Quiz # 3

1. Use a surface integral to find the surface area of a sphere of radius a centered at $(0, 0, 0)$.

   Solution. We use spherical coordinates to parameterize the sphere,

$$S\colon \begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases},$$

   where $0 \le \varphi \le \pi$ and $0 \le \theta \le 2\pi$. Notice that the **radius a is fixed**.

   We define $\vec{\Phi}(\varphi, \theta) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$, where $\mathcal{D} = \{(\varphi, \theta)\,|\,0 \le \varphi \le \pi\,,\, 0 \le \theta \le 2\pi\}$. We want to compute

$$A(S) = \iint_S dS = \iint_{\mathcal{D}} \|\vec{T}_\varphi \times \vec{T}_\theta\|\, d\varphi\, d\theta.$$

   We have

$$\begin{aligned} \vec{T}_\varphi &= (a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi) \\ \vec{T}_\theta &= (-a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0) \end{aligned},$$

   then

$$\vec{T}_\varphi \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix} = (a^2\sin^2\varphi\cos\theta, -a^2\sin^2\varphi\sin\theta, a^2\sin\varphi\cos\varphi).$$

   Thus,

$$\|\vec{T}_\varphi \times \vec{T}_\theta\| = a^2|\sin\varphi| = a^2\sin\varphi,$$

   since $0 \le \varphi \le \pi$. Hence,

$$A(S) = \int_0^{2\pi}\int_0^{\pi} a^2\sin\varphi\, d\varphi\, d\theta = 4\pi a^2.$$

2. Find $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F}(x, y, z) = (0, y, -z)$ and S is the surface bounded by the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ and the disk $x^2 + z^2 \le 1$, $y = 1$.

   Solution. Let S_1 be the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ and S_2 the disk $x^2 + z^2 \le 1$, $y = 1$. Since $S = S_1 \cup S_2$ is a closed surface, we use the outward orientation. On S_1: $\vec{F}(\vec{r}(x, z)) = (0, x^2 + z^2, -z)$, where $\vec{r}$ is a parametrization of S_1. We have $\vec{r}_x \times \vec{r}_z = (2x, -1, 2z)$. Then

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{x^2+z^2\le 1} [-(x^2 + z^2) - 2z^2]\, dA = -\int_0^{2\pi}\int_0^1 (r^2 + 2r^2\cos^2\theta) r\, dr\, d\theta = -\pi.$$

   On S_2: $\vec{F}(\vec{r}(x, z)) = (0, 1, -z)$ and $\vec{r}_x \times \vec{r}_z = (0, 1, 0)$. Then

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_{x^2+z^2\le 1} 1\, dA = \pi.$$

   Hence,

$$\iint_S \vec{F}\cdot d\vec{S} = -\pi + \pi = 0.$$

   Question: Is F a curl field?