Review Final

Sections from the textbook to be studied for the final: 5.1, 5.2, 5.3, 5.4, 5.5, 6.1, 6.2, 6.3, 6.4, 7.1, 7.2, 7.3, 7.4, 7.5, 7.6, 8.1, 8.2, 8.3 and 8.4. Review also your class notes, homework problems, quizzes, the review midterm and the midterm. In addition, I have prepared the following material to help you grasp vector analysis better. **Enjoy!**

1. If $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$, evaluate $\int_S \vec{F} \cdot d\vec{A}$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ ($a > 0$ constant.)

2. Evaluate $\int_S \vec{F} \cdot d\vec{A}$ where $\vec{F} = \frac{b}{a}x\vec{i} + \frac{a}{b}y\vec{j}$ and S is the surface of the elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $|z| \leq c$.

3. Compute the flux of the radius vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ through the lateral surface of the circular cylinder $x^2 + y^2 = 1$ bounded below by the plane $x + y + z = 1$ and from above by the plane $x + y + z = 2$.

4. Find the area of the ellipse cut on the plane $2x + y + z = 2$ by the circular cylinder $x^2 + y^2 = 2x$.

5. Evaluate $\int_S \vec{F} \cdot d\vec{A}$, where $\vec{F} = x^2\vec{i} + y^2 j + z^2\vec{k}$ and S is the triangle whose vertices are $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

6. valuate $\int_S \vec{F} \cdot d\vec{A}$, where $\vec{F} = x^2\vec{i} + y^2 j + z^2\vec{k}$ and S is the sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = d^2$.

7. Evaluate $\int_S \vec{F} \cdot d\vec{A}$, where $\vec{F} = x^3\vec{i} + y^3 j + z^3\vec{k}$ and S is the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

8. Calculate the flux of $F(x, y, z) = 3xy^2\vec{i} + xe^z\vec{j} + z^3\vec{k}$ across the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.

9. Evaluate $\int\int_S \vec{F} \cdot d\vec{S}$ where $F(x, y, z) = (z^2 x, \frac{1}{3}y^3 + \tan z, x^2 z + y^2)$ and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$. [*Hint*: Note that S is not a closed surface. First compute integrals over S_1 and S_2, where S_1 is the disk $x^2 + y^2 \leq 1$ provided with the outward orientation, and $S_2 = S \cup S_1$.]

10. Use the divergence theorem to evaluate
$$\int\int_S (2x + 2y + z^2)\, dS$$
where S is the sphere $x^2 + y^2 + z^2 = 1$. [Note that we are integrating a scalar-valued function.]

11. Determine whether the statement is true or false. Justify your answer.
    i) If $\vec{F}$ is a vector field, then $\operatorname{div}\vec{F}$ is a vector field.
    ii) If $\vec{F}$ is a vector field, then $\operatorname{curl}\vec{F}$ is a vector field.
    iii) If f is a scalar field with continuous partial derivatives of all orders on $\mathbb{R}^3$, then $\operatorname{div}(\operatorname{curl}\nabla f) = 0$.
    iv) If $\vec{F} = P\vec{i} + Q\vec{j}$ and $P_y = Q_x$ in an open region D, then $\vec{F}$ is conservative.
    v) There is a vector field F such that $\operatorname{curl}\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

12. Show that $\nabla(\vec{F}\cdot\vec{G}) = (\operatorname{div}\vec{F})\,\vec{G} + \vec{F}\,(\operatorname{div}\vec{G})$, where $\vec{F} = (F_1, F_2, F_3)$ and $\vec{G} = (G_1, G_2, G_3)$ are vector fields in $\mathbb{R}^3$.

13. Show that if $g(x, y, z)$ is a scalar valued function and $\vec{F}(x, y, z)$ is a vector field, then

$$\operatorname{div}(g\vec{F}) = (\nabla g)\,\vec{F} + g\,(\operatorname{div}\vec{F})$$

14. Show that $\operatorname{curl}(f\vec{G}) = (\nabla f) \times \vec{G} + f\,(\operatorname{curl}\vec{G})$, where f is a function of three variables and $\vec{G} = (G_1, G_2, G_3)$ is a vector field in $\mathbb{R}^3$.

15. True or false? Explain your answer. If $\int\int_S \vec{F}\cdot d\vec{A} = 12$ and S is a flat disc of area 4π, then $\operatorname{div}\vec{F} = 3/\pi$.

16. If V is a volume surrounded by a closed surface S, show that

$$\frac{1}{3}\int\int_S \vec{r}\cdot d\vec{A} = V.$$

17. Use the result of Problem 16 to compute the volume of a sphere, given that the surface area of a sphere or radius R is $4\pi R^2$.

18. Use the result of Problem 16 to compute the volume of a cone of base radius b and height h. [*Hint*: Stand the cone with its point downward and its axis along the positive z-axis.]

19. Can you use Stokes' theorem to compute the line integral $\int_C (2x\vec{i} + 2y\vec{j} + 2z\vec{k})\cdot d\vec{s}$ where C is the straight line from the point $(1, 2, 3)$ to the point $(4, 5, 6)$? Why or why not?

20. Is there a vector field G such that $\operatorname{curl}\vec{G} = y\vec{i} + x\vec{j}$? How do you know?

21. Let $\vec{F} = (0, -z, y)$ be a vector field in $\mathbb{R}^3$ and let C be the circle of radius R in the YZ-plane oriented clockwise as viewed from the positive x-axis, and S is the disk in the YZ-plane enclosed by C, oriented in the positive x-direction. See Figure 1 below.

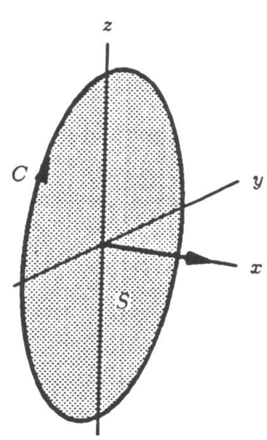

Figure 1: circle oriented clockwise

(a) Evaluate directly $\int_C \vec{F} \cdot d\vec{s}$.

(b) Evaluate directly $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{A}$.

(c) The answers in parts (a) and (b) are not equal. Explain why this does not contradict Stokes' theorem.

22. What is a vector field? Give examples.

23. What is a conservative vector field?

24. What is a scalar potential?

25. What is a vector potential?

26. State the fundamental theorem for line integrals.

27. State Green's theorem

28. Write expressions for the area enclosed by a curve C in terms of line integrals around C.

29. Suppose $\vec{F}$ is a vector field on $\mathbb{R}^3$.

    (a) Define $\operatorname{curl} \vec{F}$.

    (b) Define $\operatorname{div} \vec{F}$.

    (c) If $\vec{F}$ is a velocity field in fluid flow, what are the physical interpretations of $\operatorname{curl} \vec{F}$ and $\operatorname{div} \vec{F}$.

30. Of the six planar vector fields shown below, four have zero divergence in the regions indicated and three have zero curl. Can you categorize each vector field?

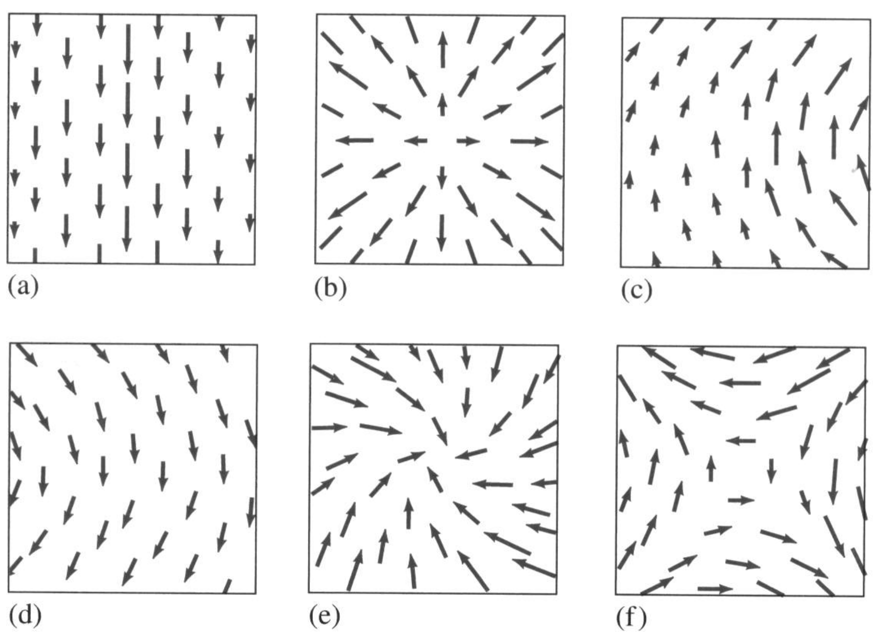

Figure 2: planar vector fields

31. (a) What is an oriented surface?
   (b) Define the surface integral (or flux) of a vector field $\vec{F}$ over an oriented surface S with unit normal vector $\vec{n}$.
   (c) How do you evaluate such an integral if S is a parametric surface given by a vector function $\varphi(u, v)$?
   (d) What if S is given by an equation $z = g(x, y)$?

32. State Stokes' theorem.

33. State the Divergence theorem.

34. Imagine the following vector fields $\vec{F} = F_1(x, y)\vec{\imath} + F_2(x, y)\vec{\jmath}$ and $\vec{G} = G_1(x, y)\vec{\imath} + G_2(x, y)\vec{\jmath}$ sketched in Figure 3 below

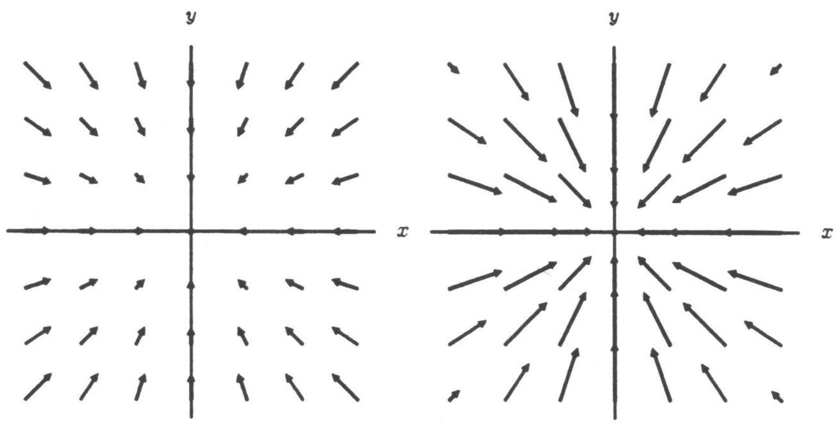

Figure 3: Cross-section of $\vec{F}$ on the left and of $\vec{G}$ on the right

   (a) What can you say about $\operatorname{div} \vec{F}$ and $\operatorname{div} \vec{G}$ at the origin?
   (b) What can you say about $\operatorname{curl} \vec{F}$ and $\operatorname{curl} \vec{G}$ at the origin?
   (c) Can you draw a closed surface around the origin such that $\vec{F}$ has a non-zero flux through it?
   (d) Repeat part (c) for $\vec{G}$.
   (e) Can you draw a closed curve around the origin such that $\vec{F}$ has a non-zero circulation around it?
   (f) Repeat part (e) for $\vec{G}$.

35. A vector field $\vec{F}$ is defined everywhere except on the z-axis, and $\operatorname{curl} \vec{F} = \vec{0}$ where $\vec{F}$ is defined. What can you say about $\oint_C \vec{F} \cdot d\vec{s}$ if C is a circle of radius 1 in the XY-plane, and if the center of C is at (a) the origin, (b) the point $(2, 0)$?

36. The next three problems concern the vector fields in Figure 4
   (i) Three of the vector fields have zero curl. Which are they? How do you know?
   (ii) Three of the vector fields have zero divergence. Which are they? How do you know?
   (iii) Four of the line integrals $\int_{C_i} \vec{F} \cdot d\vec{s}$ are zero. Which are they? How do you know?

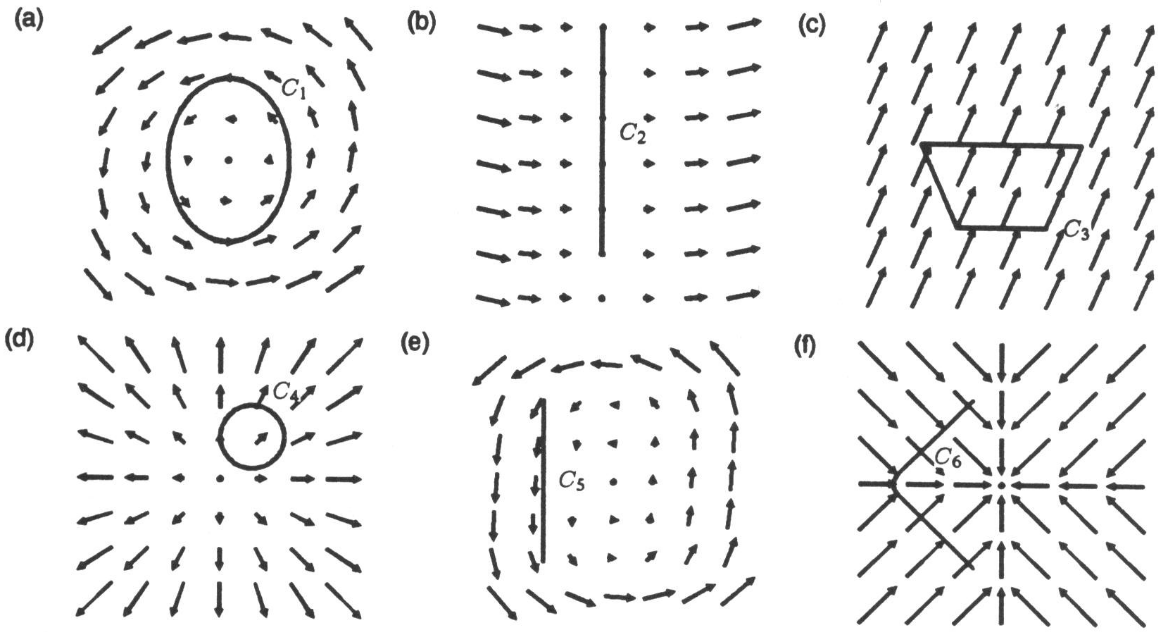

Figure 4: vector fields