A Note on the Moduli Space of Polygons

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Abstract. For any $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, let M_r be the space of polygons in the Euclidean space of fixed side lengths $r_1, \dots, r_n$ modulo rigid motions. In this note we give an overview of some classical descriptions of these spaces and we present formulas for their cobordism classes and symplectic volume. For a special case, with $n = 5$, the volume formula is applied to calculate the cohomology ring $H^*(M_r, \mathbb{Q})$.

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1. AN INTRODUCTION TO THE MODULI SPACE M_r

A polygon in $\mathbb{R}^3$ is determined by its vertices $v_1, \dots, v_n$ and its oriented edges $e_1, \dots e_n$. For any vector $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, M_r will denote the space of polygons with edge lengths $r_1, \dots r_n$ modulo rigid motions (i.e. modulo rotations and translations). The study of the geometry of these spaces is related with many areas of mathematics, including the study of symplectic reduced spaces ([4], [9]) and of quotients by means of Geometric Invariant Theory ([11], [6]).

These moduli spaces have been extensively studied in recent years. As a contribution to these proceedings, we will briefly review some of the results in the symplectic setting, taking the occasion to announce original research contained in the Ph.D. thesis [13], which will appear elsewhere.

For any $r \in \mathbb{R}^n_+$, the product $\prod_{j=1}^n S^2_{r_j}$ of n spheres of radii $r_1, \dots, r_n$ respectively is a compact symplectic manifold. The moduli space M_r of polygons of fixed side lengths r is, by definition, the symplectic quotient $\prod_{j=1}^n S^2_{r_j} // SO(3) =: M_r$ with respect to the diagonal $SO(3)$-action by rotation on each sphere. A polygon is said to be *degenerate* if it lies completely on a line. There is such a polygon in M_r if and only if the scalar quantity $\varepsilon_I(r) := \sum_{i \in I} r_i - \sum_{i \in I^c} r_i$ is zero for some subset I of $\{1, \dots, n\}$. When $\varepsilon_I(r) \neq 0$ for every subset I of $\{1, \dots, n\}$ we say that r is generic. The space $M(r)$ is a smooth manifold if and only if r is generic. Note that M_r is diffeomorphic to $M_{\lambda r}$ for each λ in $\mathbb{R}_+$. For more details on this construction see the original paper [9] and also [12].

Hausmann and Knutson [4] proved that M_r can also be described as the symplectic quotient relative to the natural action of U_1^n, the maximal torus of diagonal matrices in

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the unitary group U_n, on the Grassmannian $Gr(2,\mathbb{C}^n)$ of 2-planes in $\mathbb{C}^n$.

Theorem 1.1. *[4] Let Φ be the moment map relative to the U_1^n-action on $Gr(2,\mathbb{C}^n)$. Then*

$$\Phi(Gr(2,\mathbb{C}^n)) := \Xi = \{r \in \mathbb{R}_+^n \mid 0 \leq r_i \leq 1, \sum_{i=1}^n r_i = 2\}$$

and

$$M_r = U_1^n \backslash \Phi^{-1}(r) = Gr(2,\mathbb{C}^n)//U_1^n(r).$$

2. SOME RESULTS ON THE TOPOLOGY OF M_r

In [12, 13] an explicit characterization of the cobordism class of M_r is given, generalizing a previous result of Kamiyama [7]. (See [12] for a relation between the two.)

Denote by $M \sim N$ the existence of a cobordism between two manifolds M and N and by $\bigsqcup$ the disjoint union. A subset $J \subseteq \{1,\ldots,n-2\}$ is said to be r-admissible (or triangular) if the triple $(\sum_{j\in J} \sigma_j r_j, r_{n-1}, r_n)$ satisfies the triangle inequalities, with $\sigma_j = 1$ if $j \in J$, $\sigma_j = -1$ otherwise.

Theorem 2.1. *[12] For $r \in \mathbb{R}_+^n$ generic,*

$$M_r \sim \bigsqcup_{J\ r\text{-admissible}} (-1)^{n-|J|}\mathbb{CP}^{n-3}.$$

Example 2.2. *Take $r = \frac{2}{17}(2,1,8,2,4)$. Then r is generic and the only r-admissible set is $\{3\}$. Therefore*

$$M_r \sim \mathbb{CP}^2.$$

The cohomology ring $H^*(M_r,\mathbb{Q})$ has been computed by Hausmann and Knutson [5] and, independently, by Goldin [2]. In [13] this ring is described by using different techniques, as an application of the volume formula and of the Duistermaat–Heckman theorem.

Martin [14] and (independently) Kamiyama and Tezuka [8] calculated the volume of M_r for odd n and $r = (1,\ldots,1)$. In [13] it is shown that Martin's techniques generalize to generic $r \in \mathbb{R}_+^n$, both for even and odd n.

Theorem 2.3. *[13] For $r \in \mathbb{R}_+^n$ generic,*

$$vol(M_r) = -\frac{(2\pi)^{n-3}}{2(n-3)!} \sum_{I\in\mathscr{I}} (-1)^{n-|I|}\, \varepsilon_I(r)^{n-3},$$

where

$$\mathscr{I} = \{I \subset \{1,\ldots,n\} : \varepsilon_I(r) = \sum_{i\in I} r_i - \sum_{i\in I^c} r_i > 0\}.$$

A similar volume formula was previously proven by Khoi [10] using a different approach.

Example 2.4. *In the set of regular values for the moment map Φ, let Δ be the region delimited by the following walls:*

$$\sum_{i\neq j} r_i > r_j \quad \textit{for all} \;\; j = 1,\dots,5;$$

$$\sum_{i\in\{3,j\}} r_i > \sum_{i\in\{3,j\}^c} r_i \quad \textit{for all} \;\; j = 1,2,4,5;$$

$$\sum_{i\in\{3,j,k\}} r_i > \sum_{i\in\{3,j,k\}^c} r_i \quad \textit{for all} \;\; j,k = 1,2,4,5.$$

Note also that the vector r from Example 2.2 lies in Δ. For all r's in Δ, the set $\mathscr{I}$ is given by

$$\mathscr{I} = \big\{\{3,j\},\{3,j,k\} : j,k = 1,2,4,5\big\} \cup \big\{I \subseteq \{1,\dots,5\} : |I| = 4,5\big\}.$$

Then, applying Theorem 2.3, it can be checked that the volume for the associated symplectic quotient M_r is

$$vol(M_r) = 2\pi^2(r_1 + r_2 - r_3 + r_4 + r_5)^2$$

and, because the perimeter $\sum_{i=1}^n r_i = 2$ is fixed, we obtain

$$vol(M_r) = 2\pi^2(2 - 2r_3)^2.$$

The study of the cohomology ring structure of a reduced space $M/\!\!/G$ (even in the well behaved case of a compact connected Lie group G acting on a compact manifold M) is one of the foremost topics in symplectic topology. Although many beautiful results have been achieved, the problem is still not completely closed. Let $c = (c_1,\dots,c_n)$ be the Chern class of the fibration $\mu^{-1}(\xi) \to M/\!\!/_\xi G$ for a regular value $\xi \in (\mathfrak{g}^*)^G$ so that the symplectic quotient $M/\!\!/_\xi G$ is a symplectic manifold. Guillemin and Sternberg [3] observed that if the volume function of the symplectic reduction is known, then its cohomology ring $H^*(M/\!\!/G,\mathbb{Q})$ can be deduced from the Duistermaat–Heckman theorem whenever the c_i's generate $H^*(M/\!\!/G,\mathbb{Q})$.

In [13] it is proved that this is the case for moduli spaces of polygons, and the cohomology ring is explicitly described in terms of the c_i's. A nice characterization of these Chern classes is given by Agapito and Godinho [1] as follows. For each $1 \leq j \leq n$ define

$$V_j(r) = \Big\{(v_1,\dots,v_n) \in \prod_{i=1}^n S^2_{r_i} \mid \sum_{i=1}^n v_i = 0 \;\; \textit{and} \;\; v_j = (0,0,r_j)\Big\}.$$

For a generic r the S^1-action by rotation around the z axis is free and $V_j(r)/S^1 = M_r$. Then $c_j := c_1(V_j(r))$ is the first Chern class of the principal circle bundle $V_j(r) \to M_r$.

By the Duistermaat–Heckman theorem, for r and r^0 in the same region of regular values,

$$vol(M_r) = \int_{M_r} \exp([\omega_r]) = \int_{M_{r^0}} \exp([\omega_{r^0}] + \sum_{i=1}^n (r_i - r_i^0)c_i).$$

Thus, for any multindex α with $|\alpha| = n-3 = dim_{\mathbb{C}}(M_r)$,

$$\frac{\partial^\alpha}{\partial r^\alpha} vol(M_r)_{|_{r0}} = \int_{M_{r0}} c_1^{\alpha_1} \cdots c_n^{\alpha_n}. \tag{1}$$

If the $c_1, \ldots, c_n$ generate $H^*(M/\!\!/G, \mathbb{Q})$, then the cohomology pairings (1) determine the multiplicative relations between the generators. (See [3] for the general argument.) In the following example the cohomology ring of M_r for $r \in \Delta$ as in Example 2.4 is explicitly calculated.

Example 2.5. *For $n = 5$, the moduli space M_r is toric for each generic r [9], and thus the c_i's generate the cohomology ring. Let Δ be as in Example 2.4 and let r be any length vector in Δ. Since $vol(M_r) = 2\pi^2(2-2r_3)^2$, all the second partial derivatives of the volume function vanish except*

$$\frac{\partial^2}{\partial^2 r_3} \frac{vol(M_r)}{4\pi^2} = \frac{\partial}{\partial r_3}(-2(2-2r_3)) = 4.$$

By (1) *this implies that $c_i = 0$ for $i = 1,2,4,5$ and therefore*

$$H^*(M_r, \mathbb{Q}) = \frac{\mathbb{Q}[c_3]}{(c_3^3)}.$$

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