Math 24, Ordinary Differential Equations
UCSC Summer Session II, 2003
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Phase Portrait, $\vec{x}' = A\vec{x}$, case $detA = 0$

If $detA = 0$, then at least one eigenvalue is zero. There are basically three cases to study. This does not mean that a matrix with determinant equal to zero is going to be one of these cases but its reduced echelon matrix will be similar to one of the matrices to be shown below or a combination of them.

I) Real and distinct eigenvalues $\lambda_1 = \alpha \neq 0, \lambda_2 = 0$

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{matrix} x_1' & = & \alpha x_1 \\ x_2' & = & 0 \end{matrix}$$

Equilibrium points. Solve simultaneously $\alpha x_1 = 0$ and $0 = 0$. We obtain that $x_1 = 0$. This means that each point on the line $x_1 = 0$ is an equilibrium point. At any other point (x_1, x_2), there is no vertical direction ($x_2' = 0$). Therefore, the phase portrait has a vertical line of equilibrium points and horizontal arrows pointing towards or away from that vertical line. This will depend on the sign of α. Positive sign means pointing outwards, negative sign means pointing inwards.

To find the general solution to this system, we have a short way and a long way. Let's do it the long way. First, we compute the eigenvectors corresponding to these eigenvalues; i.e. you solve for (u_1, u_2) in the system

$$(A - \lambda I)\vec{u} = \vec{0} \Longleftrightarrow \left(\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\lambda_1 = \alpha$: $\quad \begin{pmatrix} 0 & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow u_2 = 0$, u_1 is free. We can then pick, for instance, $\vec{v}_1 = (1, 0)$ as an eigenvector corresponding to λ_1.

$\lambda_1 = \alpha$: $\quad \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow u_1 = 0$, u_2 is free. We can then pick, for instance, $\vec{v}_2 = (0, 1)$ as an eigenvector corresponding to λ_2.

Now, the general solution to the system above is

$$\vec{x}(t) = c_1 e^{\alpha t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{0t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 \end{pmatrix}.$$

What is the short way?

II) Both eigenvalues are zero $\lambda_1 = 0, \lambda_2 = 0$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{matrix} x_1' & = & 0 \\ x_2' & = & 0 \end{matrix}$$

Then, all points are equilibrium points. The general solution is (again, using the long way)

$$\vec{x}(t) = c_1 e^{0t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{0t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

What is the short way?

III) Both eigenvalues are zero, but A is nilpotent $\lambda_1 = 0, \lambda_2 = 0$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{array}{rcl} x_1' & = & x_2 \\ x_2' & = & 0 \end{array}$$

Equilibrium points. Solve simultaneously $x_2 = 0$ and $0 = 0$. We obtain that $x_1 = 0$. This means that each point on the line $x_2 = 0$ is an equilibrium point. At any other point (x_1, x_2), there is no vertical direction $(x_2' = 0)$. Therefore, the phase portrait has a horizontal line of equilibrium points and horizontal arrows pointing left or right depending whether they are above or below $x_2 = 0$.

To find the general solution to this system, we have a short way and a long way. Let's do it the long way. Again, we compute the eigenvectors corresponding to these eigenvalues; i.e. you solve for (u_1, u_2) in the system

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow u_2 = 0$, u_1 is free. We can then pick, for instance, $\vec{v}_1 = (1, 0)$ as an eigenvector corresponding to $\lambda_1 = \lambda_2 = 0$. There is no way to compute a second non parallel eigenvector $\vec{v}_2$. In this case, one computes what is called a generalized eigenvector. That is, one looks for a solution to the system

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1.$$

We find that $\vec{v}_2$ can be taken to be $(0, 1)$. Then, the general solution is

$$\vec{x}(t) = c_1 e^{0t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \left[t e^{0t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{0t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix}$$

What is the short way?