Score: ____________

Name:

Solution Midterm

All questions are worth **6** points. Maximum score: **30**.

1. Evaluate $\iiint_{\mathcal{E}} (x^3 + xy^2)\, dx\, dy\, dz$ where $\mathcal{E}$ is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$.

   Solution.

$$I = \iiint_{\mathcal{E}} (x^3 + xy^2)\, dx\, dy\, dz = \iint_{\mathcal{D}} \left(\int_{z=0}^{z=1-x^2-y^2} (x^3 + xy^2)\, dz \right) dx\, dy,$$

   where $\mathcal{D}$ is the region in the XY-plane bounded by $x^2 + y^2 = 1$, $x \geq 0$ and $y \geq 0$. Then, using cylindrical coordinates, we have

$$\begin{aligned}
\iint_{\mathcal{D}} \left(\int_{z=0}^{z=1-x^2-y^2} (x^3 + xy^2)\, dz \right) dx\, dy &= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r^2} (r^3 \cos\theta) r\, dz\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 r^4 \cos\theta (1 - r^2)\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos\theta \left(\frac{r^5}{5} - \frac{r^7}{7} \right) \Bigg|_0^1 d\theta \\
&= \left(\frac{1}{5} - \frac{1}{7} \right) \sin\theta \Bigg|_0^{\frac{\pi}{2}} = \frac{2}{35}.
\end{aligned}$$

   □

2. Let $\vec{F}(x, y, z) = (z^3 + 2xy, x^2, 3xz^2)$ be a vector field in $\mathbb{R}^3$. Compute the line integral $\int_\Gamma \vec{F} \cdot d\vec{s}$, where Γ is the square with vertices $(\pm 1, \pm 1, 0)$ oriented in the counterclockwise direction.

   Solution. Let us start at the point $(1, -1, 0)$ and continue in the counterclockwise direction. A parametrization of Γ is given by

$$\vec{\alpha_1}(t) : \begin{cases} x(t) = 1 \\ y(t) = -1 + 2t \\ z(t) = 0 \end{cases} \quad \vec{\alpha_2}(t) : \begin{cases} x(t) = 1 - 2t \\ y(t) = 1 \\ z(t) = 0 \end{cases}$$

$$\vec{\alpha_3}(t) : \begin{cases} x(t) = -1 \\ y(t) = 1 - 2t \\ z(t) = 0 \end{cases} \quad \vec{\alpha_4}(t) : \begin{cases} x(t) = -1 + 2t \\ y(t) = -1 \\ z(t) = 0 \end{cases},$$

   where $0 \leq t \leq 1$ and $\vec{\alpha_1}$ corresponds to the segment joining $(1, -1, 0)$ and $(1, 1, 0)$, $\vec{\alpha_1}$ corresponds to the segment joining $(1, 1, 0)$ and $(-1, 1, 0)$ and so on. Now, we have

$$\int_\Gamma \vec{F} \cdot d\vec{s} = \int_0^1 \vec{F}(\vec{\alpha_1}(t)) \cdot \vec{\alpha_1}'(t) dt + \int_0^1 \vec{F}(\vec{\alpha_2}(t)) \cdot \vec{\alpha_2}'(t) dt + \int_0^1 \vec{F}(\vec{\alpha_3}(t)) \cdot \vec{\alpha_3}'(t) dt + \int_0^1 \vec{F}(\vec{\alpha_4}(t)) \cdot \vec{\alpha_4}'(t) dt$$

After the corresponding substitutions and simplifications, we obtain

$$\int_{\Gamma} \vec{F} \cdot d\vec{s} = \int_0^1 2\, dt + \int_0^1 -4(1-2t)\, dt + \int_0^1 -2\, dt + \int_0^1 -4(-1+2t)\, dt = 0.$$

□

3. Let $\mathcal{D}$ be the region bounded by the $x = 1$, $y = 0$ and $y = x$. Is the integral $\displaystyle\iint_{\mathcal{D}} xy\mathrm{e}^{-(x^2+y^2)}\, dx\, dy$ convergent?

   Solution. There are three possible regions **delimited** by $x = 1$, $y = 0$ and $y = x$, but only one is actually **bounded** (in the sense that it can be put inside a rectangle). The other two are **unbounded**. On the other hand, the question asks for convergence. Usually, we ask for convergence when the integral we deal with is **improper**. There is no problem with the integrand $xy\mathrm{e}^{-(x^2+y^2)}$; it is continuous everywhere on the plane, which leads us to think that for the integral to be improper, the domain has to be unbounded. When I posed this question, the domain I had in mind was the unbounded region delimited by the equations above with $x \to \infty$. However, if you worked over the bounded region, it is all right too. (In any case, an integral is convergent if the result is a number and divergent otherwise.) If the domain is bounded, the region is a triangle. **Without doing any computation** we can immediately conclude that the integral is convergent since the domain is **compact** (**closed and bounded**) and the integrand is **continuous** over the domain. You can come to this conclusion by using, for instance, the mean value inequality. But we can **go further** and actually do the computation. After all, it is not a hard task since the domain is a triangle and the integrand can be expressed as a product of two functions, one depending on x and the other one on y. Check that the answer in this case is $\frac{1}{8} - \frac{1}{4}\mathrm{e}^{-1} + \frac{1}{8}\mathrm{e}^{-2}$. For the unbounded domain I had in mind, here is the computation,

$$\begin{aligned}
\iint_{\mathcal{D}} xy\mathrm{e}^{-(x^2+y^2)}\, dx\, dy &= \lim_{b\to\infty} \int_1^b \int_0^x xy\mathrm{e}^{-(x^2+y^2)}\, dy dx \\
&= \lim_{b\to\infty} \int_1^b x\mathrm{e}^{-x^2} \left[-\frac{1}{2}\mathrm{e}^{-y^2}\Big|_0^x \right] dx \\
&= \lim_{b\to\infty} \left(\frac{1}{2}\int_1^b x\mathrm{e}^{-x^2}\, dx - \frac{1}{2}\int_1^b x\mathrm{e}^{-2x^2}\, dx \right) \\
&= \lim_{b\to\infty} \left[\frac{1}{4}(\mathrm{e}^{-1} - \mathrm{e}^{-b^2}) + \frac{1}{8}(\mathrm{e}^{-2b^2} - \mathrm{e}^{-2}) \right] \\
&= \frac{1}{4}\mathrm{e}^{-1} - \frac{1}{8}\mathrm{e}^{-2}.
\end{aligned}$$

   This is a number, therefore the integral $\displaystyle\iint_{\mathcal{D}} xy\mathrm{e}^{-(x^2+y^2)}\, dx\, dy$ is convergent. □

4. Suppose you have a rectangular region. How does a generic linear map transforms this region?

   Solution. To determine how a generic linear map acts on a region, it is enough to know how it transforms its boundary. **A generic linear map takes lines into lines** (prove this). Therefore, a generic linear map transforms a rectangular region into a parallelogram. □

5. Evaluate $\displaystyle\iiint_U x^2\, dx\, dy\, dz$, where U is the solid bounded by $y^2 + z^2 = 4ax$, $y^2 = ax$ and $x = 3a$. Assume that a is a positive constant. *Hint*: There are two possible solids, one is convex and the other one is not. We analyzed the non-convex solid in class. Work out the convex case.

Solution. We have

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid -\sqrt{4ax - y^2} \leq z \leq \sqrt{4ax - y^2},\, -\sqrt{ax} \leq y \leq \sqrt{ax},\, 0 \leq x \leq 3a\}$$

Then

$$\begin{aligned} \iiint_U x^2\, dx\, dy\, dz &= \int_0^{3a} \int_{-\sqrt{ax}}^{\sqrt{ax}} \int_{-\sqrt{4ax-y^2}}^{\sqrt{4ax-y^2}} x^2\, dz\, dy\, dx \\ &= \int_0^{3a} \int_{-\sqrt{ax}}^{\sqrt{ax}} 2x^2 \sqrt{4ax - y^2}\, dy\, dx \\ &= \int_0^{3a} \frac{6\sqrt{3}a + 4\pi a}{3}\, x^3\, dx = 27a^5 \left(\frac{3\sqrt{3} + 2\pi}{2} \right). \end{aligned}$$

$\square$