, J(s) = k\}} ds. \quad (65) \]

If \( k \in E_0 \), then \( x(k)^{-1} = 0 \) and \( I_k(t) = 0 \). If \( k \in E_1 \), we have the following important result for the performance analysis of the system.

**Corollary 3** If \( \pi < 0 \) and \( k \in E_1 \), then

\[ \lim_{t \to \infty} \frac{I_k(t)}{t} = -x(k) \pi_k E\left[ e^{-\frac{\pi_k}{x(k)} Z_\infty^*} \middle| J_\infty^* = k \right] \quad (66) \]

and

\[ \lim_{t \to \infty} \frac{I_k(t)}{I(t)} = \frac{x(k) \pi_k E\left[ e^{-\frac{\pi_k}{x(k)} Z_\infty^*} \middle| J_\infty^* = k \right]}{\sum_{j \in E_1} x(j) \pi_j E\left[ e^{-\frac{\pi_j}{x(j)} Z_\infty^*} \middle| J_\infty^* = j \right]}. \quad (67) \]

**Proof:** Using Theorem 6 we conclude that

\[ \frac{1}{t} \int_{0}^{t} 1_{\{Z(s) = 0, J(s) = k\}} ds \to P\{Z_\infty = 0, J_\infty = k\}. \]

This implies (66) in view of (64)-(65). Also, since \( I(t) = \sum_{j \in E_1} I_j(t) \), (67) follows by using (66). \( \square \)

**Example 1** (**Continuation**) We note that \( \pi_0^* = \pi_1^* = 1/2, \zeta_{01}(1, 0) = 1, \eta_{01}(1, 0) = 1 \) and \( \eta_{00}(1, 0) = \rho \), with \( \rho = \sigma_1 / \sigma_0 \). We now assume \( \rho < 1 \). Since

\[ \pi^*[I - \chi(1, 0, 0)] [I - \chi(1, 0, \omega)]^{-1} = \frac{1}{2} \left[ (1 - \rho) + \rho \frac{\sigma_0 - \sigma_1}{(\sigma_0 - \sigma_1) - i\omega} \right] \frac{\sigma_0 - \sigma_1}{(\sigma_0 - \sigma_1) - i\omega}, \]

we conclude from Lemma 5 that \( P\{Z_\infty^* > z \mid J_\infty^* = 0\} = \rho e^{-(\sigma_0 - \sigma_1)z} \) for \( z \geq 0 \), and similarly \( P\{Z_\infty^* > z \mid J_\infty^* = 1\} = e^{-(\sigma_0 - \sigma_1)z} \). With \( \pi_0 = \lambda/(\lambda + \mu) \) and \( \pi_1 = \mu/(\lambda + \mu) \) we conclude using Theorem 6 that for \( z \geq 0 \),

\[ P\{Z_\infty > z; J_\infty = 0\} = \pi_0 e^{-(\sigma_0 - \sigma_1)z}, \quad P\{Z_\infty > z; J_\infty = 1\} = \pi_1 \rho e^{-(\sigma_0 - \sigma_1)z}. \]
Finally using Corollary 3 we conclude that a.s.

\[ I_0(t) = 0, \quad \forall t \quad \text{and} \quad \lim_{t \to \infty} \frac{I_1(t)}{t} = c_1 \pi_1 (1 - \rho). \]

We note that this example has been considered also by Chen and Yao [4], Gaver and Miller [6], and Kella and Whitt [8] (in the context of storage models for which the net input is alternately nonincreasing and non-decreasing, and by Karandikar and Kulkarni [7] (Case 1 of Example 1 - Section 6) with the storage level being a particular case of a Markov-modulated reflected Brownian motion. \(\square\)

References


