Solution homework # 4

§ 8.1.

3) Verify Green's theorem for the disk D with center $(0,0)$ and radius R and the functions

(a) $P(x,y) = xy^2$, $Q(x,y) = -yx^2$

$$\left.\begin{aligned} Q_x &= -2xy \\ P_y &= 2xy \end{aligned}\right\rangle \quad \iint_D (Q_x - P_y)\,dx\,dy = -\iint_D 4xy\,dx\,dy$$

polar coordinates

$x = r\cos\theta \qquad \theta \in [0, 2\pi]$

$y = r\sin\theta \qquad r \in [0, R]$

circle with counter-clockwise orientation

Jacobian $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$

extra remark: notice that $\vec{F} = (P,Q)$ is not a gradient vector field Why?

$$\iint_D (Q_x - P_y)\,dx\,dy = -\int_0^{2\pi}\int_0^R (4r^2\cos\theta\sin\theta)\, r\,dr\,d\theta$$

$$= -\left(\int_0^R r^3\,dr\right)\left(2\int_0^{2\pi} \sin 2\theta\,d\theta\right)$$

$2\sin\theta\cos\theta = \sin 2\theta$

$$= -\left(\frac{r^4}{4}\Big|_0^R\right)\left(\frac{2}{2}(-\cos 2\theta)\Big|_0^{2\pi}\right)$$

$$= +\frac{R^4}{4}\underbrace{(\cos 4\pi - \cos 0)}_{0} = 0$$

On the other hand,

$$\oint_{\partial D^+} P\,dx + Q\,dy = \int_0^{2\pi} \Big[(R\cos\theta)(R\sin\theta)^2(-R\sin\theta) + $$

$$- (R\sin\theta)(R\cos\theta)^2(R\cos\theta)\Big]\,d\theta$$

$$= -\int_0^{2\pi} R^4 \sin\theta\cos\theta\,\underbrace{(\sin^2\theta + \cos^2\theta)}_{1}\,d\theta = -\frac{R^4}{2}\int_0^{2\pi} \sin 2\theta\,d\theta$$

$$= +\frac{R^4}{2}\,\frac{1}{2}\cos 2\theta\Big|_0^{2\pi} = \frac{R^4}{4}\underbrace{(\cos 4\pi - \cos 0)}_{0} = 0$$

Solution homework #4

(b) $P(x,y) = x+y$, $Q(x,y) = y$

$\left.\begin{matrix} Q_x = 0 \\ P_y = 1 \end{matrix}\right\rangle \iint_D (Q_x - P_y)\,dx\,dy = -\iint_D dx\,dy$

$= -\text{area}(D) = -\pi(R)^2$.

On the other hand, (polar coord. parametrization)

$$\oint_{\partial D^+} P\,dx + Q\,dy = \int_0^{2\pi} [(R\cos\theta + R\sin\theta)(-R\sin\theta) + R\sin\theta(R\cos\theta)]\,d\theta$$

$$= \int_0^{2\pi} R^2 [-\sin\theta\cos\theta - \sin^2\theta + \sin\theta\cos\theta]\,d\theta$$

$$= \int_0^{2\pi} R^2 \left(\frac{\cos 2\theta - 1}{2}\right) d\theta = \frac{R^2}{2}\left[\frac{\sin 2\theta}{2}\Big|_0^{2\pi} - \theta\Big|_0^{2\pi}\right]$$

$$= -\frac{R^2}{2} 2\pi = -\pi R^2. \quad \square$$

(c) $P(x,y) = xy = Q(x,y)$

$\left.\begin{matrix} Q_x = y \\ P_y = x \end{matrix}\right\rangle \iint_D (Q_x - P_y)\,dx\,dy = \iint_D (y-x)\,dx\,dy$

(polar coordinates again)

$$\int_0^{2\pi}\int_0^R (r\sin\theta - r\cos\theta)\, r\, dr\,d\theta$$

$$= \left(\int_0^R r^2 dr\right)\left(\int_0^{2\pi} (\sin\theta - \cos\theta)\,d\theta\right) = \left(\frac{r^3}{3}\Big|_0^R\right)\left(-\cos\theta - \sin\theta\Big|_0^{2\pi}\right)$$

$$= -\frac{R^3}{3}(0) = 0$$

On the other hand, $\oint_{\partial D^+} P\,dx + Q\,dy = \int_0^{2\pi} [R^2\sin\theta\cos\theta(-R\sin\theta) + R^2\sin\theta\cos\theta(R\cos\theta)]\,d\theta$

$$= R^3 \int_0^{2\pi} \sin\theta\cos\theta(\cos\theta - \sin\theta)\,d\theta = -\frac{R^3}{3}\left(\cos^3\theta\Big|_0^{2\pi} + \sin^3\theta\Big|_0^{2\pi}\right) = 0$$

($\cos^3\theta\Big|_0^{2\pi} = 0$, $\sin^3\theta\Big|_0^{2\pi} = 0$) $\square$

(d) $P(x,y) = 2y$, $Q(x,y) = x$

$Q_x = 1$, $P_y = 2$ $\Rightarrow$ $\iint_D (Q_x - P_y)\,dx\,dy = -\iint_D dx\,dy$

$= -\pi R^2$.

On the other hand,

$$\oint_{\partial D^+} P\,dx + Q\,dy = \int_0^{2\pi} 2R\sin\theta(-R\sin\theta)\,d\theta + R\cos\theta(R\cos\theta)\,d\theta$$

$$= R^2\int_0^{2\pi} (-\sin^2\theta + \sin^2\theta + \cos^2\theta)\,d\theta$$

$$= R^2\left(\int_0^{2\pi} -\sin^2\theta\,d\theta + \int_0^{2\pi} \cos 2\theta\,d\theta\right)$$

$$= R^2\left(\int_0^{2\pi} \frac{\cos 2\theta - 1}{2}\,d\theta + \int_0^{2\pi} \cos 2\theta\,d\theta\right)$$

$$= R^2\left(\frac{3}{2}\int_0^{2\pi} \cos 2\theta\,d\theta - \frac{1}{2}\int_0^{2\pi} d\theta\right)$$

$$= R^2\left[\frac{3}{2}\left(\frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} - \frac{1}{2}2\pi\right] = -\pi R^2.$$

4) Using the divergence theorem, show that $\oint_{\partial D} \vec{F}\cdot\vec{n}\,ds = 0$ where $F(x,y) = (y,-x)$ and D is the unit disk.

By the divergence theorem, $\oint_{\partial D} \vec{F}\cdot\vec{n}\,ds = \iint_D (\operatorname{div}\vec{F})\,dA$ (theorem 4 page 527)

$= \iint_D 0\,dA = 0$. Notice that the hypotheses of theorem 4 are satisfied.

Solution homework #4

7) Evaluate the line integral

$$I = \oint_C (2x^3 - y^3)\,dx + (x^3 + y^3)\,dy$$

where C is the unit circle, and verify Green's theorem for this case.

$$\left.\begin{array}{l} P = 2x^3 - y^3 \\ Q = x^3 + y^3 \end{array}\right\rangle \begin{array}{l} Q_x = 3x^2 \\ P_y = -3y^2 \end{array} \Rightarrow \iint_D (Q_x - P_y)\,dx\,dy$$

D: $x^2 + y^2 \le 1$

$$= 3\iint_{x^2+y^2\le 1} (x^2 + y^2)\,dx\,dy$$

polar coord.

$$= 3\int_0^{2\pi}\int_0^1 r^2 \cdot r\,dr\,d\theta = \left(3\frac{r^4}{4}\Big|_0^1\right)(2\pi)$$

$$= \frac{3}{2}\pi \quad //$$

On the other hand,

$$I = \int_0^{2\pi} \left[(2\cos^3\theta - \sin^3\theta)(-\sin\theta) + (\cos^3\theta + \sin^3\theta)\cos\theta\right] d\theta$$

$$= \int_0^{2\pi} -2\sin\theta\cos^3\theta\,d\theta + \int_0^{2\pi} \sin^4\theta\,d\theta + \int_0^{2\pi} \cos^4\theta\,d\theta + \int_0^{2\pi} \sin^3\theta\cos\theta\,d\theta$$

use tables

$$= -2\left[\frac{\cos^2\theta\sin^2\theta}{4}\Big|_0^{2\pi} + \frac{2}{4}\int_0^{2\pi} \cos\theta\sin\theta\,d\theta\right] +$$

$$\left[-\frac{\sin^3\theta\cos\theta}{4}\Big|_0^{2\pi} + \frac{3}{4}\int_0^{2\pi} \sin^2\theta\,d\theta\right] + \left[\frac{\cos^3\theta\sin\theta}{4}\Big|_0^{2\pi} + \frac{3}{4}\int_0^{2\pi} \cos^2\theta\,d\theta\right]$$

$$\left[-\frac{\cos^2\theta\sin^2\theta}{4}\Big|_0^{2\pi} + \frac{2}{4}\int_0^{2\pi} \cos\theta\sin\theta\,d\theta\right]$$

Solution homework #4

$$= -2\left[\frac{1}{4}\int_0^{2\pi} \sin 2\theta \, d\theta\right] + \frac{3}{4}\int_0^{2\pi}(\sin^2\theta + \cos^2\theta)\,d\theta + \frac{1}{4}\int_0^{2\pi} \sin 2\theta \, d\theta$$

$$= +\frac{1}{2}\frac{1}{2}\cos 2\theta\Big|_0^{2\pi} + \frac{3}{4}(2\pi) + \frac{1}{4}\frac{(-\cos 2\theta)}{2}\Big|_0^{2\pi}$$

$$= \frac{3}{2}\pi$$

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15) Use theorem (2) to compute the area inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\text{Area}(\text{Ellipse}) = \frac{1}{2}\oint_{\partial D}(x\,dy - y\,dx) \quad \ldots \text{(Theorem 2)}$$

Parametrization of the ellipse

$$x = a\cos\theta \qquad \theta \in [0, 2\pi]$$
$$y = b\sin\theta$$

$$\text{Area} = \frac{1}{2}\int_0^{2\pi}[a\cos\theta(b\cos\theta) + b\sin\theta(a\sin\theta)]\,d\theta$$

$$= \frac{1}{2}\,ab\,2\pi = \pi ab$$

Solution homework #4

16) Use theorem (2) to recover the formula $A = \frac{1}{2}\int_a^b r^2 d\theta$ for a region in polar coordinates.

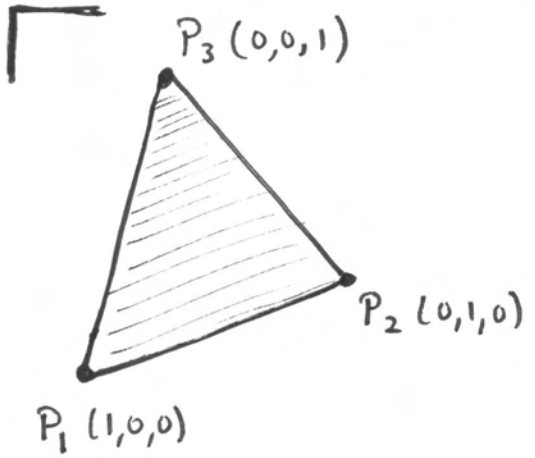
$$A = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx = \frac{1}{2}\int_a^b \left[\bar{r}(\theta)\cos\theta\, r(\theta)(\cos\theta + r^2(\theta)\sin^2\theta\right] d\theta$$

$$= \frac{1}{2}\int_a^b r^2(\theta)\,d\theta$$

$x = r(\theta)\cos\theta$
$y = r(\theta)\sin\theta$
$\theta \in [a,b]$

§ 8.2.

6) Let $\vec{C}$ consist of straight lines joining $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ and let S be the triangle with these vertices. Verify Stokes' theorem directly with $F = (yz, xz, xy)$



$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial_x & \partial_y & \partial_z \\ yz & xz & xy \end{vmatrix}$$

$$= (x - x, -y + y, z - z)$$

$$= \vec{0}$$

$$\therefore \iint \nabla \times \bar{F} \cdot d\bar{S} = \iint \vec{0} \cdot d\bar{S} = 0$$

On the other hand,

$$\oint_{\partial\triangle} \vec{F}\cdot d\vec{s} = \int_{\overrightarrow{P_1P_2}} \vec{F}\cdot d\vec{s} + \int_{\overrightarrow{P_2P_3}} \vec{F}\cdot d\vec{s} + \int_{\overrightarrow{P_3P_1}} \vec{F}\cdot d\vec{s}$$

Parametrizations of

$\overrightarrow{P_1P_2}$ $\quad \vec{\alpha}_1 :$

$$\begin{cases} x = (1-t)\,1 + t\,0 = 1-t \\ y = (1-t)\,0 + t\,1 = t \\ z = (1-t)\,0 + t\,0 = 0 \end{cases}$$

$\overrightarrow{P_2P_3}$ $\quad \vec{\alpha}_2 :$

$$\begin{cases} x = (1-t)\,0 + t\,0 = 0 \\ y = (1-t)\,1 + t\,0 = 1-t \\ z = (1-t)\,0 + t\,1 = t \end{cases}$$

$\overrightarrow{P_3P_1}$ $\quad \vec{\alpha}_3 :$

$$\begin{cases} x = (1-t)\,0 + t\,1 = t \\ y = (1-t)\,0 + t\,0 = 0 \\ z = (1-t)\,1 + t\,0 = 1-t \end{cases}$$

$0 \le t \le 1$

Then,

$$\oint_{\partial\triangle} \vec{F}\cdot d\vec{s} = \int_0^1 \underbrace{(t\cdot 0, (1-t)\,0, (1-t)\,t)\cdot(-1,1,0)}_{0}\, dt$$

$$+ \int_0^1 \underbrace{((1-t)\,t, 0\,t, 0\,(1-t))\cdot(0,-1,1)}_{0}\, dt$$

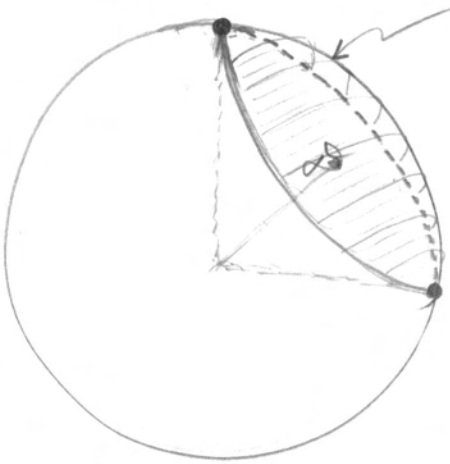
$$+ \int_0^1 \underbrace{(0\,(1-t), t\,(1-t), t\,0)\cdot(1,0,-1)}_{0}\, dt$$

$$= 0$$

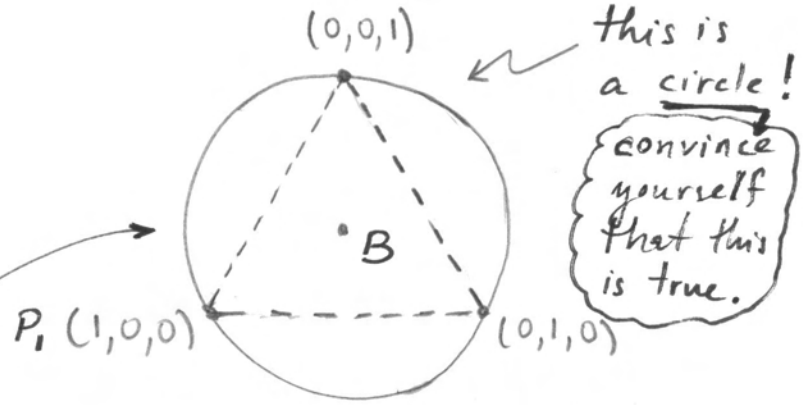
Solution homework #4

8) See problem (7). Basically, we need to evaluate $\iint_S (\nabla \times \bar{F}) \cdot d\bar{S}$, where S is the portion of the surface of a sphere defined by $x^2+y^2+z^2=1$ and $x+y+z \geq 1$, and where $\bar{F} = \bar{r} \times (1,1,1)$, $\bar{r} = (x,y,z)$.

Finally, I got the picture right.



Looking at S from here we see



The center of this circle is the baricenter of the equilateral triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. This is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ ← the average of the vertices

Then, we can compute the radius of that circle by simply calculating $d(B,P_1)$, the distance between B and P_1. This is $\frac{\sqrt{6}}{3}$.

Now, $\bar{F} = (x,y,z) \times (1,1,1) = (y-z, z-x, x-y)$ ← class C^∞ in $\mathbb{R}^3$

$\nabla \times \bar{F} = (-2,-2,-2)$

We can use Stokes'. The surface S and the flat slanted disk D are in the domain of $\bar{F}$ so, we have

$$\iint_S \nabla \times \bar{F} \cdot d\bar{S} \overset{\text{Stokes'}}{=} \oint_{\partial S} \vec{F} \cdot d\vec{s}$$

I tried to compute this surface integral directly and I had problems with finding the limits of integration for a certain parametrization

It is better to compute the line integral. Let's see ...

Notice that the vector $(1,1,1)$ is perpendicular to the plane $x+y+z=1$. The nice thing about this problem is that our computation does not depend on the surface but on its boundary. So, we take D instead of S with orientation compatible with that of S. We can do this because $\bar{F}$ is C^∞ in all of $\mathbb{R}^3$.

We have,

$$\iint_S \nabla \times \bar{F} \cdot d\bar{S} = \iint_D \nabla \times \bar{F} \cdot dS = \oint_{\partial S = \partial D} \vec{F} \cdot d\vec{s}$$

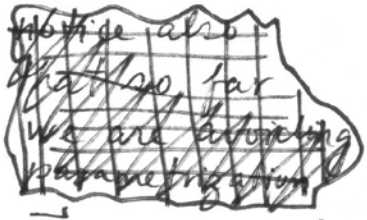
Solution homework # 4

Notice that to compute $\oint_{\partial D} \vec{F} \cdot d\vec{s}$

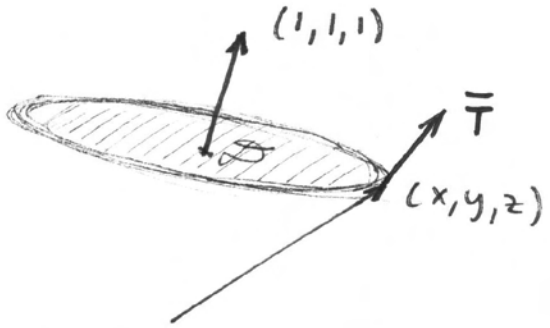
we can take a vector $\vec{T}$, tangent to ∂D, where $\vec{T}$ is of length 1. So,

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \oint_{\partial D} (\vec{F} \cdot \vec{T})\, ds$$

notice this is the arc length parameter



★ Now, to find $\vec{T}$, notice that $\vec{T}$ must be parallel to $(1,1,1) \times (x,y,z) = -(y-z, z-x, x-y)$



Then,

$$\vec{T} = -\frac{(y-z, z-x, x-y)}{\|(y-z, z-x, x-y)\|}$$

$$\|(y-z), (z-x), (x-y)\| = \sqrt{2\underbrace{(x^2+y^2+z^2)}_{1} - 2(xy+xz+yz)}$$

1 because $(x,y,z) \in$ sphere

Also, $\underbrace{(x+y+z)}_{1}^2 = \underbrace{x^2+y^2+z^2}_{1} + 2(xy+xz+yz)$

because $(x,y,z) \in$ plane $x+y+z=1$

Then $\boxed{xy+xz+yz=0}$

Then $\| (y-z, z-x, x-y) \| = \sqrt{2}$

So $\oint_{\partial D} (\bar{F} \cdot \bar{T})\, ds = \oint_{\partial D} \left(\bar{F} \cdot \frac{-\bar{F}}{\|\bar{F}\|} \right) ds = -\oint_{\partial D} \|\bar{F}\|\, ds$

$= -\sqrt{2} \oint_{\partial D} ds = -\sqrt{2} \underbrace{\left(2\pi \frac{\sqrt{6}}{3} \right)}_{\text{perimeter of } \partial D}$

$= -4\pi \frac{\sqrt{3}}{3}$ ∎

10) Find $\iint_S (\nabla \times \bar{F}) \cdot d\bar{S}$, where S is the ellipsoid $x^2 + y^2 + 2z^2 = 10$ and $\bar{F} = (\sin xy, e^x, -yz)$

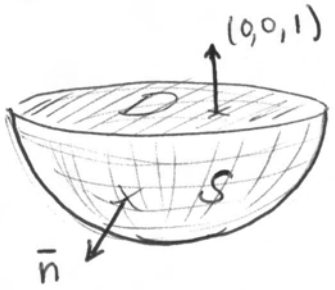
Since F is of class C^2 (actually C^∞) in $\mathbb{R}^3$ we can use Stokes',

$$\iint_S \nabla \times \bar{F} \cdot d\bar{S} = \oint_{\partial S} \bar{F} \cdot d\bar{s} = 0$$

$\partial S = \emptyset$ because the ellipsoid is closed.

Solution homework #4

11) $\bar{F} = (y, -x, zx^3y^2)$. Evaluate $\iint_S (\nabla \times \bar{F}) \cdot \vec{n}\, dA$, where S is the surface defined by $x^2 + y^2 + z^2 = 1$, $z \le 0$



By Stokes' (verify hypotheris)

$$\iint_S (\nabla \times \bar{F}) \cdot \vec{n}\, dA = \iint_D (\nabla \times \bar{F}) \cdot (0,0,1)\, dA$$

$$\iint_D (2zx^3y, -3zx^2y^2, -2) \cdot (0,0,1)\, dA = \iint_D -2\, dA$$

$$\nabla \times \bar{F} = (2zx^3y, -3zx^2y^2, -2)$$

$$= -2\pi(1)^2$$

$$= -2\pi \quad //$$

13) Prove that Faraday's law implies $\nabla \times \bar{E} = -\frac{\partial H}{\partial t}$.

Faraday's law: "the voltage around C (the boundary of a surface S in $\mathbb{R}^3$) equals the negative rate of change of magnetic flux through S."

$$\oint_C \vec{E} \cdot d\vec{s} = \text{voltage around } C$$

$$\iint_S \vec{H} \cdot d\vec{S} = \text{magnetic flux across } S.$$

$\bar{E}, \bar{H}$: time dependent electric and magnetic fields in $\mathbb{R}^3$.

Solution homework #4

$$\oint_C \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \iint_S \vec{H} \cdot d\vec{S} = \iint_S -\frac{\partial}{\partial t} \vec{H} \cdot d\vec{S}$$

// Stokes'

$$\iint_S (\nabla \times \vec{E}) \cdot d\vec{S} = \iint_S -\frac{\partial}{\partial t} \vec{H} \cdot d\vec{S}$$

$$\Rightarrow \iint_S \left(\nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{H}\right) \cdot \vec{n} \, dS = 0$$

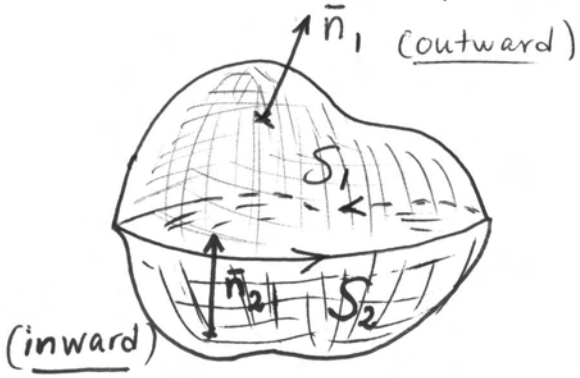
If $\left(\nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{H}\right) \cdot \vec{n} \geq 0$ and

$\nabla \times \vec{E}$ and $\frac{\partial}{\partial t} \vec{H}$ are parallel to $\vec{n}$ then

$$\nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{H} = 0 \quad \blacksquare$$

15) Consider two surfaces S_1, S_2 with the same boundary ∂S. Describe with sketches how S_1 and S_2 must be oriented to ensure that

$$\iint_{S_1} (\nabla \times \bar{F}) \cdot d\bar{S} = \iint_{S_2} (\nabla \times \bar{F}) \cdot d\bar{S}$$



Solution homework #4

18) If C is a closed curve that is the boundary of a surface S, and f and g are C^2 functions, show that

a) $\oint_C f \nabla g \cdot d\bar{s} = \iint_S (\nabla f \times \nabla g) \cdot d\vec{S}$

b) $\oint_C (f \nabla g + g \nabla f) \cdot d\bar{s} = 0$

(b) is easy because $f \nabla g + g \nabla f = \nabla(fg)$ (3)

so $\oint_C \nabla(fg) \cdot d\bar{s} = 0$.

(a) By Stokes,

$$\oint_C f \nabla g \cdot d\bar{s} = \iint_S (\nabla \times (f \nabla g)) \cdot d\vec{S}$$

$$\nabla \times (f \nabla g) = f \underbrace{\nabla \times \nabla g}_{0} + \nabla f \times \nabla g \quad (10)$$

see basic identities of vector analysis page 306

(a) follows from here.

§8.3

1) Show that any two potential functions for a vector field on $\mathbb{R}^3$ differ at most by a constant

Suppose $\bar{F} = \nabla f = \nabla g$ for some $f, g : \mathbb{R}^3 \to \mathbb{R}$.

Then $\nabla(f-g) = 0 \iff \frac{\partial}{\partial x}(f-g) = 0$, $\frac{\partial}{\partial y}(f-g) = 0$, $\frac{\partial}{\partial z}(f-g) = 0$ $\Rightarrow f - g = C$ (constant) since $\mathbb{R}^3$ is open and connected.

Solution homework #4

2) (a) Let $F(x,y) = (xy, y^2)$ and let $\vec{c}$ be the path $y = 2x^2$ joining $(0,0)$ to $(1,2)$ in $\mathbb{R}^2$. Evaluate $\int_C \vec{F} \cdot d\vec{s}$.

$$\int_C \vec{F} \cdot d\vec{s} = \int_0^1 F(x(t), y(t)) \cdot (x'(t), y'(t))\, dt$$

for some parametrization $(x(t), y(t))$ of C, $t \in [0,1]$

We can take $x(t) = t$, $y(t) = 2t^2$, $t \in [0,1]$

Then, $\int_0^1 (t \cdot 2t^2, (2t^2)^2) \cdot (1, 4t)\, dt$

$$= \int_0^1 (2t^3 + 16t^5)\, dt = \left(\frac{2t^4}{4} + \frac{16t^6}{6} \right) \Big|_0^1 = \frac{1}{2} + \frac{16}{6} = \frac{19}{6}$$

(b) Does the integral in part (a) depend on the path joining $(0,0)$ to $(1,2)$ in $\mathbb{R}^2$.

If the integral does not depend on the path, then using the path $(x(t), y(t)) = (t, 2t)$, $t \in [0,1]$ (whose image is entirely contained in the domain of $\vec{F}$) we must get the same answer. However, we have

$$\int_0^1 (t \cdot 2t, (2t)^2) \cdot (1,2)\, dt = \int_0^1 (2t^2 + 8t^2)\, dt = \frac{10}{3}$$

∴ The integral does depend on the path joining $(0,0)$ to $(1,2)$ in $\mathbb{R}^2$.

Solution homework #4

3) Let $\bar{F}(x,y,z) = (2xyz + \sin x, x^2z, x^2y)$.
Find a function f such that $\bar{F} = \nabla f$.

First, we want to make sure $\bar{F}$ is conservative.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz+\sin x & x^2z & x^2y \end{vmatrix} = (x^2 - x^2, -(2xy - 2xy), 2xz - 2xz)$$

$$= (0,0,0). \quad \text{Yes!}$$

Want to find $f(x,y,z)$ such that

$$\frac{\partial f}{\partial x} = 2xyz + \sin x \quad \cdots (1)$$

$$\frac{\partial f}{\partial y} = x^2z \quad \cdots (2)$$

$$\frac{\partial f}{\partial z} = x^2y \quad \cdots (3)$$

Integrate (1) with respect to x. We have

$$f(x,y,z) = x^2yz - \cos x + g(y,z) \quad \cdots (4)$$

Differentiate (4) with respect to y and compare with (2). We have

$$\frac{\partial f}{\partial y} = x^2z + \frac{\partial}{\partial y} g(y,z) = x^2z \Rightarrow \frac{\partial}{\partial y} g(y,z) = 0$$

This means, g depends only on z.

So $f(x,y,z) = x^2yz - \cos x + g(z) \quad \cdots (5)$

Differentiate (5) with respect to z and compare with (3). We have

$$\frac{\partial f}{\partial z} = x^2y + \frac{d}{dz} g(z) = x^2y \Rightarrow \frac{d}{dz} g(z) = 0 \Rightarrow g(z) = C \text{ (constant)}$$

$$\therefore \boxed{f(x,y,z) = x^2yz - \cos x + C}$$

Solution homework #4

7) Let $F(x,y,z) = (xy, y, z)$. Can there exist a function f such that $\bar{F} = \nabla f$?

If there existed such an f, then

$$\nabla \times \underbrace{(\nabla f)}_{\bar{F}} = \bar{0}.$$

However,

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy & y & z \end{vmatrix} = (0, 0, -x) \neq \vec{0}$$

12) See lecture notes.

20) Let $\vec{F} = (xe^y, -x\cos z, -ze^y)$. Find a G such that $\vec{F} = \nabla \times G$ (such a $\vec{G}$ is called vector potential)

First, we want to check that $\operatorname{div}(\vec{F}) = 0$

$$\operatorname{div}(\bar{F}) = \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial y}(-x\cos z) + \frac{\partial}{\partial z}(-ze^y)$$

$$= e^y + 0 - e^y = 0 \quad \checkmark$$

Then, there exists a C^1 vector field G with $\vec{F} = \nabla \times \vec{G}$. To find such a G, let's use exercise 16, §8.3, from the textbook. Define $\vec{G} = (G_1, G_2, G_3)$ where $G_3 = 0$ and

$$G_1 = \int_0^z F_2(x,y,t)\,dt - \int_0^y F_3(x,t,0)\,dt$$

$$G_2 = -\int_0^z F_1(x,y,t)\,dt$$

Solution homework #4

$F_1 = xe^y$, $F_2 = -x\cos z$, $F_3 = -ze^y$

$$G_2 = -\int_0^z F_1(x,y,t)\,dt = -\int_0^z xe^y\,dt = -xe^y z$$

$$G_1 = \int_0^z F_2(x,y,t)\,dt - \int_0^y F_3(x,t,0)\,dt$$

$$= \int_0^z -x\cos t\,dt - \cancel{\int_0^y -0e^t\,dt}^{\,0} = -x\sin t\Big|_0^z$$

$$= -x\sin z$$

$\therefore\ \vec{G} = (-x\sin z, -xe^y z, 0)$. ⌟

§8.4.

1) Use the divergence theorem to calculate the flux of $\vec{F} = (x-y, y-z, z-x)$ out of the unit sphere.

⌜ Notice that $\vec{F}$ is of class C^1 on $\mathbb{R}^3$.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_B \operatorname{div} F\,dV = 3\iiint_B dV$$

S: "unit" sphere; B: solid unit ball; $\iiint_B dV$: vol of solid unit ball

$$= 3\left(\frac{4}{3}\pi(1)^3\right) = 4\pi\,.$$ ⌟