A logical view on Tao's finitizations in analysis

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Outline

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II Two logical points

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Part I

Tao's finitizations

What are finitizations

In 2007 and 2008, Terence Tao wrote essays about finitization of statements in analysis.

Soft analysis

Deals with:

- infinite objects (example: sequences);
- qualitative properties (example: convergence).

Hard analysis

Deals with:

- finite objects (example: finite sets);
- quantitative properties (example: bounds).

Finitization

A finitization of a soft analysis statement is an equivalent hard analysis statement.

Motivation for finitizations

Exact relations between quantities

- Soft analysis is hard analysis with the relations hidden.
- Hard analysis is soft analysis with the relations made explicit.

Finitizations make the relations explicit.

Example

Consider $x_n := \frac{1}{n^2} \to 0$.

soft analysis	hard analysis
$\forall \varepsilon > 0, \exists N : \forall n > N, \|x_n - 0\| < \varepsilon$	$\forall \varepsilon > 0, \forall n > \frac{1}{\sqrt{\varepsilon}}, \|x_n - 0\| < \varepsilon$
no relation between N and ε	$N = \frac{1}{\sqrt{\varepsilon}}$

Motivation for finitizations

Long and short range mathematics

- Soft analysis is good for long range mathematics (it moves faster by ignoring exact quantities).
- Hard analysis is good for short range mathematics (refining existing results by relating the exact quantities).

Finitizations refine long range mathematics.

Example

In Green-Tao theorem (establishing arbitrarily long arithmetic progressions of primes), they used

soft analysis	hard analysis
intuitions from ergodic theory to know how to proceed	finitizations of ergodic theory arguments to actually prove the result

Motivation for finitizations

Best of both worlds

There are connections between soft analysis and hard analysis that allow:

- to use soft analysis in hard analysis;
- to use hard analysis in soft analysis.

Finitizations can be used in both soft and hard analysis.

Examples

soft analysis	hard analysis	connection
ergodic theory	combinatorial number theory	Furstenberg correspondence principle
ergodic graph theory	graph theory	graph correspondence principle

Examples of finitizations

Infinite convergence principle

Every monotone bounded sequence of real numbers is convergent.

Finite convergence principle

Every long enough (length M) bounded monotone sequence has arbitrary high-quality (error tolerance ε) long (length $F(N)$) amounts of stability:

$$\forall \varepsilon > 0,\ F : \mathbb{N} \to \mathbb{N},$$
$$\exists M \in \mathbb{N} : \forall (x_n)_{n=1,\ldots,M} \subseteq [0,1] \text{ monotone},$$
$$\exists N \leq M : \underbrace{\forall m, n \in [N, N+F(N)],\ |x_m - x_n| \leq \varepsilon}_{x_n \text{ is stable with error } \varepsilon \text{ in a interval of length } F(N)}\ .$$

Examples of finitizations

- Denote $\{1, \ldots, k\}$ by k.
- A sequence $(A_n) \subseteq \mathcal{P}_{fin}(\mathbb{N})$ *weakly converges* to $I \in \mathcal{P}_{inf}(\mathbb{N})$ if for all $k \in \mathbb{N}$ eventually we have $A_n \cap k = I \cap k$.
- A function $F : \mathcal{P}_{fin}(\mathbb{N}) \to \mathbb{N}$ is *asymptotically stable near infinite sets ($F \in ASNIS$)* if for all all weakly convergent sequences (A_n), $F(A_n)$ eventually becomes constant.

Infinite pigeonhole principle IPP

Every colouring of $\mathbb{N}$ with finitely many colours has an infinite colour class.

Tao's "finitary" infinite pigeonhole principle $\mathrm{FIPP_T}$

Every colouring f of a large enough initial segment k of $\mathbb{N}$ with finitely many colours n has a big colour class A:

$$\forall n \in \mathbb{N},\, F \in ASNIS,\, \exists k \in \mathbb{N} :$$

$$\forall f : k \to n,\, \underbrace{\exists c < n,\, A = f^{-1}(c)}_{\text{exists a colour class}} : \underbrace{|A| > F(A)}_{\text{that is big}}.$$

Examples of finitizations

IPP

Every colouring of $\mathbb{N}$ with finitely many colours has an infinite colour class.

$\mathrm{FIPP_T}$

$\forall n \in \mathbb{N}$, $F \in ASNIS$,
$\exists k \in \mathbb{N} : \forall f : k \to n$,
$\exists c < n$, $A = f^{-1}(c) : |A| > F(A)$.

Proof of IPP $\Rightarrow$ $\mathrm{FIPP_T}$.

- Assume IPP and, by contradiction, $\sim\mathrm{FIPP_T}$. We have $n \in \mathbb{N}$, $F \in ASNIS$ and a sequence $f_k : k \to n$ such that:
  (∗) no $A_k = (f_k)^{-1}(c)$ verifies $|A_k| > F(A_k)$.
- Extend f_k to $\bar{f}_k : \mathbb{N} \to n$. The $\bar{f}_k$'s are in the sequentially compact $n^{\mathbb{N}}$, so (a subsequence of) $\bar{f}_k$ converges to some $f : \mathbb{N} \to n$.
- By IPP, f has an infinite colour class $f^{-1}(c)$.
- Unfolding $\bar{f}_k \to f$ we see that $A_k = (f_k)^{-1}(c)$ weakly converges to $f^{-1}(c)$. Then F stabilizes over (A_k) but $|A_k| \to \infty$. So $\exists k \in \mathbb{N} : |A_k| > F(A_k)$, contradicting (∗). □

Summary

- Tao's finitizations: soft analysis $\rightarrow$ hard analysis.
- Examples:
  - infinite convergence principle $\rightarrow$ finite convergence principle;
  - IPP $\rightarrow$ FIPP_T.
- Contradiction and sequential compactness argument.

Part II

Two logical points

Point 1: Gödel functional interpretation

- PA^ω is a Peano arithmetic that deals with $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$, ...
- The *axiom of choice* AC is $\forall x, \exists y : A(x,y) \Rightarrow \exists f : \forall x, A(x, f(x))$ (A without $\forall, \exists$).

Gödel functional interpretation

Is a function $A \mapsto A^G = \exists x : \forall y, A_G(x,y)$ (A_G without $\forall, \exists$) such that:

- if $\mathsf{PA}^\omega + \mathsf{AC}$ proves A, then PA^ω proves $A_G(f,y)$, for suitable functions f extracted from a proof of A;
- $\mathsf{PA}^\omega + \mathsf{AC}$ proves $A \Leftrightarrow A^G$.

$A_G(f,y)$ is essentially a finitization of A

- $A_G(f,y)$ is a hard analysis statement.
- $A \Leftrightarrow A^G$.
- PA^ω proves $A_G(f,y)$ if and only if PA^ω proves A^G with $x = f$.

Point 1

Gödel functional interpretation finitizes systematically.

Point 2: Heine-Borel compactness

Sequential compactness

Every sequence has a convergent subsequence.

Heine-Borel compactness

Every continuous function is bounded.
Every open cover has a finite subcover.

Reverse mathematics

Seeks to find which axioms are need to prove theorems. The axioms considered are the *big five* subsystems of second order arithmetic:

$S_1 \longrightarrow$	$S_2 \longrightarrow$	$S_3 \longrightarrow \cdots$
PA with induction for $\exists x : A(x, n)$ (A without $\forall, \exists$)	PA with induction for $\exists x : A(x, n)$ (A without $\forall, \exists$)	PA with all induction
only computable sets	some non-computable sets	all arithmetical sets $\{n \in \mathbb{N} : A(n)\}$
	Heine-Borel compactness	sequential compactness

Point 2: Heine-Borel compactness

$$S_1 \longrightarrow S_2 \longrightarrow S_3 \longrightarrow \cdots$$

S_2: Heine-Borel compactness; S_3: sequential compactness

Assume that AC is true. Then AC finitizes into $1+1=2$:

- AC $\Leftrightarrow$ $1+1=2$ (they are both true);
- $1+1=2$ is a hard analysis statement.

The semantic equivalence doesn't discriminate between true statements.

Let "equivalence" mean "equivalence provable in set theory ZF". Now AC $\not\Leftrightarrow$ $1+1=2$. We have more discrimination.

S_2 is weaker than S_3, so S_2 discriminates more than S_3.

Point 2

We should prefer Heine-Borel compactness.

Summary

- Point 1: Gödel functional interpretation finitizes systematically.
- Point 2: we should prefer Heine-Borel compactness.

Part III

A case study

Point 1: Gödel functional interpretation

A function $F : \mathcal{P}_{fin}(\mathbb{N}) \to \mathbb{N}$ is *asymptotically stable ($F \in AS$)* if for all chains $A_1 \subseteq A_2 \subseteq \cdots$ in $\mathcal{P}_{fin}(\mathbb{N})$, $F(A_n)$ eventually becomes constant.

IPP^G is essentially:

Kohlenbach's "finitary" infinite pigeonhole principle FIPP_K

Every colouring of a large enough initial segment of $\mathbb{N}$ with finitely many colours has a big subset of a colour class:

$$\forall n \in \mathbb{N},\, F \in AS,\, \exists k \in \mathbb{N} :$$
$$\forall f : k \to n,\, \exists c < n,\, A \subseteq f^{-1}(c) : |A| > F(A).$$

Tao's "finitary" infinite pigeonhole principle FIPP_T

Every colouring of a large enough initial segment of $\mathbb{N}$ with finitely many colours has a big colour class:

$$\forall n \in \mathbb{N},\, F \in ASNIS,\, \exists k \in \mathbb{N} :$$
$$\forall f : k \to n,\, \exists c < n,\, A = f^{-1}(c) : |A| > F(A).$$

Point 2: Heine-Borel compactness

IPP

Every colouring of $\mathbb{N}$ with finitely many colours has an infinite colour class.

FIPP_K

$\forall n \in \mathbb{N}, F \in AS,$
$\exists k \in \mathbb{N} : \forall f : k \to n,$
$\exists c < n, A \subseteq f^{-1}(c) : |A| > F(A).$

Theorem

S_2 proves IPP $\Leftrightarrow$ FIPP_K.

Proof of IPP $\Rightarrow$ FIPP_K.

- Prove $\forall f : \mathbb{N} \to n, \exists i \in \mathbb{N} : \overbrace{\exists c < n, A = f^{-1}(c) \cap i : |A| > F(A)}^{B(f,i)}$.
- Heine-Borel compactness: $\phi(f) := \min i : B(f, i)$ is total and continuous on the compact $n^{\mathbb{N}}$, so it has an upper bound k.
- f only appears in $f^{-1}(c) \cap i$ with $i \leq k$, so $f|_k$ suffices. $\square$

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Two logical points

- Point 1: Gödel functional interpretation finitizes systematically.
- Point 2: we should prefer Heine-Borel compactness.

A case study

- Point 1: IPP $\rightarrow$ FIPP_K by Gödel functional interpretation.
- Point 2: S_2 proves IPP $\Leftrightarrow$ FIPP_K by Heine-Borel compactness.

References

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