Math 24 - Ordinary Differential Equations

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Final

You are free to use any combination of paper/pencil and/or packages like *Mathematica*, `Maple` or Matlab for calculations, graphs, reports, etc. If you know LaTeX, feel free to use it to present your final work. However you decide to present your work, please be always neat and clear. Remember, in preparing your solutions, keep in mind the style I mentioned last time as a guide: your job is to convince me, with an argument that is neither longer nor shorter than necessary, of the correctness of your position. If the right answer is 3 it is never enough to say "3", but brevity will be rewarded.

1. (a) Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$. Also draw a direction field and plot a few trajectories of the system. Remember to justify your work.

$$(i) \quad \vec{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \vec{x}, \quad (ii) \quad \vec{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \vec{x}, \quad (iii) \quad \vec{x}' = \begin{pmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{pmatrix} \vec{x}$$

(b) Solve the given value problem and describe the behavior of the solution as $t \to \infty$.

$$\vec{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

(c) Find the general solution of the given system of equations

$$\begin{aligned} x_1' &= -\tfrac{5}{4}x_1 + \tfrac{3}{4}x_2 + 2t \\ x_2' &= \tfrac{3}{4}x_1 - \tfrac{5}{4}x_2 + e^t \end{aligned}$$

2. (Taken from *The Boston University Ordinary Differential Equations Project*. ***This is the default project***) Here you will study a nonlinear, first-order system known as the **Predator-Prey** model. The equations are

$$\begin{aligned} \frac{dx}{dt} &= (9 - \alpha x - 3y)x \\ \frac{dy}{dt} &= (-2 + x)y \end{aligned}$$

where $\alpha \geq 0$ is a parameter. In other words, for different values of α we have different systems. The variable x is the population (in some scaled units) of prey (rabbits), and y is the population of predators (foxes). For a given value of α, we want to understand what happens to these populations as $t \to \infty$.

Your job is to investigate the phase portraits of these equations for various values of α in the interval $0 \le \alpha \le 5$. To get started, you might want to try $\alpha = 0, 1, 2, 3, 4$ and 5. Think about what the phase portrait means in terms of the evolution of the x and y populations. Where are the equilibrium points? What types are they? What happens to a typical solution curve? Also, consider the behavior of the special solutions where either $x = 0$ or $y = 0$ (that is, solution curves lying on the x- or y-axes).

Determine the bifurcation values of α. Namely, determine the values of α where nearby α's lead to *different* behaviors in the phase portrait. For example, $\alpha = 0$ is a bifurcation value because, for $\alpha = 0$, the long term behavior of the populations is dramatically different from the long term behavior of the populations if α is slightly positive. (*Hint*: If the type of an equilibrium point changes at a certain value of α, then that value of α may be a bifurcation value.)

After you have determined all of the bifurcation values for α in the interval $0 \le \alpha \le 5$, study enough specific values of α to be able to discuss all of the various population evolution scenarios for these systems. In your report, you should describe these scenarios using the phase portraits, tx- and ty-graphs. Give an interpretation of your graphs in terms of the populations. Fundamental calculations should be included in your essay. Your report should include:

a) A brief discussion of the significance of the various terms in the system. For example, what does the 9 represent? What does the $3y$ term represent?

b) A discussion of all bifurcations including the bifurcation at $\alpha = 0$. What does these bifurcations mean for the predator population?

c) All fundamental calculations.

You may provide as many illustrations from the computer or your personal art work as you wish, but the relevance of each illustration must be explained in your report. To get further information on nonlinear system analysis (using *Mathematica*) click *here*.

3. (***Extra credit***) Solve Problem 18 of Section §7.8 of the textbook. Those of you that did not do well in the midterm may want to solve this problem for sure.