Math 23B, Multivariable Calculus II
Summer 2004, José Agapito

Final

Name:	**Solution**

All 5 questions and the bonus problem must be answered on this exam using the backs of the sheets if necessary.
Show all your work.

Do your best!! ◡

Presentation	2	/2
1	8	/8
2	7	/7
3	8	/8
4	7	/7
5	8	/8
Bonus	5	/5
Total	40 + 5	/40+5

1. (8 points) For each of the questions below, indicate if the statement is *true* or *false*. Briefly justify your answer. Each correct answer is worth 1 point.

a	F
b	T
c	T
d	F

e	T
f	F
g	F
h	T

(a) The flux of a tangent vector field across the sphere $x^2 + y^2 + z^2 = a^2$ is positive.

False. A vector field $\vec{F}$ tangent to the sphere $S : x^2 + y^2 + z^2 = a^2$ is perpendicular to the normal vector field $\vec{n}$ representing the orientation of the sphere; namely, $\vec{F} \cdot \vec{n} = 0$ which implies $\int\int_S \vec{F} \cdot d\vec{S} = \int\int_S \vec{F} \cdot \vec{n}\, dS = 0$.

(b) The divergence of a vector field at a point is a number that can be interpreted as the outflow of the vector field per unit volume at that point.

True. By definition, the divergence of a vector field $\vec{F}$ at a point p is

$$\text{div}\, \vec{F}(p) = \lim_{vol \to 0} \frac{\text{flux of } \vec{F} \text{ across } S}{\text{volume of } S},$$

where S is a sphere centered at p contained in the domain of $\vec{F}$.

(c) Let $\vec{a} = (a_1, a_2, a_3)$ be a constant vector and $\vec{F} = \vec{a} \times \vec{r}$, where $\vec{r}$ is the usual position vector (x, y, z). Then $\vec{F}$ is conservative.

True. The vector field $\vec{F} = \vec{a} \times \vec{r} = (a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x)$ has curl $\nabla \times \vec{F} = (a_1 + a_1, a_2 + a_2, a_3 + a_3) = 2(a_1, a_2, a_3)$. Therefore, $\vec{F}$ is not conservative.

(d) There is a vector field $\vec{F}$ such that $\operatorname{curl}\vec{F} = (x^2, z, y)$.

False. We have $\operatorname{div}(\operatorname{curl}\vec{F}) = \operatorname{div}(x^2, z, y) = 2x \neq 0$. Recall that the divergence of the curl of any vector field is always zero.

(e) The Jacobian of the transformation $x = \frac{\rho}{bc}\sin\varphi\cos\theta$, $y = \frac{\rho}{ac}\sin\varphi\sin\theta$, $z = \frac{\rho}{ab}\cos\varphi$ is $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = -\frac{\rho^2}{a^2b^2c^2}\sin\varphi$.

True. We have

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \frac{\sin\varphi\cos\theta}{bc} & -\frac{\rho}{bc}\sin\varphi\sin\theta & \frac{\rho}{bc}\cos\varphi\cos\theta \\ \frac{\sin\varphi\sin\theta}{ac} & \frac{\rho}{ac}\sin\varphi\cos\theta & \frac{\rho}{ac}\cos\varphi\sin\theta \\ \frac{\cos\varphi}{ab} & 0 & -\frac{\rho}{ab}\sin\varphi \end{vmatrix} = -\frac{\rho^2}{a^2b^2c^2}\sin\varphi$$

(f) Let D be a y-simple (type I) region in $\mathbb{R}^2$; namely, $D = \{(x,y)\,|\,a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x)\}$. Then $\int\int_D f(x)g(y)\,dx\,dy = \left(\int_a^b f(x)\,dx\right)\left(\int_{\phi_1(a)}^{\phi_2(b)} g(y)\,dy\right)$

False. See Lecture notes from Week #2, Wednesday.

(g) Let D be a simple region in $\mathbb{R}^2$ and let $F = (P,Q)$ be an arbitrary vector field of class C^1 in $\mathbb{R}^2$. Then the area of D is equal to $\frac{1}{2}\oint_{\partial D} Pdy - Qdx$.

False. Only when $\vec{F} = (P,Q) = (x,y)$ we have $\text{Area}(D) = \frac{1}{2}\oint_{\partial D} Pdy - Qdx$.

(h) The area of the ellipse $x^2 + y^2/4 = 1$ is given by the iterated integral

$$\int_{-1}^{1}\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} dy\,dx.$$

True. Since $y^2 = 4(1-x^2)$, we conclude that the integration above is the right set-up to get the area of the ellipse.

2. (7 points) Prove that the surface area of a sphere of radius R centered at $(0,0,0)$ is $4\pi R^2$. Show in detail your computations.

See question 1 from Quiz #3.

3. (8 points) Evaluate $\int\int_S \vec{F}\cdot d\vec{S}$, where $F(x,y,z) = \vec{i} + \vec{j} + z(x^2+y^2)\vec{k}$ and S is the surface of the cylinder $x^2+y^2 \le 1$, $0 \le z \le 1$ including the top and the bottom.

I will show two ways of doing the computation.

First way. The vector field $\vec{F} = (1, 1, z(x^2+y^2))$ is C^1 everywhere in $\mathbb{R}^3$. We can use Gauss' theorem without any problem. Let B denote the solid cylinder whose boundary ∂B is S. We have

$$\int\int_S \vec{F}\cdot d\vec{S} = \int\int\int_B \operatorname{div}\vec{F}\,dV = \int\int\int_B (x^2+y^2)\,dV = \int\int_D\int_0^1 (x^2+y^2)\,dz\,dx\,dy,$$

where $D = \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 \leq 1\}$. Then

$$\int\int_S \vec{F}\cdot d\vec{S} = \int\int_D (x^2+y^2)\,dx\,dy.$$

It is convenient to use polar coordinates at this point. We have

$$\int\int_D (x^2+y^2)\,dx\,dy = \int_0^{2\pi}\int_0^1 r^2 \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = \int_0^{2\pi}\int_0^1 r^2\,r\,dr\,d\theta = \frac{\pi}{2}.$$

Second way. We also have

$$\int\int_S \vec{F}\cdot d\vec{S} = \int\int_{side} \vec{F}\cdot d\vec{S} + \int\int_{top} \vec{F}\cdot d\vec{S} + \int\int_{bottom} \vec{F}\cdot d\vec{S},$$

where *side*, *top* and *bottom* are the three obvious parts of S. Now, providing S with the outward orientation, the lateral side of the cylinder has normal vector $\vec{n}_1 = (x,y,0)$, the top part has normal vector $\vec{n}_2 = (0,0,1)$ and the bottom part normal vector $\vec{n}_3 = (0,0,-1)$. Thus,

$$\begin{aligned}
\int\int_S \vec{F}\cdot d\vec{S} &= \int\int_{side} \vec{F}\cdot d\vec{S} + \int\int_{top} \vec{F}\cdot d\vec{S} + \int\int_{bottom} \vec{F}\cdot d\vec{S} \\
&= \int\int_{side} \vec{F}\cdot \vec{n}_1\,dS + \int\int_{top} \vec{F}\cdot \vec{n}_2\,dS + \int\int_{bottom} \vec{F}\cdot \vec{n}_3\,dS \\
&= \int\int_{side} (x+y)\,dS + \int\int_{top} 1(x^2+y^2)\,dS + \int\int_{bottom} -0(x^2+y^2)\,dS \\
&= \int_0^{2\pi}\int_0^1 \cos\theta + \sin\theta)\,dz\,d\theta + \int_0^{2\pi}\int_0^1 r^2\,r\,dr\,d\theta + 0 \\
&= 0 + \frac{\pi}{2} = \frac{\pi}{2}\,.
\end{aligned}$$

4. (a) (3 points) Let $f(x,y,z) = 3xye^{z^2}$ and let $\vec{c}(t) = (3\cos^3 t, \sin^2 t, e^t)$, with $0 \leq t \leq \pi$. Evaluate $\int_c \nabla f \cdot d\vec{s}$.

   By the fundamental theorem of line integrals,

$$\int_c \nabla f \cdot d\vec{s} = f(\vec{c}(\pi)) - f(\vec{c}(0)) = f(-3,0,e^{\pi}) - f(3,0,1) = 0 - 0 = 0.$$

   (b) (4 points) Evaluate $\int\int_D \sin(x^2+y^2)\,dx\,dy$, where D is the region in the first quadrant lying between the arcs of the circles $x^2+y^2 = \frac{\pi}{2}$ and $x^2+y^2 = \pi$.

   It is convenient to use polar coordinates, with $\sqrt{\pi/2} \leq r \leq \sqrt{\pi}$ and $0 \leq \theta \leq \pi/2$. We have

$$\int\int_D \sin(x^2+y^2)\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \sin(r^2)\,r\,dr\,d\theta = \frac{\pi}{2}\left(-\frac{\cos(r^2)}{2}\right)\Bigg|_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} = \frac{\pi}{4}.$$

5. (8 points) Is there a function $h(x, y, z)$ such that the vector field $\vec{F} = (3x^2y^2z + z, 2x^3yz, h(x, y, z))$ is conservative? If so, then find one such h.

If $\vec{F}$ is conservative, then $\vec{F} = \nabla f$ for some function f. Therefore,

$$\frac{\partial f}{\partial x} = 3x^2y^2z + z, \tag{1}$$

$$\frac{\partial f}{\partial y} = 2x^3yz, \tag{2}$$

$$\frac{\partial f}{\partial z} = h(x, y, z). \tag{3}$$

We integrate (1) with respect to x and get

$$f(x, y, z) = x^3y^2z + xz + g(y, z), \tag{4}$$

for some function g which does not depend on x. Now, we differentiate (4) with respect to y and compare with (2). We have

$$\frac{\partial f}{\partial y} = 2x^3yz + \frac{\partial g}{\partial y}(y, z) = 2x^3yz.$$

Then $\frac{\partial g}{\partial y}(y, z) = 0$, which implies that g does not depend on y either. Thus, $g = g(z)$ and

$$f(x, y, z) = x^3y^2z + xz + g(z). \tag{5}$$

Finally, we differentiate (5) with respect to z and compare with (3). We have

$$\frac{\partial f}{\partial z}(x, y, z) = x^3y^2 + x + \frac{d}{dz}g(z) = h(x, y, z).$$

Therefore, we can choose $g(z) \equiv 0$ (or any constant) and set $h(x, y, z) = x^3y^2 + x$. This function h makes $\vec{F}$ a conservative vector field. (You can also check that $\nabla \times \vec{F} = \vec{0}$.)

6. (Bonus 5 points) Let S be the surface given by $z = \mathrm{e}^{-(x^2+y^2)}$ and $z \geq \mathrm{e}^{-1}$. Let $\vec{F} = (\mathrm{e}^{y+z} - 2y)\vec{i} + (x\mathrm{e}^{y+z} + y)\vec{j} + \mathrm{e}^{x+y}\vec{k}$. Compute the circulation $\oint_{\partial S} \vec{F} \cdot d\vec{s}$ of $\vec{F}$ around ∂S.

In this problem we need to use Stokes' theorem and work with the flat disk $x^2 + y^2 \leq 1, z = e^{-1}$, instead of S. The answer is $\oint_{\partial S} \vec{F} \cdot d\vec{s} = 2\pi$. See Lecture Notes from Week #4, Wednesday.