A Note on the Moduli Space of Polygons

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Abstract. For any \( r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n \), let \( M_r \) be the space of polygons in the Euclidean space of fixed side lengths \( r_1, \ldots, r_n \) modulo rigid motions. In this note we give an overview of some classical descriptions of these spaces and we present formulas for their cobordism classes and symplectic volume. For a special case, with \( n = 5 \), the volume formula is applied to calculate the cohomology ring \( H^*(M_r, \mathbb{Q}) \).

Keywords: Polygons, Cobordism, Symplectic Volume, Cohomology Ring.

PACS: 02.40.K.

1. AN INTRODUCTION TO THE MODULI SPACE \( M_r \)

A polygon in \( \mathbb{R}^3 \) is determined by its vertices \( v_1, \ldots, v_n \) and its oriented edges \( e_1, \ldots, e_n \). For any vector \( r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n \), \( M_r \) will denote the space of polygons with edge lengths \( r_1, \ldots, r_n \) modulo rigid motions (i.e. modulo rotations and translations). The study of the geometry of these spaces is related with many areas of mathematics, including the study of symplectic reduced spaces ([4], [9]) and of quotients by means of Geometric Invariant Theory ([11], [6]).

These moduli spaces have been extensively studied in recent years. As a contribution to these proceedings, we will briefly review some of the results in the symplectic setting, taking the occasion to announce original research contained in the Ph.D. thesis [13], which will appear elsewhere.

For any \( r \in \mathbb{R}_+^n \), the product \( \prod_{j=1}^n S^2_{r_j} \) of \( n \) spheres of radii \( r_1, \ldots, r_n \) respectively is a compact symplectic manifold. The moduli space \( M_r \) of polygons of fixed side lengths \( r \) is, by definition, the symplectic quotient \( \prod_{j=1}^n S^2_{r_j} // SO(3) =: M_r \) with respect to the diagonal \( SO(3) \)-action by rotation on each sphere. A polygon is said to be degenerate if it lies completely on a line. There is such a polygon in \( M_r \) if and only if the scalar quantity \( \varepsilon_I(r) := \sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i \) is zero for some subset \( I \) of \( \{1, \ldots, n\} \). When \( \varepsilon_I(r) \neq 0 \) for every subset \( I \) of \( \{1, \ldots, n\} \) we say that \( r \) is generic. The space \( M(r) \) is a smooth manifold if and only if \( r \) is generic. Note that \( M_r \) is diffeomorphic to \( M_{\lambda r} \) for each \( \lambda \) in \( \mathbb{R}_+ \). For more details on this construction see the original paper [9] and also [12].

Hausmann and Knutson [4] proved that \( M_r \) can also be described as the symplectic quotient relative to the natural action of \( U_1^n \), the maximal torus of diagonal matrices in...
the unitary group $U_n$, on the Grassmannian $Gr(2, \mathbb{C}^n)$ of 2-planes in $\mathbb{C}^n$.

**Theorem 1.1.** [4] Let $\Phi$ be the moment map relative to the $U_1^n$-action on $Gr(2, \mathbb{C}^n)$. Then

\[
\Phi(Gr(2, \mathbb{C}^n)) := \Xi = \{ r \in \mathbb{R}_+^n \mid 0 \leq r_i \leq 1, \sum_{i=1}^n r_i = 2 \}
\]

and

\[
M_r = U_1^n \backslash \Phi^{-1}(r) = Gr(2, \mathbb{C}^n) / U_1^n(r).
\]

### 2. SOME RESULTS ON THE TOPOLOGY OF $M_r$

In [12, 13] an explicit characterization of the cobordism class of $M_r$ is given, generalizing a previous result of Kamiyama [7]. (See [12] for a relation between the two.)

Denote by $M \sim N$ the existence of a cobordism between two manifolds $M$ and $N$ and by $\sqcup$ the disjoint union. A subset $J \subseteq \{1, \ldots, n-2\}$ is said to be $r$-admissible (or triangular) if the triple $(\sum_{j \in J} \sigma_j r_j, r_{n-1}, r_n)$ satisfies the triangle inequalities, with $\sigma_j = 1$ if $j \in J$, $\sigma_j = -1$ otherwise.

**Theorem 2.1.** [12] For $r \in \mathbb{R}_+^n$ generic,

\[M_r \sim \bigcup_{J \text{ r-admissible}} (-1)^{n-|J|} \mathbb{C}P^{n-3} \]

**Example 2.2.** Take $r = \frac{1}{17} (2, 1, 8, 2, 4)$. Then $r$ is generic and the only $r$-admissible set is $\{3\}$. Therefore

\[M_r \sim \mathbb{C}P^2.\]

The cohomology ring $H^*(M_r, \mathbb{Q})$ has been computed by Hausmann and Knutson [5] and, independently, by Goldin [2]. In [13] this ring is described by using different techniques, as an application of the volume formula and of the Duistermaat–Heckman theorem.

Martin [14] and (independently) Kamiyama and Tezuka [8] calculated the volume of $M_r$ for odd $n$ and $r = (1, \ldots, 1)$. In [13] it is shown that Martin’s techniques generalize to generic $r \in \mathbb{R}_+^n$, both for even and odd $n$.

**Theorem 2.3.** [13] For $r \in \mathbb{R}_+^n$ generic,

\[\text{vol}(M_r) = \frac{(2\pi)^{n-3}}{2(n-3)!} \sum_{I \in \mathcal{J}} (-1)^{n-|I|} \varepsilon_I(r)^{n-3},\]

where

\[\mathcal{J} = \{ I \subseteq \{1, \ldots, n\} : \varepsilon_I(r) = \sum_{i \in I} r_i - \sum_{i \in I} r_i > 0 \}.\]

A similar volume formula was previously proven by Khoi [10] using a different approach.
Example 2.4. In the set of regular values for the moment map \( \Phi \), let \( \Delta \) be the region delimited by the following walls:

\[
\sum_{i \neq j} r_i > r_j \text{ for all } j = 1, \ldots, 5;
\]

\[
\sum_{i \in \{3,j\}} r_i > \sum_{i \in \{3,j\}^c} r_i \text{ for all } j = 1, 2, 4, 5;
\]

\[
\sum_{i \in \{3,j,k\}} r_i > \sum_{i \in \{3,j,k\}^c} r_i \text{ for all } j, k = 1, 2, 4, 5.
\]

Note also that the vector \( r \) from Example 2.2 lies in \( \Delta \). For all \( r \)'s in \( \Delta \), the set \( J \) is given by

\[
J = \{ \{3,j\}, \{3,j,k\} : j, k = 1, 2, 4, 5 \} \cup \{ I \subseteq \{1, \ldots, 5\} : |I| = 4, 5 \}.
\]

Then, applying Theorem 2.3, it can be checked that the volume for the associated symplectic quotient \( M_r \) is

\[
\text{vol}(M_r) = 2\pi^2(r_1 + r_2 - r_3 + r_4 + r_5)^2
\]

and, because the perimeter \( \sum_{i=1}^n r_i = 2 \) is fixed, we obtain

\[
\text{vol}(M_r) = 2\pi^2(2 - 2r_3)^2.
\]

The study of the cohomology ring structure of a reduced space \( M \sslash G \) (even in the well behaved case of a compact connected Lie group \( G \) acting on a compact manifold \( M \)) is one of the foremost topics in symplectic topology. Although many beautiful results have been achieved, the problem is still not completely closed. Let \( c = (c_1, \ldots, c_n) \) be the Chern class of the fibration \( \mu^{-1}(\xi) \rightarrow M \sslash \xi G \) for a regular value \( \xi \in (\mathfrak{g}^*)^G \) so that the symplectic quotient \( M \sslash \xi G \) is a symplectic manifold. Guillemin and Sternberg [3] observed that if the volume function of the symplectic reduction is known, then its cohomology ring \( H^*(M \sslash G, \mathbb{Q}) \) can be deduced from the Duistermaat–Heckman theorem whenever the \( c_i \)'s generate \( H^*(M \sslash G, \mathbb{Q}) \).

In [13] it is proved that this is the case for moduli spaces of polygons, and the cohomology ring is explicitly described in terms of the \( c_i \)'s. A nice characterization of these Chern classes is given by Agapito and Godinho [1] as follows. For each \( 1 \leq j \leq n \) define

\[
V_j(r) = \left\{ (v_1, \ldots, v_n) \in \prod_{i=1}^n S_{r_i}^2 \mid \sum_{i=1}^n v_i = 0 \text{ and } v_j = (0,0,r_j) \right\}.
\]

For a generic \( r \) the \( S^1 \)-action by rotation around the \( z \) axis is free and \( V_j(r)/S^1 = M_r \). Then \( c_j := c_1(V_j(r)) \) is the first Chern class of the principal circle bundle \( V_j(r) \rightarrow M_r \).

By the Duistermaat–Heckman theorem, for \( r \) and \( r^0 \) in the same region of regular values,

\[
\text{vol}(M_r) = \int_{M_r} \exp([\omega_r]) = \int_{M_r^0} \exp([\omega_{r^0}] + \sum_{i=1}^n (r_i - r_i^0)c_i).
\]
Thus, for any multindex \( \alpha \) with \( |\alpha| = n - 3 = \text{dim}_C(M_r) \),

\[
\frac{\partial^\alpha}{\partial r^\alpha} \text{vol}(M_r)|_{r_0} = \int_{M_{r_0}} c_1^{\alpha_1} \cdots c_n^{\alpha_n}.
\]

(1)

If the \( c_1, \ldots, c_n \) generate \( H^*(M//G, \mathbb{Q}) \), then the cohomology pairings (1) determine the multiplicative relations between the generators. (See [3] for the general argument.) In the following example the cohomology ring of \( M_r \) for \( r \in \Delta \) as in Example 2.4 is explicitly calculated.

**Example 2.5.** For \( n = 5 \), the moduli space \( M_r \) is toric for each generic \( r \) [9], and thus the \( c_i \)'s generate the cohomology ring. Let \( \Delta \) be as in Example 2.4 and let \( r \) be any length vector in \( \Delta \). Since \( \text{vol}(M_r) = 2\pi^2 (2 - 2r_3)^2 \), all the second partial derivatives of the volume function vanish except

\[
\frac{\partial^2}{\partial^2 r_3} \text{vol}(M_r) = \frac{\partial}{\partial r_3}(-2(2 - 2r_3)) = 4.
\]

By (1) this implies that \( c_i = 0 \) for \( i = 1, 2, 4, 5 \) and therefore

\[
H^*(M_r, \mathbb{Q}) = \mathbb{Q}[c_3] / (c_3^4).
\]

**ACKNOWLEDGMENTS**

The results announced in this note have been developed during my Ph.D. under the direction of Luca Migliorini, to whom I am grateful for suggesting the study of these moduli spaces and for his help and support through these years. Also, I would like to thank Leonor Godinho for her suggestions regarding this note.

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