Introduction to Symplectic and Hamiltonian Geometry

Notes for a Short Course at IMPA
Rio de Janeiro, February 2002

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Revised November 2007

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Prefácio

A geometria simpléctica é a geometria de variedades equipadas com uma forma simpléctica, ou seja, com uma forma de grau 2 fechada e não-degenerada. A geometria hamiltoniana é a geometria de variedades (simplécticas) equipadas com uma aplicação momento, ou seja, com uma colecção de quantidades conservadas por simetrias.

Há cerca de dois séculos, a geometria simpléctica fornecia a linguagem para a mecânica clássica; pelo seu rápido crescimento recente, conquistou um rico território, estabelecendo-se como um ramo central da geometria e da topologia diferenciais. Além da sua actividade como disciplina independente, a geometria simpléctica é significativamente estimulada por interacções importantes com sistemas dinâmicos, análise global, física-matemática, topologia em baixas dimensões, teoria de representações, análise microlocal, equações diferenciais parciais, geometria algébrica, geometria riemanniana, análise combinatórica geométrica, co-homologia equivariante, etc.

Este texto cobre fundamentos da geometria simpléctica numa linguagem moderna. Começa-se por descrever as variedades simplécticas e as suas transformações e por explicar ligações a topologia e outras geometrias. Seguidamente estudam-se campos hamiltonianos, acções hamiltonianas e algumas das suas aplicações práticas no âmbito da mecânica e dos sistemas dinâmicos. Ao longo do texto fornecem-se exemplos simples e exercícios relevantes. Pressupõem-se conhecimentos prévios de geometria de variedades diferenciáveis, se bem que os principais factos requeridos estejam coleccionados em apêndices.

Estas notas reproduzem aproximadamente o curso curto de geometria simpléctica, constituído por cinco lições dirigidas a estudantes de pós-graduação e investigadores, integrado no programa de Verão do Instituto de Matemática Pura e Aplicada, no Rio de Janeiro, em Fevereiro de 2002. Alguns trechos deste texto são rearranjos do Lectures on Symplectic Geometry (Springer LNM 1764).

Fico grata ao IMPA pelo acolhimento muito proveitoso, e em especial ao Marcelo Viana por me ter gentilmente proporcionado a honra e o prazer desta visita, e à Suely Torres de Melo pela sua inestimável ajuda perita com os preparativos locais.

Lisboa, Janeiro de 2002 e Rio de Janeiro, Fevereiro de 2002

Foreword

Symplectic geometry is the geometry of manifolds equipped with a symplectic form, that is, with a closed nondegenerate 2-form. Hamiltonian geometry is the geometry of (symplectic) manifolds equipped with a moment map, that is, with a collection of quantities conserved by symmetries.

About two centuries ago, symplectic geometry provided a language for classical mechanics; through its recent fast development, it conquered a rich territory, asserting itself as a central branch of differential geometry and topology. Besides its activity as an independent subject, symplectic geometry is significantly stimulated by important interactions with dynamical systems, global analysis, mathematical physics, low-dimensional topology, representation theory, microlocal analysis, partial differential equations, algebraic geometry, riemannian geometry, geometric combinatorics, equivariant cohomology, etc.

This text covers foundations of symplectic geometry in a modern language. We start by describing symplectic manifolds and their transformations, and by explaining connections to topology and other geometries. Next we study hamiltonian fields, hamiltonian actions and some of their practical applications in the context of mechanics and dynamical systems. Throughout the text we provide simple examples and relevant exercises. We assume previous knowledge of the geometry of smooth manifolds, though the main required facts are collected in appendices.

These notes approximately transcribe the short course on symplectic geometry, delivered in five lectures mostly for graduate students and researchers, held at the summer program of Instituto de Matemática Pura e Aplicada, Rio de Janeiro, in February of 2002. Many stretches of this text are rearrangements from the book Lectures on Symplectic Geometry (Springer LNM 1764).

I am grateful to IMPA for the very rewarding hospitality, and specially to Marcelo Viana for kindly providing me the honour and the pleasure of this visit, and to Suely Torres de Melo for her invaluable expert help with local arrangements.

Lisbon, January 2002 and Rio de Janeiro, February 2002

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Lecture 1

Symplectic Forms

A symplectic form is a 2-form satisfying an algebraic condition – non-degeneracy – and an analytical condition – closedness. In this lecture we define symplectic forms, describe some of their basic properties, and introduce the first examples. We conclude by exhibiting a major technique in the symplectic trade, namely the so-called Moser trick, which takes advantage of the main features of a symplectic form in order to show the equivalence of symplectic structures.

1.1 Skew-Symmetric Bilinear Maps

Let V be an m-dimensional vector space over $\mathbb{R}$, and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map. The map Ω is **skew-symmetric** if $\Omega(u, v) = -\Omega(v, u)$, for all $u, v \in V$.

Theorem 1.1 (Standard Form for Skew-symmetric Bilinear Maps) *Let Ω be a skew-symmetric bilinear map on V. Then there is a basis*
$u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n$ *of V such that*

$$\begin{array}{ll} \Omega(u_i, v) = 0 \ , & \textit{for all } i \textit{ and all } v \in V, \\ \Omega(e_i, e_j) = 0 = \Omega(f_i, f_j) \ , & \textit{for all } i, j, \textit{ and} \\ \Omega(e_i, f_j) = \delta_{ij} \ , & \textit{for all } i, j. \end{array}$$

Remarks.

1. The basis in Theorem 1.1 is not unique, though it is traditionally also called a "canonical" basis.

2. In matrix notation with respect to such basis, we have

$$\Omega(u,v) = [\text{---}u\text{---}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & -\mathrm{Id} & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix} .$$

$\diamondsuit$

Proof. This induction proof is a skew-symmetric version of the Gram-Schmidt process.

Let $U := \{u \in V \mid \Omega(u,v) = 0 \text{ for all } v \in V\}$. Choose a basis $u_1, \ldots, u_k$ of U, and choose a complementary space W to U in V,

$$V = U \oplus W \ .$$

Take any nonzero $e_1 \in W$. Then there is $f_1 \in W$ such that $\Omega(e_1, f_1) \neq 0$. Assume that $\Omega(e_1, f_1) = 1$. Let

$$\begin{aligned} W_1 &= \text{span of } e_1, f_1 \\ W_1^\Omega &= \{w \in W \mid \Omega(w,v) = 0 \text{ for all } v \in W_1\} \ . \end{aligned}$$

Claim. $W_1 \cap W_1^\Omega = \{0\}$.

Suppose that $v = ae_1 + bf_1 \in W_1 \cap W_1^\Omega$.

$$\left.\begin{aligned} 0 = \Omega(v, e_1) &= -b \\ 0 = \Omega(v, f_1) &= a \end{aligned}\right\} \quad \Longrightarrow \quad v = 0 \ .$$

Claim. $W = W_1 \oplus W_1^\Omega$.

Suppose that $v \in W$ has $\Omega(v, e_1) = c$ and $\Omega(v, f_1) = d$. Then

$$v = \underbrace{(-cf_1 + de_1)}_{\in W_1} + \underbrace{(v + cf_1 - de_1)}_{\in W_1^\Omega} \ .$$

Go on: let $e_2 \in W_1^\Omega$, $e_2 \neq 0$. There is $f_2 \in W_1^\Omega$ such that $\Omega(e_2, f_2) \neq 0$. Assume that $\Omega(e_2, f_2) = 1$. Let W_2 = span of e_2, f_2. Etc.

This process eventually stops because $\dim V < \infty$. We hence obtain

$$V = U \oplus W_1 \oplus W_2 \oplus \ldots \oplus W_n$$

where all summands are orthogonal with respect to Ω, and where W_i has basis e_i, f_i with $\Omega(e_i, f_i) = 1$. □

The dimension of the subspace $U = \{u \in V \mid \Omega(u, v) = 0, \text{ for all } v \in V\}$ does not depend on the choice of basis. Hence, $k := \dim U$ is an invariant of (V, Ω).

Since $k+2n = m = \dim V$, we have that n is an invariant of (V, Ω); $2n$ is called the **rank** of Ω.

1.2 Symplectic Vector Spaces

Let V be an m-dimensional vector space over $\mathbb{R}$, and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map.

Definition 1.2 *The map $\widetilde{\Omega} : V \to V^*$ is the linear map defined by $\widetilde{\Omega}(v)(u) = \Omega(v, u)$.*

The kernel of $\widetilde{\Omega}$ is the subspace U in the previous section.

Definition 1.3 *A skew-symmetric bilinear map Ω is* **symplectic** *(or* **nondegenerate***) if $\widetilde{\Omega}$ is bijective, i.e., $U = \{0\}$. The map Ω is then called a* **linear symplectic structure** *on V, and (V, Ω) is called a* **symplectic vector space***.*

The following are immediate properties of a symplectic map Ω:

- **Duality**: the map $\widetilde{\Omega} : V \overset{\simeq}{\to} V^*$ is a bijection.
- By Theorem 1.1, we must have that $k = \dim U = 0$, so $\dim V = 2n$ is **even**.

- Also by Theorem 1.1, a symplectic vector space (V, Ω) has a basis
  $e_1, \ldots, e_n, f_1, \ldots, f_n$ satisfying

  $$\Omega(e_i, f_j) = \delta_{ij} \quad \text{and} \quad \Omega(e_i, e_j) = 0 = \Omega(f_i, f_j) \ .$$

  Such a basis is called a **symplectic basis** of (V, Ω). With respect to a symplectic basis, we have

  $$\Omega(u, v) = [-u-] \begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix} .$$

The **prototype of a symplectic vector space** is $(\mathbb{R}^{2n}, \Omega_0)$ with Ω_0 such that the basis

$$\begin{array}{lll} e_1 = (1, 0, \ldots, 0), & \ldots, & e_n = (0, \ldots, 0, \overbrace{1}^{n}, 0, \ldots, 0), \\ f_1 = (0, \ldots, 0, \underbrace{1}_{n+1}, 0, \ldots, 0), & \ldots, & f_n = (0, \ldots, 0, 1) \ , \end{array}$$

is a symplectic basis. The map Ω_0 on other vectors is determined by its values on a basis and bilinearity.

Definition 1.4 *A* **symplectomorphism** φ *between symplectic vector spaces* (V, Ω) *and* (V', Ω') *is a linear isomorphism* $\varphi : V \xrightarrow{\simeq} V'$ *such that* $\varphi^*\Omega' = \Omega$*. (By definition,* $(\varphi^*\Omega')(u, v) = \Omega'(\varphi(u), \varphi(v))$*.) If a symplectomorphism exists,* (V, Ω) *and* (V', Ω') *are said to be* **symplectomorphic**.

The relation of being symplectomorphic is clearly an equivalence relation in the set of all even-dimensional vector spaces. Furthermore, by Theorem 1.1, every $2n$-dimensional symplectic vector space (V, Ω) is symplectomorphic to the prototype $(\mathbb{R}^{2n}, \Omega_0)$; a choice of a symplectic basis for (V, Ω) yields a symplectomorphism to $(\mathbb{R}^{2n}, \Omega_0)$. Hence, nonnegative even integers classify equivalence classes for the relation of being symplectomorphic.

1.3 Special Subspaces

Given a linear subspace Y of a symplectic vector space (V, Ω), its **symplectic orthogonal** Y^Ω is the linear subspace defined by

$$Y^\Omega := \{v \in V \mid \Omega(v, u) = 0 \text{ for all } u \in Y\} .$$

By considering the kernel and image of the map

$$\begin{array}{ccc} V & \longrightarrow & Y^* = \mathrm{Hom}(Y, \mathbb{R}) \\ v & \longmapsto & \Omega(v, \cdot)|_Y \end{array}$$

we obtain that

$$\dim Y + \dim Y^\Omega = \dim V .$$

By nondegeneracy of Ω, we have that $(Y^\Omega)^\Omega = Y$. It is also easily checked that, if Y and W are subspaces, then

$$Y \subseteq W \iff W^\Omega \subseteq Y^\Omega .$$

Not all subspaces W of a symplectic vector space (V, Ω) look the same:

- A subspace Y is called **symplectic** if $\Omega|_{Y \times Y}$ is nondegenerate. This is the same as saying that $Y \cap Y^\Omega = \{0\}$, or, by counting dimensions, that $V = Y \oplus Y^\Omega$.

- A subspace Y is called **isotropic** if $\Omega|_{Y \times Y} \equiv 0$. If Y is isotropic, then $\dim Y \leq \frac{1}{2} \dim V$. Every one-dimensional subspace is isotropic.

- A subspace is called **coisotropic** if its symplectic orthogonal is isotropic. If Y is coisotropic, then $\dim Y \geq \frac{1}{2} \dim V$. Every codimension 1 subspace is coisotropic.

For instance, if $e_1, \ldots, e_n, f_1, \ldots, f_n$ is a symplectic basis of (V, Ω), then:

- the span of e_1, f_1 is symplectic,

- the span of e_1, e_2 is isotropic, and

- the span of $e_1, \ldots, e_n, f_1, f_2$ is coisotropic.

An isotropic subspace Y of (V, Ω) is called **lagrangian** when $\dim Y = \frac{1}{2} \dim V$. We have that

$$Y \text{ is lagrangian} \iff Y \text{ is isotropic and coisotropic} \iff Y = Y^\Omega \; .$$

Exercise 1
Show that, if Y is a lagrangian subspace of (V, Ω), then any basis $e_1, \ldots, e_n$ of Y can be extended to a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of (V, Ω).

Hint: Choose f_1 in W^Ω, where W is the linear span of $\{e_2, \ldots, e_n\}$.

If Y is a lagrangian subspace, then (V, Ω) is symplectomorphic to the space $(Y \oplus Y^*, \Omega_0)$, where Ω_0 is determined by the formula

$$\Omega_0(u \oplus \alpha, v \oplus \beta) = \beta(u) - \alpha(v) \; .$$

Moreover, for any vector space E, the direct sum $V = E \oplus E^*$ has a canonical symplectic structure determined by the formula above. If $e_1, \ldots, e_n$ is a basis of E, and $f_1, \ldots, f_n$ is the dual basis, then $e_1 \oplus 0, \ldots, e_n \oplus 0, 0 \oplus f_1, \ldots, 0 \oplus f_n$ is a symplectic basis for V.

1.4 Symplectic Manifolds

Let ω be a de Rham 2-form on a manifold M, that is, for each $p \in M$, the map $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is skew-symmetric bilinear on the tangent space to M at p, and ω_p varies smoothly in p. We say that ω is closed if it satisfies the differential equation $d\omega = 0$, where d is the de Rham differential (i.e., exterior derivative).

Definition 1.5 *The 2-form ω is* **symplectic** *if ω is closed and ω_p is symplectic for all $p \in M$.*

If ω is symplectic, then $\dim T_pM = \dim M$ must be even.

Definition 1.6 *A* **symplectic manifold** *is a pair (M, ω) where M is a manifold and ω is a symplectic form.*

Examples.

1. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The form
$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$
is symplectic as can be easily checked; the set
$$\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \ldots, \left(\frac{\partial}{\partial y_n}\right)_p \right\}$$
is a symplectic basis of T_pM.

2. Let $M = \mathbb{C}^n$ with linear coordinates $z_1, \ldots, z_n$. The form
$$\omega_0 = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$$
is symplectic. In fact, this form equals that of the previous example under the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $z_k = x_k + iy_k$.

3. Let $M = S^2$ regarded as the set of unit vectors in $\mathbb{R}^3$. Tangent vectors to S^2 at p may then be identified with vectors orthogonal to p. The **standard symplectic form on** S^2 is induced by the inner and exterior products:
$$\omega_p(u, v) := \langle p, u \times v \rangle \ , \qquad \text{for } u, v \in T_pS^2 = \{p\}^\perp \ .$$
This form is closed because it is of top degree; it is nondegenerate because $\langle p, u \times v \rangle \neq 0$ when $u \neq 0$ and we take, for instance, $v = u \times p$.

◊

Exercise 2
Consider cylindrical polar coordinates (θ, h) on S^2 away from its poles, where $0 \leq \theta < 2\pi$ and $-1 \leq h \leq 1$. Show that, in these coordinates, the form of the previous example is
$$\omega = d\theta \wedge dh \ .$$

1.5 Symplectic Volume

Given a vector space V, the exterior algebra of its dual space is

$$\wedge^*(V^*) = \bigoplus_{k=0}^{\dim V} \wedge^k(V^*) \ ,$$

where $\wedge^k(V^*)$ is the set of maps $\alpha : \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}$ which are linear in each entry, and for any permutation π, $\alpha(v_{\pi_1}, \ldots, v_{\pi_k}) = (\operatorname{sign} \pi) \cdot \alpha(v_1, \ldots, v_k)$. The elements of $\wedge^k(V^*)$ are known as **skew-symmetric k-linear maps** or **k-forms** on V.

Exercise 3
Show that any $\Omega \in \wedge^2(V^*)$ is of the form $\Omega = e_1^* \wedge f_1^* + \ldots + e_n^* \wedge f_n^*$, where $u_1^*, \ldots, u_k^*, e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^*$ is a basis of V^* dual to the standard basis $(k + 2n = \dim V)$.

In this language, a symplectic map $\Omega : V \times V \to \mathbb{R}$ is just a nondegenerate 2-form $\Omega \in \wedge^2(V^*)$, called a **symplectic form** on V. By the previous exercise, if Ω is any symplectic form on a vector space V of dimension $2n$, then the nth exterior power $\Omega^n = \underbrace{\Omega \wedge \ldots \wedge \Omega}_{n}$ does not vanish. Conversely, given a 2-form $\Omega \in \wedge^2(V^*)$, if $\Omega^n \neq 0$, then Ω is symplectic.

We conclude that the nth exterior power ω^n of any symplectic form ω on a $2n$-dimensional manifold M is a volume form.[1] Hence, any symplectic manifold (M, ω) is canonically oriented by the symplectic structure, and any nonorientable manifold cannot be symplectic. The form $\frac{\omega^n}{n!}$ is called the **symplectic volume** of (M, ω).

Let (M, ω) be a $2n$-dimensional symplectic manifold, and let ω^n be the volume form obtained by wedging ω with itself n times. By Stokes' theorem., if M is compact, the de Rham cohomology class $[\omega^n] \in H^{2n}(M; \mathbb{R})$ is non-zero. Hence, $[\omega]$ itself is non-zero (in other words, ω is not exact). This reveals a necessary topological condition for a compact $2n$-dimensional manifold to be symplectic: there must

[1] A **volume form** is a nonvanishing form of top degree.

exist a degree 2 cohomology class whose nth power is a volume form. In particular, for $n > 1$ there are no symplectic structures on the sphere S^{2n}.

1.6 Equivalence for Symplectic Structures

Let M be a $2n$-dimensional manifold with two symplectic forms ω_0 and ω_1, so that (M, ω_0) and (M, ω_1) are two symplectic manifolds.

Definition 1.7 *A* **symplectomorphism** *between* (M_1, ω_1) *and* (M_2, ω_2) *is a diffeomorphism* $\varphi : M_1 \to M_2$ *such that* $\varphi^*\omega_2 = \omega_1$.[2]

We would like to classify symplectic manifolds up to symplectomorphism. The Darboux theorem (stated and proved in Section 1.9) takes care of this classification locally: the dimension is the only local invariant of symplectic manifolds up to symplectomorphisms. Just as any n-dimensional manifold looks locally like $\mathbb{R}^n$, any $2n$-dimensional *symplectic* manifold looks locally like $(\mathbb{R}^{2n}, \omega_0)$. More precisely, any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Definition 1.8 *We say that*

- (M, ω_0) *and* (M, ω_1) *are* **symplectomorphic** *if there is a diffeomorphism* $\varphi : M \to M$ *with* $\varphi^*\omega_1 = \omega_0$*;*

- (M, ω_0) *and* (M, ω_1) *are* **strongly isotopic** *if there is an isotopy* $\rho_t : M \to M$ *such that* $\rho_1^*\omega_1 = \omega_0$*;*

- (M, ω_0) *and* (M, ω_1) *are* **deformation-equivalent** *if there is a smooth family* ω_t *of symplectic forms joining* ω_0 *to* ω_1*;*

- (M, ω_0) *and* (M, ω_1) *are* **isotopic** *if they are deformation-equivalent with* $[\omega_t]$ *independent of* t.

[2] Recall that, by definition of pullback, at tangent vectors $u, v \in T_pM_1$, we have $(\varphi^*\omega_2)_p(u, v) = (\omega_2)_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v))$.

Clearly, we have

$$\text{strongly isotopic} \implies \text{symplectomorphic}\ , \qquad \text{and}$$

$$\text{isotopic} \implies \text{deformation-equivalent}\ .$$

We also have

$$\text{strongly isotopic} \implies \text{isotopic}$$

because, if $\rho_t : M \to M$ is an isotopy such that $\rho_1^*\omega_1 = \omega_0$, then the set $\omega_t := \rho_t^*\omega_1$ is a smooth family of symplectic forms joining ω_1 to ω_0 and $[\omega_t] = [\omega_1]$, $\forall t$, by the homotopy invariance of de Rham cohomology. As we will see below, the Moser theorem states that, on a compact manifold,

$$\text{isotopic} \implies \text{strongly isotopic}\ .$$

The remainder of this lecture concerns the following problem:

Problem. Given a $2n$-dimensional manifold M, a k-dimensional submanifold X, neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of X, and symplectic forms ω_0, ω_1 on $\mathcal{U}_0, \mathcal{U}_1$, does there exist a symplectomorphism preserving X? More precisely, does there exist a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ with $\varphi^*\omega_1 = \omega_0$ and $\varphi(X) = X$?

At the two extremes, we have:
Case $X = \textit{point}$: Darboux theorem – see Section 1.9.
Case $X = M$: Moser theorem – see Section 1.7.

Inspired by the elementary normal form in symplectic linear algebra (Theorem 1.1), we will go on to describe normal neighborhoods of a point (the Darboux theorem) and of a lagrangian submanifold (the Weinstein theorems), inside a symplectic manifold. The main tool is the Moser trick, explained below, which leads to the crucial Moser theorems and which is at the heart of many arguments in symplectic geometry. We need some (non-symplectic) ingredients discussed in Appendix A; for more on these topics, see, for instance, [12, 25, 41].

1.7 Moser Trick

Let M be a *compact* manifold with symplectic forms ω_0 and ω_1. Moser asked whether we can find a symplectomorphism $\varphi : M \to M$ which is homotopic to id_M. A necessary condition is $[\omega_0] = [\omega_1] \in H^2(M;\mathbb{R})$ because: if $\varphi \sim \mathrm{id}_M$, then, by the homotopy formula, there exists a homotopy operator Q such that

$$\begin{array}{rc} & \mathrm{id}_M^*\omega_1 - \varphi^*\omega_1 = dQ\omega_1 + Q\underbrace{d\omega_1}_{0} \\ \Longrightarrow & \omega_1 = \varphi^*\omega_1 + d(Q\omega_1) \\ \Longrightarrow & [\omega_1] = [\varphi^*\omega_1] = [\omega_0]\ . \end{array}$$

Suppose now that $[\omega_0] = [\omega_1]$. Moser [37] proved that the answer to the question above is yes, with a further hypothesis as in Theorem 1.9. McDuff showed that, in general, the answer is no; for a counterexample, see Example 7.23 in [35].

Theorem 1.9 (Moser Theorem – Version I) *Suppose that $[\omega_0] = [\omega_1]$ on a compact manifold M and that the 2-form $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic for each $t \in [0,1]$. Then there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^*\omega_t = \omega_0$ for all $t \in [0,1]$.*

In particular, $\varphi = \rho_1 : M \stackrel{\simeq}{\longrightarrow} M$, satisfies $\varphi^*\omega_1 = \omega_0$. The following argument, due to Moser, is extremely useful; it is known as the **Moser trick**.

Proof. Suppose that there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^*\omega_t = \omega_0$, $0 \le t \le 1$. Let

$$v_t = \frac{d\rho_t}{dt} \circ \rho_t^{-1}\ , \qquad t \in \mathbb{R}\ .$$

Then

$$\begin{array}{rc} & 0 = \frac{d}{dt}(\rho_t^*\omega_t) = \rho_t^*\big(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\big) \\ \Longleftrightarrow & \mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt} = 0\ . \qquad (\star) \end{array}$$

Suppose conversely that we can find a smooth time-dependent vector field v_t, $t \in \mathbb{R}$, such that $(\star)$ holds for $0 \le t \le 1$. Since M is compact, we can integrate v_t to an isotopy $\rho : M \times \mathbb{R} \to M$ with

$$\frac{d}{dt}(\rho_t^*\omega_t) = 0 \quad \Longrightarrow \quad \rho_t^*\omega_t = \rho_0^*\omega_0 = \omega_0\ .$$

So everything boils down to solving $(\star)$ for v_t.

First, from $\omega_t = (1-t)\omega_0 + t\omega_1$, we conclude that

$$\frac{d\omega_t}{dt} = \omega_1 - \omega_0 \ .$$

Second, since $[\omega_0] = [\omega_1]$, there exists a 1-form μ such that

$$\omega_1 - \omega_0 = d\mu \ .$$

Third, by the Cartan magic formula, we have

$$\mathcal{L}_{v_t}\omega_t = d\imath_{v_t}\omega_t + \imath_{v_t}\underbrace{d\omega_t}_{0} \ .$$

Putting everything together, we must find v_t such that

$$d\imath_{v_t}\omega_t + d\mu = 0 \ .$$

It is sufficient to solve $\imath_{v_t}\omega_t + \mu = 0$. By the nondegeneracy of ω_t, we can solve this pointwise, to obtain a unique (smooth) v_t. □

Theorem 1.10 (Moser Theorem – Version II) *Let M be a compact manifold with symplectic forms ω_0 and ω_1. Suppose that ω_t, $0 \le t \le 1$, is a smooth family of closed 2-forms joining ω_0 to ω_1 and satisfying:*

(1) cohomology assumption: $[\omega_t]$ *is independent of* t, *i.e.,* $\frac{d}{dt}[\omega_t] = \left[\frac{d}{dt}\omega_t\right] = 0$,

(2) nondegeneracy assumption: ω_t *is nondegenerate for* $0 \le t \le 1$.

Then there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^*\omega_t = \omega_0$, $0 \le t \le 1$.

Proof. *(Moser trick)* We have the following implications from the hypotheses:

(1) $\Longrightarrow$ There is a family of 1-forms μ_t such that

$$\frac{d\omega_t}{dt} = d\mu_t \ , \quad 0 \le t \le 1 \ .$$

We can indeed find a *smooth* family of 1-forms μ_t such that $\frac{d\omega_t}{dt} = d\mu_t$. The argument involves the Poincaré lemma for compactly-supported forms, together with the Mayer-Vietoris sequence in order to use induction on the number of charts in a good cover of M. For a sketch of the argument, see page 95 in [35].

(2) $\Longrightarrow$ There is a unique family of vector fields v_t such that

$$\imath_{v_t}\omega_t + \mu_t = 0 \qquad \textbf{(Moser equation)} \ .$$

Extend v_t to all $t \in \mathbb{R}$. Let ρ be the isotopy generated by v_t (ρ exists by compactness of M). Then we indeed have

$$\frac{d}{dt}(\rho_t^*\omega_t) = \rho_t^*(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}) = \rho_t^*(d\imath_{v_t}\omega_t + d\mu_t) = 0 \ .$$

□

The compactness of M was used to be able to integrate v_t for all $t \in \mathbb{R}$. If M is *not* compact, we need to check the existence of a solution ρ_t for the differential equation $\frac{d\rho_t}{dt} = v_t \circ \rho_t$ for $0 \leq t \leq 1$.

Picture. Fix $c \in H^2(M)$. Define $S_c = \{$symplectic forms ω in M with $[\omega] = c\}$. The Moser theorem implies that, on a compact manifold, all symplectic forms on the same path-connected component of S_c are symplectomorphic.

Exercises 4

Any oriented 2-dimensional manifold with an area form is a symplectic manifold.

(a) Show that convex combinations of two area forms ω_0 and ω_1 that induce the same orientation are symplectic.

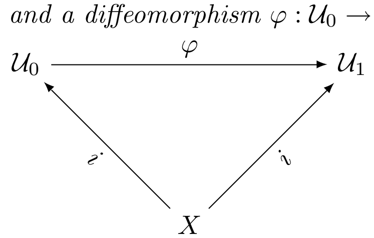
This is wrong in dimension 4: find two symplectic forms on the vector space $\mathbb{R}^4$ that induce the same orientation, yet some convex combination of which is degenerate. Find a path of symplectic forms that connect them.

(b) Suppose that we have two area forms ω_0, ω_1 on a compact 2-dimensional manifold M representing the same de Rham cohomology class, i.e., $[\omega_0] = [\omega_1] \in H^2_{\text{deRham}}(M)$. Prove that there is a 1-parameter family of diffeomorphisms $\varphi_t : M \to M$ such that $\varphi_1^*\omega_0 = \omega_1$, $\varphi_0 = \text{id}$, and $\varphi_t^*\omega_0$ is symplectic for all $t \in [0,1]$. Such a 1-parameter family φ_t is a strong isotopy between ω_0 and ω_1. In this language, this exercise shows that, up to strong isotopy, there is a unique symplectic representative in each non-zero 2-cohomology class of M.

1.8 Moser Relative Theorem

Theorem 1.11 (Moser Theorem – Relative Version) *Let M be a manifold, X a compact submanifold of M, $i : X \hookrightarrow M$ the inclusion map, ω_0 and ω_1 symplectic forms in M.*

Hypothesis: $\omega_0|_p = \omega_1|_p$, $\forall p \in X$.
Conclusion: *There exist neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of X in M , and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that*



commutes

and $\varphi^*\omega_1 = \omega_0$.

Proof.

1. Pick a tubular neighborhood $\mathcal{U}_0$ of X. The 2-form $\omega_1 - \omega_0$ is closed on $\mathcal{U}_0$, and $(\omega_1 - \omega_0)_p = 0$ at all $p \in X$. By the homotopy formula on the tubular neighborhood, there exists a 1-form μ on $\mathcal{U}_0$ such that $\omega_1 - \omega_0 = d\mu$ and $\mu_p = 0$ at all $p \in X$.

2. Consider the family $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + td\mu$ of closed 2-forms on $\mathcal{U}_0$. Shrinking $\mathcal{U}_0$ if necessary, we can assume that ω_t is symplectic for $0 \le t \le 1$.

3. Solve the Moser equation: $\imath_{v_t}\omega_t = -\mu$. Notice that $v_t = 0$ on X.

4. Integrate v_t. Shrinking $\mathcal{U}_0$ again if necessary, there exists an isotopy $\rho : \mathcal{U}_0 \times [0,1] \to M$ with $\rho_t^*\omega_t = \omega_0$, for all $t \in [0,1]$. Since $v_t|_X = 0$, we have $\rho_t|_X = \mathrm{id}_X$. Set $\varphi = \rho_1$, $\mathcal{U}_1 = \rho_1(\mathcal{U}_0)$.

□

1.9 Darboux Theorem

We will apply the relative version of the Moser theorem to $X = \{p\}$ in order to prove:

Theorem 1.12 (Darboux) *Let (M, ω) be a $2n$-dimensional symplectic manifold, and let p be any point in M. Then there is a coordinate chart $(\mathcal{U}, x_1, \dots, x_n, y_1, \dots, y_n)$ centered at p such that on $\mathcal{U}$*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \ .$$

As a consequence of Theorem 1.12, if we show for $(\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$ a local assertion which is invariant under symplectomorphisms, then that assertion holds for any symplectic manifold.

Proof. Use any symplectic basis for T_pM to construct coordinates $(x'_1, \dots, x'_n, y'_1, \dots y'_n)$ centered at p and valid on some neighborhood $\mathcal{U}'$, so that

$$\omega_p = \sum dx'_i \wedge dy'_i \Big|_p \ .$$

There are two symplectic forms on $\mathcal{U}'$: the given $\omega_0 = \omega$ and $\omega_1 = \sum dx'_i \wedge dy'_i$. By the Moser theorem (Theorem 1.11) applied to $X = \{p\}$, there are neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of p, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that

$$\varphi(p) = p \quad \text{and} \quad \varphi^*(\sum dx'_i \wedge dy'_i) = \omega \ .$$

Since $\varphi^*(\sum dx'_i \wedge dy'_i) = \sum d(x'_i \circ \varphi) \wedge d(y'_i \circ \varphi)$, we only need to set new coordinates $x_i = x'_i \circ \varphi$ and $y_i = y'_i \circ \varphi$. □

A chart $(\mathcal{U}, x_1, \dots, x_n, y_1, \dots, y_n)$ as in Theorem 1.12 is called a **Darboux chart**.

By Theorem 1.12, the **prototype of a local piece of a $2n$-dimensional symplectic manifold** is $M = \mathbb{R}^{2n}$, with linear coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$, and with symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i \ .$$

Exercise 5
Prove the Darboux theorem in the 2-dimensional case, using the fact that every nonvanishing 1-form on a surface can be written locally as $f\,dg$ for suitable functions f, g.

Hint: $\omega = df \wedge dg$ nondegenerate $\iff$ (f, g) local diffeomorphism.

Exercise 6
Let $\mathcal{H}$ be the vector space of $n \times n$ complex hermitian matrices. The unitary group $\mathrm{U}(n)$ acts on $\mathcal{H}$ by conjugation: $A\cdot\xi = A\xi A^{-1}$, for $A \in \mathrm{U}(n)$, $\xi \in \mathcal{H}$. For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, let $\mathcal{H}_\lambda$ be the set of all $n \times n$ complex hermitian matrices whose spectrum is λ.

(a) Show that the orbits of the $\mathrm{U}(n)$-action are the manifolds $\mathcal{H}_\lambda$.
For a fixed $\lambda \in \mathbb{R}^n$, what is the stabilizer of a point in $\mathcal{H}_\lambda$?
Hint: If $\lambda_1, \ldots, \lambda_n$ are all distinct, the stabilizer of the diagonal matrix is the torus $\mathbb{T}^n$ of all diagonal unitary matrices.

(b) Show that the symmetric bilinear form on $\mathcal{H}$, $(X, Y) \mapsto \mathrm{trace}\,(XY)$, is nondegenerate.
For $\xi \in \mathcal{H}$, define a skew-symmetric bilinear form ω_ξ on $\mathfrak{u}(n) = T_1\mathrm{U}(n) = i\mathcal{H}$ (space of skew-hermitian matrices) by

$$\omega_\xi(X, Y) = i\,\mathrm{trace}\,([X, Y]\xi)\ , \qquad X, Y \in i\mathcal{H}\ .$$

Check that $\omega_\xi(X, Y) = i\,\mathrm{trace}\,(X(Y\xi - \xi Y))$ and $Y\xi - \xi Y \in \mathcal{H}$.
Show that the kernel of ω_ξ is $K_\xi := \{Y \in \mathfrak{u}(n) \mid [Y, \xi] = 0\}$.

(c) Show that K_ξ is the Lie algebra of the stabilizer of ξ.
Hint: Differentiate the relation $A\xi A^{-1} = \xi$.
Show that the ω_ξ's induce nondegenerate 2-forms on the orbits $\mathcal{H}_\lambda$.
Show that these 2-forms are closed.
Conclude that all the orbits $\mathcal{H}_\lambda$ are compact symplectic manifolds.

(d) Describe the manifolds $\mathcal{H}_\lambda$.
When all eigenvalues are equal, there is only one point in the orbit.
Suppose that $\lambda_1 \neq \lambda_2 = \ldots = \lambda_n$. Then the eigenspace associated with λ_1 is a line, and the one associated with λ_2 is the orthogonal hyperplane. Show that there is a diffeomorphism $\mathcal{H}_\lambda \simeq \mathbb{CP}^{n-1}$. We have thus exhibited a lot of symplectic forms on $\mathbb{CP}^{n-1}$, on for each pair of distinct real numbers.
What about the other cases?
Hint: When the eigenvalues $\lambda_1 < \ldots < \lambda_n$ are all distinct, any element in $\mathcal{H}_\lambda$ defines a family of pairwise orthogonal lines in $\mathbb{C}^n$: its eigenspaces.

(e) Show that, for any skew-hermitian matrix $X \in \mathfrak{u}(n)$, the vector field on $\mathcal{H}$ generated by $X \in \mathfrak{u}(n)$ for the $\mathrm{U}(n)$-action by conjugation is $X^\#_\xi = [X, \xi]$.

Lecture 2

Cotangent Bundles

We will now construct a major class of examples of symplectic forms. The *canonical forms* on cotangent bundles are relevant for several branches, including analysis of differential operators, dynamical systems and classical mechanics.

2.1 Tautological and Canonical Forms

Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a coordinate chart for X, with associated cotangent coordinates[1] $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$. Define a 2-form ω on

[1]Let X be any n-dimensional manifold and $M = T^*X$ its cotangent bundle. If the manifold structure on X is described by coordinate charts $(\mathcal{U}, x_1, \ldots, x_n)$ with $x_i : \mathcal{U} \to \mathbb{R}$, then at any $x \in \mathcal{U}$, the differentials $(dx_1)_x, \ldots (dx_n)_x$ form a basis of T_x^*X. Namely, if $\xi \in T_x^*X$, then $\xi = \sum_{i=1}^n \xi_i (dx_i)_x$ for some real coefficients $\xi_1, \ldots, \xi_n$. This induces a map

$$\begin{array}{ccc} T^*\mathcal{U} & \longrightarrow & \mathbb{R}^{2n} \\ (x, \xi) & \longmapsto & (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) \ . \end{array}$$

The chart $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ is a coordinate chart for T^*X; the coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ are the **cotangent coordinates** associated to the coordinates $x_1, \ldots, x_n$ on $\mathcal{U}$. The transition functions on the overlaps are smooth: given two charts $(\mathcal{U}, x_1, \ldots, x_n)$, $(\mathcal{U}', x_1', \ldots, x_n')$, and $x \in \mathcal{U} \cap \mathcal{U}'$, if $\xi \in T_x^*X$, then

$$\xi = \sum_{i=1}^n \xi_i \, (dx_i)_x = \sum_{i,j} \xi_i \left(\frac{\partial x_i}{\partial x_j'} \right) (dx_j')_x = \sum_{j=1}^n \xi_j' \, (dx_j')_x$$

$T^*\mathcal{U}$ by

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i \ .$$

In order to check that this definition is coordinate-independent, consider the 1-form on $T^*\mathcal{U}$

$$\alpha = \sum_{i=1}^{n} \xi_i \ dx_i \ .$$

Clearly, $\omega = -d\alpha$.

Claim. The form α is intrinsically defined (and hence the form ω is also intrinsically defined) .

Proof. Let $(\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ and $(\mathcal{U}', x_1', \ldots, x_n', \xi_1', \ldots, \xi_n')$ be two cotangent coordinate charts. On $\mathcal{U} \cap \mathcal{U}'$, the two sets of coordinates are related by $\xi_j' = \sum_i \xi_i \left(\frac{\partial x_i}{\partial x_j'}\right)$. Since $dx_j' = \sum_i \left(\frac{\partial x_j'}{\partial x_i}\right) dx_i$, we have

$$\alpha = \sum_i \xi_i dx_i = \sum_j \xi_j' dx_j' = \alpha' \ .$$

□

The 1-form α is the **tautological form** and the 2-form ω is the **canonical symplectic form**. Next we provide an alternative proof of the intrinsic character of these forms. Let

$$\begin{array}{ccc} M = T^*X & \qquad p = (x, \xi) & \qquad \xi \in T_x^*X \\ \downarrow \pi & \downarrow & \\ X & x & \end{array}$$

be the natural projection. The **tautological 1-form** α may be defined pointwise by

$$\alpha_p = (d\pi_p)^* \xi \quad \in T_p^* M \ ,$$

---

where $\xi_j' = \sum_i \xi_i \left(\frac{\partial x_i}{\partial x_j'}\right)$ is smooth. Hence, T^*X is a $2n$-dimensional manifold.

where $(d\pi_p)^*$ is the transpose of $d\pi_p$, that is, $(d\pi_p)^*\xi = \xi \circ d\pi_p$:

$$\begin{array}{lll} p = (x,\xi) & T_pM & T_p^*M \\ \downarrow \pi & \downarrow d\pi_p & \uparrow (d\pi_p)^* \\ x & T_xX & T_x^*X \end{array}$$

Equivalently, $\alpha_p(v) = \xi\Big((d\pi_p)v\Big)$, for $v \in T_pM$.

Exercise 7
Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a chart on X with associated cotangent coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$. Show that on $T^*\mathcal{U}$, $\alpha = \sum_{i=1}^{n} \xi_i \, dx_i$.

The **canonical symplectic 2-form** ω on T^*X is defined as

$$\omega = -d\alpha \ .$$

Locally, $\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i$.

Exercise 8
Show that the tautological 1-form α is uniquely characterized by the property that, for every 1-form $\mu : X \to T^*X$, $\mu^*\alpha = \mu$. (See Section 2.4.)

2.2 Naturality of the Canonical Forms

Let X_1 and X_2 be n-dimensional manifolds with cotangent bundles $M_1 = T^*X_1$ and $M_2 = T^*X_2$, and tautological 1-forms α_1 and α_2. Suppose that $f : X_1 \to X_2$ is a diffeomorphism. Then there is a natural diffeomorphism

$$f_\sharp : M_1 \to M_2$$

which **lifts** f; namely, if $p_1 = (x_1, \xi_1) \in M_1$ for $x_1 \in X_1$ and $\xi_1 \in T_{x_1}^*X_1$, then we define

$$f_\sharp(p_1) = p_2 = (x_2, \xi_2) \ , \qquad \text{with} \left\{ \begin{array}{l} x_2 = f(x_1) \in X_2 \text{ and} \\ \xi_1 = (df_{x_1})^*\xi_2 \ , \end{array} \right.$$

where $(df_{x_1})^* : T_{x_2}^*X_2 \xrightarrow{\simeq} T_{x_1}^*X_1$, so $f_\sharp|_{T_{x_1}^*}$ is the inverse map of $(df_{x_1})^*$.

Exercise 9
Check that $f_\sharp$ is a diffeomorphism. Here are some hints:

1. $$\begin{array}{ccc} M_1 & \xrightarrow{f_\sharp} & M_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$ commutes;

2. $f_\sharp : M_1 \to M_2$ is bijective;

3. $f_\sharp$ and $f_\sharp^{-1}$ are smooth.

Proposition 2.1 *The lift $f_\sharp$ of a diffeomorphism $f : X_1 \to X_2$ pulls the tautological form on T^*X_2 back to the tautological form on T^*X_1, i.e.,*

$$(f_\sharp)^*\alpha_2 = \alpha_1 \ .$$

Proof. At $p_1 = (x_1, \xi_1) \in M_1$, this identity says that

$$(df_\sharp)^*_{p_1} (\alpha_2)_{p_2} = (\alpha_1)_{p_1} \qquad (\star)$$

where $p_2 = f_\sharp(p_1)$. Using the following facts,

- definition of $f_\sharp$:
  $p_2 = f_\sharp(p_1) \iff p_2 = (x_2, \xi_2)$ with $x_2 = f(x_1)$, $(df_{x_1})^*\xi_2 = \xi_1$,
- definition of tautological 1-form:
  $(\alpha_1)_{p_1} = (d\pi_1)^*_{p_1}\xi_1$ and $(\alpha_2)_{p_2} = (d\pi_2)^*_{p_2}\xi_2$,
- the diagram $$\begin{array}{ccc} M_1 & \xrightarrow{f_\sharp} & M_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$ commutes,

the proof of $(\star)$ is:

$$\begin{array}{rclcl} (df_\sharp)^*_{p_1}(\alpha_2)_{p_2} & = & (df_\sharp)^*_{p_1}(d\pi_2)^*_{p_2}\xi_2 & = & (d(\pi_2 \circ f_\sharp))^*_{p_1}\,\xi_2 \\ & = & (d(f \circ \pi_1))^*_{p_1}\,\xi_2 & = & (d\pi_1)^*_{p_1}(df)^*_{x_1}\xi_2 \\ & = & (d\pi_1)^*_{p_1}\xi_1 & = & (\alpha_1)_{p_1} \ . \end{array}$$

□

Corollary 2.2 *The lift $f_\sharp$ of a diffeomorphism $f : X_1 \to X_2$ is a symplectomorphism, i.e., $(f_\sharp)^*\omega_2 = \omega_1$, where ω_1, ω_2 are the canonical symplectic forms.*

In summary, a diffeomorphism of manifolds induces a canonical symplectomorphism of cotangent bundles:

$$\begin{array}{rccc} f_\sharp : & T^*X_1 & \longrightarrow & T^*X_2 \\ & & \uparrow & \\ f : & X_1 & \longrightarrow & X_2 \end{array}$$

Example. Let $X_1 = X_2 = S^1$. Then T^*S^1 is an infinite cylinder $S^1 \times \mathbb{R}$. The canonical 2-form ω is the area form $\omega = d\theta \wedge d\xi$. If $f : S^1 \to S^1$ is any diffeomorphism, then $f_\sharp : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ is a symplectomorphism, i.e., is an area-preserving diffeomorphism of the cylinder. $\diamondsuit$

If $f : X_1 \to X_2$ and $g : X_2 \to X_3$ are diffeomorphisms, then $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$. In terms of the group $\mathrm{Diff}(X)$ of diffeomorphisms of X and the group $\mathrm{Sympl}(M,\omega)$ of symplectomorphisms of (M,ω), we say that the map

$$\begin{array}{ccc} \mathrm{Diff}(X) & \longrightarrow & \mathrm{Sympl}(M,\omega) \\ f & \longmapsto & f_\sharp \end{array}$$

is a group homomorphism. This map is clearly injective. Is it surjective? Do all symplectomorphisms $T^*X \to T^*X$ come from diffeomorphisms $X \to X$? No: for instance, translation along cotangent fibers is not induced by a diffeomorphism of the base manifold. A criterion for which symplectomorphisms arise as lifts of diffeomorphisms is discussed in the next section.

2.3 Symplectomorphisms of T^*X

Let (M,ω) be a symplectic manifold, and let α be a 1-form such that

$$\omega = -d\alpha \ .$$

There exists a unique vector field v such that its interior product with ω is α, i.e., $\imath_v\omega = -\alpha$.

Proposition 2.3 *If g is a symplectomorphism which preserves α (that is, $g^*\alpha = \alpha$), then g commutes with the one-parameter group of diffeomorphisms generated by v, i.e.,*

$$(\exp tv) \circ g = g \circ (\exp tv) \ .$$

Proof. Recall that, for $p \in M$, $(\exp tv)(p)$ is the *unique* curve in M solving the ordinary differential equation

$$\begin{cases} \frac{d}{dt}(\exp tv(p)) = v(\exp tv(p)) \\ (\exp tv)(p)|_{t=0} = p \end{cases}$$

for t in some neighborhood of 0. From this is follows that $g \circ (\exp tv) \circ g^{-1}$ must be the one-parameter group of diffeomorphisms generated by $g_* v$. (The push-forward of v by g is defined by $(g_* v)_{g(p)} = dg_p(v_p)$.) Finally we have that $g_* v = v$, i.e., that g preserves v. □

Let X be an arbitrary n-dimensional manifold, and let $M = T^*X$. Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a coordinate system on X, and let $x_1, \ldots, x_n$, $\xi_1, \ldots, \xi_n$ be the corresponding coordinates on $T^*\mathcal{U}$. When α is the tautological 1-form on M (which, in these coordinates, is $\sum \xi_i \, dx_i$), the vector field v above is just the vector field $\sum \xi_i \frac{\partial}{\partial \xi_i}$. Let $\exp tv$, $-\infty < t < \infty$, be the one-parameter group of diffeomorphisms generated by v.

Exercise 10
Show that, for every point $p = (x, \xi)$ in M,

$$(\exp tv)(p) = p_t \quad \text{where} \quad p_t = (x, e^t \xi) \ .$$

If g is a symplectomorphism of $M = T^*X$ which preserves α, then we must have that

$$g(x, \xi) = (y, \eta) \quad \Longrightarrow \quad g(x, \lambda\xi) = (y, \lambda\eta)$$

for all $(x, \xi) \in M$ and $\lambda \in \mathbb{R}$. In fact, if $g(p) = q$ where $p = (x, \xi)$ and $q = (y, \eta)$, this assertion follows from a combination of the identity

$$(dg_p)^* \alpha_q = \alpha_p$$

with the identity

$$d\pi_q \circ dg_p = df_x \circ d\pi_p \ .$$

(The first identity expresses the fact that $g^*\alpha = \alpha$, and the second identity is obtained by differentiating both sides of the equation $\pi \circ g = f \circ \pi$ at p.) We conclude that g has to preserve the cotangent fibration, i.e., there exists a diffeomorphism $f : X \to X$ such that $\pi \circ g = f \circ \pi$, where $\pi : M \to X$ is the projection map $\pi(x, \xi) = x$. Moreover, $g = f_\#$, the map $f_\#$ being the symplectomorphism of M lifting f. Hence, the symplectomorphisms of T^*X of the form $f_\#$ are those which preserve the tautological 1-form α.

Here is a different class of symplectomorphisms of $M = T^*X$. Let h be a smooth function on X. Define $\tau_h : M \to M$ by setting

$$\tau_h(x, \xi) = (x, \xi + dh_x) \ .$$

Then

$$\tau_h^*\alpha = \alpha + \pi^*dh$$

where π is the projection map

$$\begin{array}{ccc} M & & (x,\xi) \\ \downarrow \pi & & \downarrow \\ X & & x \end{array}$$

Therefore,

$$\tau_h^*\omega = \omega \ ,$$

so all such τ_h are symplectomorphisms.

2.4 Lagrangian Submanifolds of T^*X

Let (M, ω) be a $2n$-dimensional symplectic manifold.

Definition 2.4 *A submanifold Y of M is a* **lagrangian submanifold** *if, at each $p \in Y$, T_pY is a lagrangian subspace of T_pM, i.e., $\omega_p|_{T_pY} \equiv 0$ and $\dim T_pY = \frac{1}{2}\dim T_pM$. Equivalently, if $i : Y \hookrightarrow M$ is the inclusion map, then Y is* **lagrangian** *if and only if $i^*\omega = 0$ and $\dim Y = \frac{1}{2}\dim M$.*

Let X be an n-dimensional manifold, with $M = T^*X$ its cotangent bundle. If $x_1, \ldots, x_n$ are coordinates on $U \subseteq X$, with associated cotangent coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ on T^*U, then the tautological 1-form on T^*X is

$$\alpha = \sum \xi_i dx_i$$

and the canonical 2-form on T^*X is

$$\omega = -d\alpha = \sum dx_i \wedge d\xi_i \ .$$

The **zero section** of T^*X,

$$X_0 := \{(x, \xi) \in T^*X \mid \xi = 0 \text{ in } T^*_x X\} \ ,$$

is an n-dimensional submanifold of T^*X whose intersection with T^*U is given by the equations $\xi_1 = \ldots = \xi_n = 0$. Clearly $\alpha = \sum \xi_i dx_i$ vanishes on $X_0 \cap T^*U$. In particular, if $i_0 : X_0 \hookrightarrow T^*X$ is the inclusion map, we have $i_0^*\alpha = 0$. Hence, $i_0^*\omega = i_0^* d\alpha = 0$, and X_0 is lagrangian.

What are all the lagrangian submanifolds of T^*X which are "C^1-close to X_0"?

Let X_μ be (the image of) another section, that is, an n-dimensional submanifold of T^*X of the form

$$X_\mu = \{(x, \mu_x) \mid x \in X, \ \mu_x \in T^*_x X\} \qquad (\star\star)$$

where the covector μ_x depends smoothly on x, and $\mu : X \to T^*X$ is a de Rham 1-form. Relative to the inclusion $i : X_\mu \hookrightarrow T^*X$ and the cotangent projection $\pi : T^*X \to X$, X_μ is of the form $(\star\star)$ if and only if $\pi \circ i : X_\mu \to X$ is a diffeomorphism.

When is such an X_μ lagrangian?

Proposition 2.5 *Let X_μ be of the form $(\star\star)$, and let μ be the associated de Rham 1-form. Denote by $s_\mu : X \to T^*X$, $x \mapsto (x, \mu_x)$, the 1-form μ regarded exclusively as a map. Notice that the image of s_μ is X_μ. Let α be the tautological 1-form on T^*X. Then*

$$s_\mu^*\alpha = \mu \ .$$

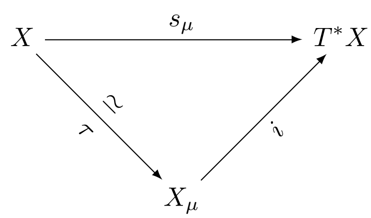
Proof. By definition of the tautological form α, $\alpha_p = (d\pi_p)^*\xi$ at $p = (x, \xi) \in M$. For $p = s_\mu(x) = (x, \mu_x)$, we have $\alpha_p = (d\pi_p)^*\mu_x$.

Then

$$\begin{aligned}(s_\mu^*\alpha)_x &= (ds_\mu)_x^*\alpha_p \\ &= (ds_\mu)_x^*(d\pi_p)^*\mu_x \\ &= (d\underbrace{(\pi\circ s_\mu)}_{\mathrm{id}_X})_x^*\mu_x = \mu_x\ .\end{aligned}$$

□

Suppose that X_μ is an n-dimensional submanifold of T^*X of the form $(\star\star)$, with associated de Rham 1-form μ. Then $s_\mu : X \to T^*X$ is an embedding with image X_μ, and there is a diffeomorphism $\tau : X \to X_\mu$, $\tau(x) := (x, \mu_x)$, such that the following diagram commutes.



We want to express the condition of X_μ being lagrangian in terms of the form μ:

$$\begin{aligned}X_\mu \text{ is lagrangian} &\iff i^*d\alpha = 0 \\ &\iff \tau^*i^*d\alpha = 0 \\ &\iff (i\circ\tau)^*d\alpha = 0 \\ &\iff s_\mu^*d\alpha = 0 \\ &\iff ds_\mu^*\alpha = 0 \\ &\iff d\mu = 0 \\ &\iff \mu \text{ is closed}\ .\end{aligned}$$

Therefore, there is a one-to-one correspondence between the set of lagrangian submanifolds of T^*X of the form $(\star\star)$ and the set of closed 1-forms on X.

When X is simply connected, $H^1_{\mathrm{deRham}}(X) = 0$, so every closed 1-form μ is equal to df for some $f \in C^\infty(X)$. Any such primitive f is then called a **generating function** for the lagrangian submanifold X_μ associated to μ. (Two functions generate the same lagrangian submanifold if and only if they differ by a locally constant function.)

On arbitrary manifolds X, functions $f \in C^\infty(X)$ originate lagrangian submanifolds as images of df.

Exercise 11
Check that, if X is compact (and not just one point) and $f \in C^\infty(X)$, then $\#(X_{df} \cap X_0) \geq 2$.

2.5 Conormal Bundles

There are lots of lagrangian submanifolds of T^*X not covered by the description in terms of closed 1-forms from the previous section, starting with the cotangent fibers.

Let S be any k-dimensional submanifold of an n-dimensional manifold X.

Definition 2.6 *The* **conormal space** *at* $x \in S$ *is*

$$N_x^* S = \{\xi \in T_x^* X \mid \xi(v) = 0 \ , \ \textit{for all } v \in T_x S\} \ .$$

The **conormal bundle** *of* S *is*

$$N^* S = \{(x, \xi) \in T^* X \mid x \in S, \ \xi \in N_x^* S\} \ .$$

By using coordinates on X adapted[2] to S, one sees that the conormal bundle N^*S is an n-dimensional submanifold of T^*X.

Proposition 2.7 *Let* $i : N^*S \hookrightarrow T^*X$ *be the inclusion, and let* α *be the tautological 1-form on* T^*X. *Then*

$$i^*\alpha = 0 \ .$$

Proof. Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a coordinate system on X centered at $x \in S$ and adapted to S, so that $\mathcal{U} \cap S$ is described by $x_{k+1} = \ldots = x_n = 0$. Let $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be the associated cotangent coordinate system. The submanifold $N^*S \cap T^*\mathcal{U}$ is then described by

$$x_{k+1} = \ldots = x_n = 0 \qquad \text{and} \qquad \xi_1 = \ldots = \xi_k = 0 \ .$$

[2] A coordinate chart $(\mathcal{U}, x_1, \ldots, x_n)$ on X is adapted to a k-dimensional submanifold S if $S \cap \mathcal{U}$ is described by $x_{k+1} = \ldots = x_n = 0$.

Since $\alpha = \sum \xi_i dx_i$ on $T^*\mathcal{U}$, we conclude that, at $p \in N^*S$,

$$(i^*\alpha)_p = \alpha_p|_{T_p(N^*S)} = \sum_{i>k} \xi_i dx_i \Big|_{\text{span}\{\frac{\partial}{\partial x_i}, i \leq k\}} = 0 \ .$$

□

Corollary 2.8 *For any submanifold $S \subset X$, the conormal bundle N^*S is a lagrangian submanifold of T^*X.*

Taking $S = \{x\}$ to be one point, the conormal bundle $L = N^*S = T_x^*X$ is a cotangent fiber. Taking $S = X$, the conormal bundle $L = X_0$ is the zero section of T^*X.

2.6 Lagrangian Complements

Normal neighborhoods of lagrangian submanifolds are described by the theorems in the following two sections. It was proved by Weinstein [44] that the conclusion of the Moser local theorem (Theorem 1.11) still holds if we assume instead

Hypothesis: X is an n-dimensional submanifold with
$i^*\omega_0 = i^*\omega_1 = 0$ where $i : X \hookrightarrow M$ is inclusion, i.e.,
X is a submanifold lagrangian for ω_0 and ω_1 .

We need some algebra for the Weinstein theorem.

Suppose that U and W are n-dimensional vector spaces, and $\Omega : U \times W \to \mathbb{R}$ is a bilinear pairing; the map Ω gives rise to a linear map $\widetilde{\Omega} : U \to W^*$, $\widetilde{\Omega}(u) = \Omega(u, \cdot)$. Then Ω is nondegenerate if and only if $\widetilde{\Omega}$ is bijective.

Proposition 2.9 *Suppose that (V, Ω) is a $2n$-dimensional symplectic vector space and U is a lagrangian subspace of (V, Ω) (i.e., $\Omega|_{U \times U} = 0$ and U is n-dimensional). Let W be any vector space complement to U, not necessarily lagrangian.*

Then from W we can canonically *build a* lagrangian *complement to U.*

Proof. The pairing Ω gives a nondegenerate pairing $U \times W \stackrel{\Omega'}{\to} \mathbb{R}$. Therefore, $\widetilde{\Omega}' : U \to W^*$ is bijective. We look for a lagrangian complement to U of the form

$$W' = \{w + Aw \mid w \in W\} ,$$

the map $A : W \to U$ being linear. For W' to be lagrangian we need

$$\forall\ w_1, w_2 \in W\ , \quad \Omega(w_1 + Aw_1, w_2 + Aw_2) = 0$$

$$\Longrightarrow\ \Omega(w_1, w_2) + \Omega(w_1, Aw_2) + \Omega(Aw_1, w_2) + \underbrace{\Omega(\underbrace{Aw_1, Aw_2}_{\in U})}_{0} = 0$$

$$\begin{aligned} \Longrightarrow\ \Omega(w_1, w_2) &= \Omega(Aw_2, w_1) - \Omega(Aw_1, w_2) \\ &= \widetilde{\Omega}'(Aw_2)(w_1) - \widetilde{\Omega}'(Aw_1)(w_2)\ . \end{aligned}$$

Let $A' = \widetilde{\Omega}' \circ A : W \to W^*$, and look for A' such that

$$\forall\ w_1, w_2 \in W\ , \quad \Omega(w_1, w_2) = A'(w_2)(w_1) - A'(w_1)(w_2)\ .$$

The canonical choice is $A'(w) = -\frac{1}{2}\Omega(w, \cdot)$. Then set $A = (\widetilde{\Omega}')^{-1} \circ A'$. $\square$

Proposition 2.10 *Let V be a $2n$-dimensional vector space, let Ω_0 and Ω_1 be symplectic forms in V, let U be a subspace of V lagrangian for Ω_0 and Ω_1, and let W be any complement to U in V. Then from W we can canonically construct a linear isomorphism $L : V \stackrel{\simeq}{\to} V$ such that $L|_U = \mathrm{Id}_U$ and $L^*\Omega_1 = \Omega_0$.*

Proof. From W we canonically obtain complements W_0 and W_1 to U in V such that W_0 is lagrangian for Ω_0 and W_1 is lagrangian for Ω_1. The nondegenerate bilinear pairings

$$\begin{matrix} W_0 \times U \stackrel{\Omega_0}{\longrightarrow} \mathbb{R} \\ W_1 \times U \stackrel{\Omega_1}{\longrightarrow} \mathbb{R} \end{matrix} \quad \text{give isomorphisms} \quad \begin{matrix} \widetilde{\Omega}_0 : W_0 \stackrel{\simeq}{\longrightarrow} U^* \\ \widetilde{\Omega}_1 : W_1 \stackrel{\simeq}{\longrightarrow} U^*\ . \end{matrix}$$

Consider the diagram

$$\begin{array}{ccc} W_0 & \stackrel{\widetilde{\Omega}_0}{\longrightarrow} & U^* \\ B \downarrow & & \downarrow \mathrm{id} \\ W_1 & \stackrel{\widetilde{\Omega}_1}{\longrightarrow} & U^* \end{array}$$

where the linear map B satisfies $\widetilde{\Omega}_1 \circ B = \widetilde{\Omega}_0$, i.e., $\Omega_0(w_0, u) = \Omega_1(Bw_0, u)$, $\forall w_0 \in W_0$, $\forall u \in U$. Extend B to the rest of V by setting it to be the identity on U:

$$L := \mathrm{Id}_U \oplus B : U \oplus W_0 \longrightarrow U \oplus W_1 \ .$$

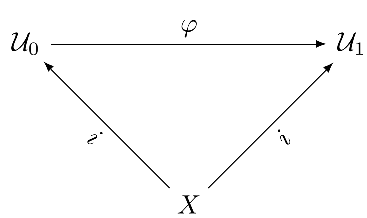
Finally, we check that $L^*\Omega_1 = \Omega_0$:

$$\begin{aligned}(L^*\Omega_1)(u \oplus w_0, u' \oplus w_0') &= \Omega_1(u \oplus Bw_0, u' \oplus Bw_0') \\ &= \Omega_1(u, Bw_0') + \Omega_1(Bw_0, u') \\ &= \Omega_0(u, w_0') + \Omega_0(w_0, u') \\ &= \Omega_0(u \oplus w_0, u' \oplus w_0') \ .\end{aligned}$$

□

2.7 Lagrangian Neighborhood Theorem

Theorem 2.11 (Weinstein Lagrangian Neighborhood Theorem [44]) *Let M be a $2n$-dimensional manifold, X a compact n-dimensional submanifold, $i : X \hookrightarrow M$ the inclusion map, and ω_0 and ω_1 symplectic forms on M such that $i^*\omega_0 = i^*\omega_1 = 0$, i.e., X is a lagrangian submanifold of both (M, ω_0) and (M, ω_1). Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that*



commutes *and* $\varphi^*\omega_1 = \omega_0$.

Proof. The proof of the Weinstein theorem uses the Whitney extension theorem (see Appendix A).

Put a riemannian metric g on M; at each $p \in M$, $g_p(\cdot, \cdot)$ is a positive-definite inner product. Fix $p \in X$, and let $V = T_pM$, $U = T_pX$ and $W = U^{\perp}$ the orthocomplement of U in V relative to $g_p(\cdot, \cdot)$.

Since $i^*\omega_0 = i^*\omega_1 = 0$, the space U is a lagrangian subspace of both $(V, \omega_0|_p)$ and $(V, \omega_1|_p)$. By symplectic linear algebra, we canonically get from $U^\perp$ a linear isomorphism $L_p : T_pM \to T_pM$, such that $L_p|_{T_pX} = \mathrm{Id}_{T_pX}$ and $L_p^*\omega_1|_p = \omega_0|_p$. L_p varies smoothly with respect to p since our recipe is canonical.

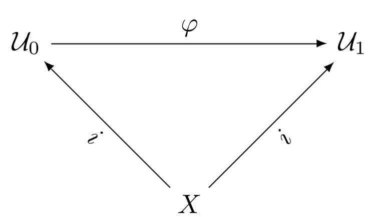
By the Whitney theorem (Theorem A.11), there are a neighborhood $\mathcal{N}$ of X and an embedding $h : \mathcal{N} \hookrightarrow M$ with $h|_X = \mathrm{id}_X$ and $dh_p = L_p$ for $p \in X$. Hence, at any $p \in X$,

$$(h^*\omega_1)_p = (dh_p)^*\omega_1|_p = L_p^*\omega_1|_p = \omega_0|_p \ .$$

Applying the Moser relative theorem (Theorem 1.11) to ω_0 and $h^*\omega_1$, we find a neighborhood $\mathcal{U}_0$ of X and an embedding $f : \mathcal{U}_0 \to \mathcal{N}$ such that $f|_X = \mathrm{id}_X$ and $f^*(h^*\omega_1) = \omega_0$ on $\mathcal{U}_o$. Set $\varphi = h \circ f$. $\square$

Theorem 2.11 has the following generalization; see, for instance, either of [22, 27, 46].

Theorem 2.12 (Coisotropic Embedding Theorem) *Let M be a manifold of dimension $2n$, X a submanifold of dimension $k \geq n$, $i : X \hookrightarrow M$ the inclusion map, and ω_0 and ω_1 symplectic forms on M, such that $i^*\omega_0 = i^*\omega_1$ and X is coisotropic for both (M, ω_0) and (M, ω_1). Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that*



commutes *and* $\varphi^*\omega_1 = \omega_0$.

2.8 Weinstein Tubular Neighborhood

Let (V, Ω) be a symplectic linear space, and let U be a lagrangian subspace.

Claim. There is a canonical nondegenerate bilinear pairing $\Omega' : V/U \times U \to \mathbb{R}$.

Proof. Define $\Omega'([v],u) = \Omega(v,u)$ where $[v]$ is the equivalence class of v in V/U. $\square$

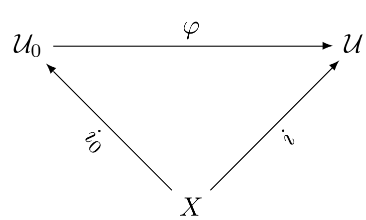
Consequently, we get that $\widetilde{\Omega}' : V/U \to U^*$ defined by $\widetilde{\Omega}'([v]) = \Omega'([v],\cdot)$ is an isomorphism, so that $V/U \simeq U^*$ are canonically identified.

In particular, if (M,ω) is a symplectic manifold, and X is a lagrangian submanifold, then T_xX is a lagrangian subspace of (T_xM, ω_x) for each $x \in X$. The space $N_xX := T_xM/T_xX$ is called the **normal space** of X at x. Since we have a canonical identification $N_xX \simeq T_x^*X$, we get:

Proposition 2.13 *The vector bundles NX and T^*X are canonically identified.*

Putting this observation together with the lagrangian neighborhood theorem, we arrive at:

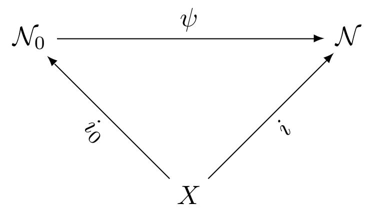
Theorem 2.14 (Weinstein Tubular Neighborhood Theorem) *Let (M,ω) be a symplectic manifold, X a compact lagrangian submanifold, ω_0 the canonical symplectic form on T^*X, $i_0 : X \hookrightarrow T^*X$ the lagrangian embedding as the zero section, and $i : X \hookrightarrow M$ the lagrangian embedding given by inclusion. Then there are neighborhoods $\mathcal{U}_0$ of X in T^*X, $\mathcal{U}$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}$ such that*



commutes and $\varphi^*\omega = \omega_0$.

Proof. This proof relies on (1) the standard tubular neighborhood theorem (see Appendix A), and (2) the Weinstein lagrangian neighborhood theorem.

1. Since $NX \simeq T^*X$, we can find a neighborhood $\mathcal{N}_0$ of X in T^*X, a neighborhood $\mathcal{N}$ of X in M, and a diffeomorphism $\psi : \mathcal{N}_0 \to \mathcal{N}$ such that

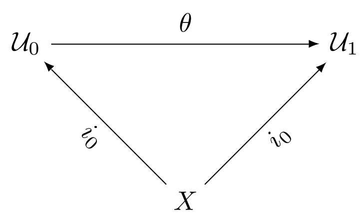


commutes .

Let $\left.\begin{array}{l}\omega_0 = \text{canonical form on } T^*X \\ \omega_1 = \psi^*\omega\end{array}\right\}$ symplectic forms on $\mathcal{N}_0$.

The submanifold X is lagrangian for both ω_0 and ω_1.

2. There exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in $\mathcal{N}_0$ and a diffeomorphism $\theta : \mathcal{U}_0 \to \mathcal{U}_1$ such that



commutes and $\theta^*\omega_1 = \omega_0$.

Take $\varphi = \psi \circ \theta$ and $\mathcal{U} = \varphi(\mathcal{U}_0)$. Check that $\varphi^*\omega = \theta^* \underbrace{\psi^*\omega}_{\omega_1} = \omega_0$.

□

Remark. Theorem 2.14 classifies lagrangian embeddings: up to local symplectomorphism, the set of lagrangian embeddings is the set of embeddings of manifolds into their cotangent bundles as zero sections.

The classification of *isotropic* embeddings was also carried out by Weinstein in [45, 46]. An **isotropic embedding** of a manifold X into a symplectic manifold (M, ω) is a closed embedding $i : X \hookrightarrow M$

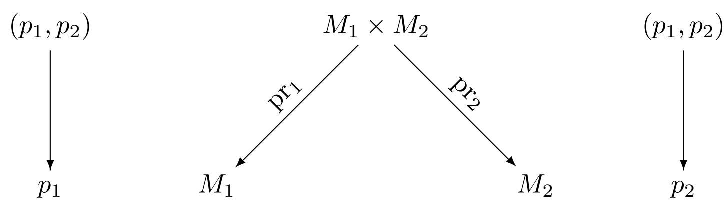
such that $i^*\omega = 0$. Weinstein showed that neighbourhood equivalence of isotropic embeddings is in one-to-one correspondence with isomorphism classes of symplectic vector bundles.

The classification of *coisotropic* embeddings is due to Gotay [22]. A **coisotropic embedding** of a manifold X carrying a closed 2-form α of constant rank into a symplectic manifold (M, ω) is an embedding $i : X \hookrightarrow M$ such that $i^*\omega = \alpha$ and $i(X)$ is coisotropic as a submanifold of M. Let E be the **characteristic distribution** of a closed form α of constant rank on X, i.e., E_p is the kernel of α_p at $p \in X$. Gotay showed that then E^* carries a symplectic structure in a neighbourhood of the zero section, such that X embeds coisotropically onto this zero section, and, moreover every coisotropic embedding is equivalent to this in some neighbourhood of the zero section. $\diamondsuit$

2.9 Symplectomorphisms as Lagrangians

Lagarangian submanifolds are important to study symplectomorphisms, as will be explored in the next lecture.

Let (M_1, ω_1) and (M_2, ω_2) be two $2n$-dimensional symplectic manifolds. Given a diffeomorphism $\varphi : M_1 \xrightarrow{\simeq} M_2$, when is it a symplectomorphism? (I.e., when is $\varphi^*\omega_2 = \omega_1$?) Consider the two projection maps



Then $\omega = (\mathrm{pr}_1)^*\omega_1 + (\mathrm{pr}_2)^*\omega_2$ is a 2-form on $M_1 \times M_2$ which is closed,

$$d\omega = (\mathrm{pr}_1)^* \underbrace{d\omega_1}_{0} + (\mathrm{pr}_2)^* \underbrace{d\omega_2}_{0} = 0 \ ,$$

and symplectic,

$$\omega^{2n} = \binom{2n}{n} \Big((\mathrm{pr}_1)^*\omega_1\Big)^n \wedge \Big((\mathrm{pr}_2)^*\omega_2\Big)^n \neq 0 \ .$$

More generally, if $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}$, then $\lambda_1(\mathrm{pr}_1)^*\omega_1 + \lambda_2(\mathrm{pr}_2)^*\omega_2$ is also a symplectic form on $M_1 \times M_2$. Take $\lambda_1 = 1$, $\lambda_2 = -1$ to obtain the **twisted product form** on $M_1 \times M_2$:

$$\widetilde{\omega} = (\mathrm{pr}_1)^*\omega_1 - (\mathrm{pr}_2)^*\omega_2 \ .$$

The graph of a diffeomorphism $\varphi : M_1 \stackrel{\simeq}{\longrightarrow} M_2$ is the $2n$-dimensional submanifold of $M_1 \times M_2$:

$$\Gamma_\varphi := \mathrm{Graph}\,\varphi = \{(p, \varphi(p)) \mid p \in M_1\} \ .$$

The submanifold Γ_φ is an embedded image of M_1 in $M_1 \times M_2$, the embedding being the map

$$\begin{array}{rccc} \gamma : & M_1 & \longrightarrow & M_1 \times M_2 \\ & p & \longmapsto & (p, \varphi(p)) \ . \end{array}$$

Proposition 2.15 *A diffeomorphism φ is a symplectomorphism if and only if Γ_φ is a lagrangian submanifold of $(M_1 \times M_2, \widetilde{\omega})$.*

Proof. The graph Γ_φ is lagrangian if and only if $\gamma^*\widetilde{\omega} = 0$. But

$$\begin{array}{rcl} \gamma^*\widetilde{\omega} & = & \gamma^*\ \mathrm{pr}_1^*\ \omega_1 - \gamma^*\ \mathrm{pr}_2^*\ \omega_2 \\ & = & (\mathrm{pr}_1 \circ \gamma)^*\omega_1 - (\mathrm{pr}_2 \circ \gamma)^*\omega_2 \end{array}$$

and $\mathrm{pr}_1 \circ \gamma$ is the identity map on M_1 whereas $\mathrm{pr}_2 \circ \gamma = \varphi$. Therefore,

$$\gamma^*\widetilde{\omega} = 0 \quad \Longleftrightarrow \quad \varphi^*\omega_2 = \omega_1 \ .$$

□

Lecture 3

Generating Functions

Generating functions provide a method for producing symplectomorphisms via lagrangian submanifolds. We will illustrate their use in riemannian geometry and dynamics. We conclude with an application to the study of the group of symplectomorphisms and to the problem of the existence of fixed points, whose first instance is the Poincaré-Birkhoff theorem.

3.1 Constructing Symplectomorphisms

Let X_1, X_2 be n-dimensional manifolds, with cotangent bundles $M_1 = T^*X_1$, $M_2 = T^*X_2$, tautological 1-forms α_1, α_2, and canonical 2-forms ω_1, ω_2.

Under the natural identification

$$M_1 \times M_2 = T^*X_1 \times T^*X_2 \simeq T^*(X_1 \times X_2) \ ,$$

the tautological 1-form on $T^*(X_1 \times X_2)$ is

$$\alpha = (\mathrm{pr}_1)^*\alpha_1 + (\mathrm{pr}_2)^*\alpha_2 \ ,$$

where $\mathrm{pr}_i : M_1 \times M_2 \to M_i$, $i = 1, 2$ are the two projections. The canonical 2-form on $T^*(X_1 \times X_2)$ is

$$\omega = -d\alpha = -d\mathrm{pr}_1^*\alpha_1 - d\mathrm{pr}_2^*\alpha_2 = \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2 \ .$$

In order to describe the twisted form $\widetilde{\omega} = \mathrm{pr}_1^*\omega_1 - \mathrm{pr}_2^*\omega_2$, we define an involution of $M_2 = T^*X_2$ by

$$\begin{array}{rccc} \sigma_2 : & M_2 & \longrightarrow & M_2 \\ & (x_2, \xi_2) & \longmapsto & (x_2, -\xi_2) \end{array}$$

which yields $\sigma_2^*\alpha_2 = -\alpha_2$. Let $\sigma = \mathrm{id}_{M_1} \times \sigma_2 : M_1 \times M_2 \to M_1 \times M_2$. Then

$$\sigma^*\widetilde{\omega} = \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2 = \omega \ .$$

If Y is a lagrangian submanifold of $(M_1 \times M_2, \omega)$, then its "twist" $Y^\sigma := \sigma(Y)$ is a lagrangian submanifold of $(M_1 \times M_2, \widetilde{\omega})$.

Recipe for producing symplectomorphisms $M_1 = T^*X_1 \to M_2 = T^*X_2$**:**

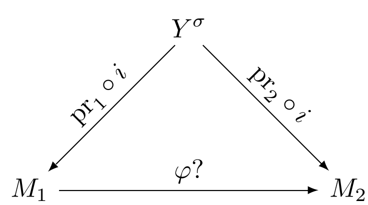
1. Start with a lagrangian submanifold Y of $(M_1 \times M_2, \omega)$.

2. Twist it to obtain a lagrangian submanifold Y^σ of $(M_1 \times M_2, \widetilde{\omega})$.

3. Check whether Y^σ is the graph of some diffeomorphism $\varphi : M_1 \to M_2$.

4. If it is, then φ is a symplectomorphism by Section 2.9.

Let $i : Y^\sigma \hookrightarrow M_1 \times M_2$ be the inclusion map



Step 3 amounts to checking whether $\mathrm{pr}_1 \circ i$ and $\mathrm{pr}_2 \circ i$ are diffeomorphisms. If yes, then $\varphi := (\mathrm{pr}_2 \circ i) \circ (\mathrm{pr}_1 \circ i)^{-1}$ is a diffeomorphism.

In order to obtain lagrangian submanifolds of $M_1 \times M_2 \simeq T^*(X_1 \times X_2)$, we can use the *method of generating functions*.

3.2 Method of Generating Functions

For any $f \in C^\infty(X_1 \times X_2)$, df is a closed 1-form on $X_1 \times X_2$. The **lagrangian submanifold generated by** f is

$$Y_f := \{((x,y),(df)_{(x,y)}) \mid (x,y) \in X_1 \times X_2\} \ .$$

We adopt the notation

$$\begin{array}{rcl} d_x f & := & (df)_{(x,y)} \text{ projected to } T^*_x X_1 \times \{0\}, \\ d_y f & := & (df)_{(x,y)} \text{ projected to } \{0\} \times T^*_y X_2 \ , \end{array}$$

which enables us to write

$$Y_f = \{(x,y,d_x f, d_y f) \mid (x,y) \in X_1 \times X_2\}$$

and

$$Y_f^\sigma = \{(x,y,d_x f, -d_y f) \mid (x,y) \in X_1 \times X_2\} \ .$$

When Y_f^σ is in fact the graph of a diffeomorphism $\varphi : M_1 \to M_2$, we call φ the **symplectomorphism generated by** f, and call f the **generating function**, of $\varphi : M_1 \to M_2$.

So when is Y_f^σ the graph of a diffeomorphism $\varphi : M_1 \to M_2$?

Let $(\mathcal{U}_1, x_1, \ldots, x_n), (\mathcal{U}_2, y_1, \ldots, y_n)$ be coordinate charts for X_1 and X_2, with associated charts $(T^*\mathcal{U}_1, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, $(T^*\mathcal{U}_2, y_1, \ldots, y_n, \eta_1, \ldots, \eta_n)$ for M_1 and M_2. The set Y_f^σ is the graph of $\varphi : M_1 \to M_2$ if and only if, for any $(x,\xi) \in M_1$ and $(y,\eta) \in M_2$, we have

$$\varphi(x,\xi) = (y,\eta) \iff \xi = d_x f \text{ and } \eta = -d_y f \ .$$

Therefore, given a point $(x,\xi) \in M_1$, to find its image $(y,\eta) = \varphi(x,\xi)$ we must solve the "Hamilton" equations

$$\begin{cases} \xi_i = \dfrac{\partial f}{\partial x_i}(x,y) & (\star) \\ \eta_i = -\dfrac{\partial f}{\partial y_i}(x,y) \ . & (\star\star) \end{cases}$$

If there is a solution $y = \varphi_1(x,\xi)$ of $(\star)$, we may feed it to $(\star\star)$ thus obtaining $\eta = \varphi_2(x,\xi)$, so that $\varphi(x,\xi) = (\varphi_1(x,\xi), \varphi_2(x,\xi))$. Now by the implicit function theorem, in order to solve ($\star$) locally for y in terms of x and ξ, we need the condition

$$\det \left[\frac{\partial}{\partial y_j} \left(\frac{\partial f}{\partial x_i} \right) \right]_{i,j=1}^{n} \neq 0 \ .$$

This is a necessary local condition for f to generate a symplectomorphism φ. Locally this is also sufficient, but globally there is the usual bijectivity issue.

Example. Let $X_1 = \mathcal{U}_1 \simeq \mathbb{R}^n$, $X_2 = \mathcal{U}_2 \simeq \mathbb{R}^n$, and $f(x,y) = -\frac{|x-y|^2}{2}$, the square of euclidean distance up to a constant.

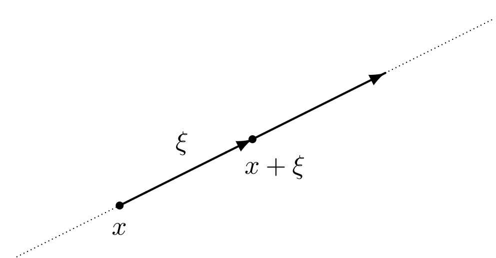
The "Hamilton" equations are

$$\begin{cases} \xi_i = \dfrac{\partial f}{\partial x_i} = y_i - x_i \\ \eta_i = -\dfrac{\partial f}{\partial y_i} = y_i - x_i \end{cases} \iff \begin{cases} y_i = x_i + \xi_i \\ \eta_i = \xi_i \ . \end{cases}$$

The symplectomorphism generated by f is

$$\varphi(x,\xi) = (x + \xi, \xi) \ .$$

If we use the euclidean inner product to identify $T^*\mathbb{R}^n$ with $T\mathbb{R}^n$, and hence regard φ as $\widetilde{\varphi} : T\mathbb{R}^n \to T\mathbb{R}^n$ and interpret ξ as the velocity vector, then the symplectomorphism φ corresponds to free translational motion in euclidean space.



$\diamondsuit$

3.3 Riemannian Distance

Let V be an n-dimensional vector space. A **positive inner product** G on V is a bilinear map $G : V \times V \to \mathbb{R}$ which is

$$\begin{array}{lll} \text{symmetric :} & G(v,w) = G(w,v) \ , & \text{and} \\ \text{positive-definite :} & G(v,v) > 0 \quad \text{when} & v \neq 0 \ . \end{array}$$

Definition 3.1 *A* **riemannian metric** *on a manifold X is a function g which assigns to each point $x \in X$ a positive inner product g_x on T_xX.*

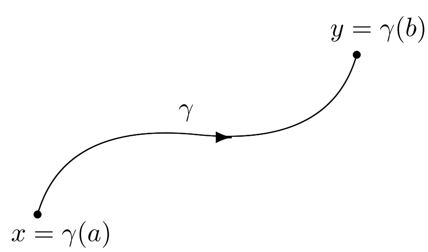
A riemannian metric g is **smooth** *if for every smooth vector field $v : X \to TX$ the real-valued function $x \mapsto g_x(v_x, v_x)$ is a smooth function on X.*

Definition 3.2 *A* **riemannian manifold** (X, g) *is a manifold X equipped with a smooth riemannian metric g.*

Let (X, g) be a riemannian manifold. The **arc-length** of a piecewise smooth curve $\gamma : [a, b] \to X$ is

$$\text{arc-length of } \gamma \ := \int_a^b \left| \frac{d\gamma}{dt} \right| dt \ , \quad \text{where} \quad \left| \frac{d\gamma}{dt} \right| \ := \ \sqrt{g_{\gamma(t)} \left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right)} \ .$$

By changing variables in the integral, we see that the arc-length of γ is independent of the parametrization of γ, i.e., if we reparametrize γ by $\tau : [a', b'] \to [a, b]$, the new curve $\gamma' = \gamma \circ \tau : [a', b'] \to X$ has the same arc-length.



A curve γ is called a curve of **constant velocity** when $\left|\frac{d\gamma}{dt}\right|$ is independent of t. Given any curve $\gamma : [a,b] \to X$ (with $\frac{d\gamma}{dt}$ never vanishing), there is a reparametrization $\tau : [a,b] \to [a,b]$ such that $\gamma \circ \tau : [a,b] \to X$ is of constant velocity. The **action** of a piecewise smooth curve $\gamma : [a,b] \to X$ is

$$\mathcal{A}(\gamma) := \int_a^b \left|\frac{d\gamma}{dt}\right|^2 dt \ .$$

Exercise 12
Show that, among all curves joining two given points, γ minimizes the action if and only if γ is of constant velocity and γ minimizes arc-length.

Hint:

(a) Let $\tau : [a,b] \to [a,b]$ be a smooth monotone map taking the endpoints of $[a,b]$ to the endpoints of $[a,b]$. Then

$$\int_a^b \left(\frac{d\tau}{dt}\right)^2 dt \geq b - a \ ,$$

with equality holding if and only if $\frac{d\tau}{dt} = 1$.

(b) Suppose that γ is of constant velocity, and let $\tau : [a,b] \to [a,b]$ be a reparametrization. Show that $\mathcal{A}(\gamma \circ \tau) \geq \mathcal{A}(\gamma)$, with equality only when $\tau = \text{identity}$.

The **riemannian distance** between two points x and y of a connected riemannian manifold (X, g) is the infimum $d(x,y)$ of the set of all arc-lengths for piecewise smooth curves joining x to y. A **geodesic** is a curve which locally minimizes distance and whose velocity is constant. A smooth curve joining x to y is a **minimizing geodesic** if its arc-length is the riemannian distance $d(x,y)$. A riemannian manifold (X, g) is **geodesically convex** if every point x is joined to every other point y by a unique (up to reparametrization) minimizing geodesic.

Example. On $X = \mathbb{R}^n$ with $TX \simeq \mathbb{R}^n \times \mathbb{R}^n$, let $g_x(v,w) = \langle v, w\rangle$, $g_x(v,v) = |v|^2$, where $\langle \cdot, \cdot \rangle$ is the euclidean inner product, and $|\cdot|$ is the euclidean norm. Then $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is a geodesically convex riemannian manifold, and the riemannian distance is the usual euclidean distance $d(x,y) = |x - y|$. ◇

3.4 Geodesic Flow

Suppose that (X, g) is a geodesically convex riemannian manifold. Assume also that (X, g) is **geodesically complete**, that is, every geodesic can be extended indefinitely. Given $(x, v) \in TX$, let $\exp(x, v) : \mathbb{R} \to X$ be the unique minimizing geodesic of constant velocity with initial conditions $\exp(x, v)(0) = x$ and $\frac{d\exp(x,v)}{dt}(0) = v$.

Consider the function

$$f : X \times X \longrightarrow \mathbb{R} \ , \qquad f(x, y) = -\frac{1}{2} \cdot d(x, y)^2 \ .$$

What is the symplectomorphism $\varphi : T^*X \to T^*X$ generated by f?

Proposition 3.3 *Under the identification of TX with T^*X by g, the symplectomorphism generated by φ coincides with the map $TX \to TX$, $(x, v) \mapsto \exp(x, v)(1)$.*

Proof. Let $d_x f$ and $d_y f$ be the components of $df_{(x,y)}$ with respect to $T^*_{(x,y)}(X \times X) \simeq T^*_x X \times T^*_y X$. The metric $g_x : T_x X \times T_x X \to \mathbb{R}$ induces an identification

$$\begin{array}{rccc} \widetilde{g}_x : & T_x X & \stackrel{\simeq}{\longrightarrow} & T^*_x X \\ & v & \longmapsto & g_x(v, \cdot) \end{array}$$

Use $\widetilde{g}$ to translate φ into a map $\widetilde{\varphi} : TX \to TX$.

Recall that, if

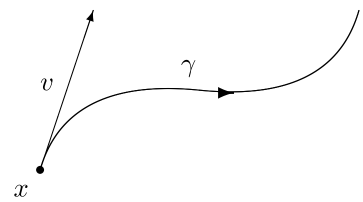
$$\Gamma^\sigma_\varphi = \{(x, y, d_x f, -d_y f) \mid (x, y) \in X \times X\}$$

is the graph of a diffeomorphism $\varphi : T^*X \to T^*X$, then φ is the symplectomorphism generated by f. In this case, $\varphi(x, \xi) = (y, \eta)$ if and only if $\xi = d_x f$ and $\eta = -d_y f$. We need to show that, given $(x, v) \in TX$, the unique solution of

$$\left\{ \begin{array}{rcrcr} \widetilde{g}_x(v) & = & \xi_i & = & d_x f(x, y) \\ \widetilde{g}_y(w) & = & \eta_i & = & -d_y f(x, y) \end{array} \right.$$

for (y, η) in terms of (x, ξ) in order to find φ, or, equivalently, for (y, w) in terms (x, v) in order to find $\widetilde{\varphi}$.

Let γ be the geodesic with initial conditions $\gamma(0) = x$, $\frac{d\gamma}{dt}(0) = v$.



By the Gauss lemma (look up [15], for instance), geodesics are orthogonal to the level sets of the distance function.

To solve the first equation of the system for y, we evaluate both sides at $v = \frac{d\exp(x,v)}{dt}(0)$, to conclude that

$$y = \exp(x, v)(1) \ .$$

Check that $d_x f(v') = 0$ for vectors $v' \in T_x X$ orthogonal to v (that is, $g_x(v, v') = 0$); this is a consequence of $f(x, y)$ being the square of the arc-length of a *minimizing* geodesic, and it suffices to check locally.

The vector w is obtained from the second equation of the system. Compute $-d_y f(\frac{d\exp(x,v)}{dt}(1))$. Then evaluate $-d_y f$ at vectors $w' \in T_y X$ orthogonal to $\frac{d\exp(x,v)}{dt}(1)$; this pairing is again 0 because $f(x, y)$ is the /square of the) arc-length of a minimizing geodesic. Conclude, using the nondegeneracy of g, that

$$w = \frac{d\exp(x,v)}{dt}(1) \ .$$

For both steps above, recall that, given a function $f : X \to \mathbb{R}$ and a tangent vector $v \in T_x X$, we have $d_x f(v) = \dfrac{d}{du}\left[f(\exp(x, v)(u))\right]_{u=0}$. □

In summary, the symplectomorphism φ corresponds to the map

$$\begin{array}{rccc} \widetilde{\varphi}: & TX & \longrightarrow & TX \\ & (x, v) & \longmapsto & (\gamma(1), \frac{d\gamma}{dt}(1)) \ , \end{array}$$

which is called the **geodesic flow** on X.

3.5 Periodic Points

Let X be an n-dimensional manifold. Let $M = T^*X$ be its cotangent bundle with canonical symplectic form ω.

Suppose that we are given a smooth function $f : X \times X \to \mathbb{R}$ which generates a symplectomorphism $\varphi : M \to M$, $\varphi(x, d_x f) = (y, -d_y f)$, by the recipe of Section sec:method.

What are the fixed points of φ?

Define $\psi : X \to \mathbb{R}$ by $\psi(x) = f(x, x)$.

Proposition 3.4 *There is a one-to-one correspondence between the fixed points of φ and the critical points of ψ.*

Proof. At $x_0 \in X$, we have that $d_{x_0}\psi = (d_x f + d_y f)|_{(x,y)=(x_0,x_0)}$. Let $\xi = d_x f|_{(x,y)=(x_0,x_0)}$. Now

$$x_0 \text{ is a critical point of } \psi \Leftrightarrow d_{x_0}\psi = 0 \Leftrightarrow d_y f|_{(x,y)=(x_0,x_0)} = -\xi.$$

So, the point in Γ_f^σ corresponding to $(x, y) = (x_0, x_0)$ is (x_0, x_0, ξ, ξ). But Γ_f^σ is the graph of φ, so $\varphi(x_0, \xi) = (x_0, \xi)$ is a fixed point. This argument also works backwards. □

Consider the iterates of φ,

$$\varphi^{(N)} = \underbrace{\varphi \circ \varphi \circ \ldots \circ \varphi}_{N} : M \longrightarrow M \ , \qquad\qquad N = 1, 2, \ldots \ ,$$

each of which is a symplectomorphism of M. According to the previous proposition, if $\varphi^{(N)} : M \to M$ is generated by $f^{(N)}$, then there is a one-to-one correspondence

$$\left\{\text{fixed points of } \varphi^{(N)}\right\} \ \leftrightarrow \ \left\{ \begin{array}{c} \text{critical points of} \\ \psi^{(N)} : X \to \mathbb{R}, \psi^{(N)}(x) = f^{(N)}(x,x) \end{array} \right\}$$

Knowing that φ is generated by f, does $\varphi^{(2)}$ have a generating function? The answer is a partial yes:

Fix $x, y \in X$. Define a map

$$\begin{array}{rcl} X & \longrightarrow & \mathbb{R} \\ z & \longmapsto & f(x,z) + f(z,y) \ . \end{array}$$

Suppose that this map has a unique critical point z_0, and that z_0 is nondegenerate. Let

$$f^{(2)}(x,y) := f(x,z_0) + f(z_0,y) \ .$$

Proposition 3.5 *The function $f^{(2)} : X \times X \to \mathbb{R}$ is smooth and is a generating function for $\varphi^{(2)}$.*

Proof. The point z_0 is given implicitly by $d_y f(x,z_0) + d_x f(z_0,y) = 0$. The nondegeneracy condition is

$$\det \left[\frac{\partial}{\partial z_i} \left(\frac{\partial f}{\partial y_j}(x,z) + \frac{\partial f}{\partial x_j}(z,y) \right) \right] \neq 0 \ .$$

By the implicit function theorem, $z_0 = z_0(x,y)$ is smooth.

As for the second assertion, $f^{(2)}(x,y)$ is a generating function for $\varphi^{(2)}$ if and only if

$$\varphi^{(2)}(x, d_x f^{(2)}) = (y, -d_y f^{(2)})$$

(assuming that, for each $\xi \in T_x^* X$, there is a unique $y \in X$ for which $d_x f^{(2)} = \xi$). Since φ is generated by f, and z_0 is critical, we obtain

$$\begin{aligned} \varphi^{(2)}(x, d_x f^{(2)}(x,y)) &= \varphi(\varphi(x, \underbrace{d_x f^{(2)}(x,y)}_{=d_x f(x,z_0)}) &&= \varphi(z_0, -d_y f(x,z_0)) \\ &= \varphi(z_0, d_x f(z_0,y)) &&= (y, \underbrace{-d_y f(z_0,y)}_{=-d_y f^{(2)}(x,y)}) \ . \end{aligned}$$

□

Exercise 13
What is a generating function for $\varphi^{(3)}$?

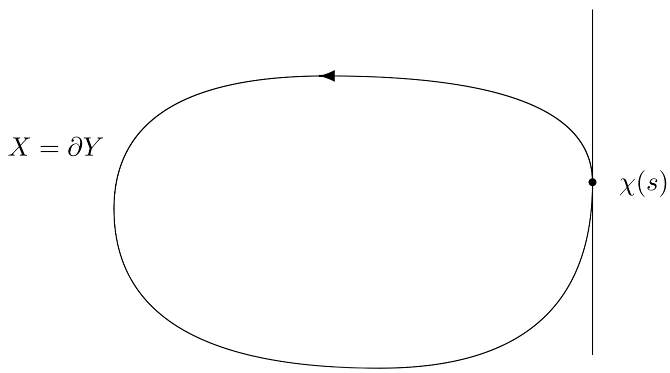
Hint: Suppose that the function

$$\begin{array}{ccc} X \times X & \longrightarrow & \mathbb{R} \\ (z,u) & \longmapsto & f(x,z) + f(z,u) + f(u,y) \end{array}$$

has a unique critical point (z_0, u_0), and that it is a nondegenerate critical point. Let $f^{(3)}(x,y) = f(x,z_0) + f(z_0,u_0) + f(u_0,y)$.

3.6 Billiards

Let $\chi : \mathbb{R} \to \mathbb{R}^2$ be a smooth plane curve which is 1-periodic, i.e., $\chi(s+1) = \chi(s)$, and parametrized by arc-length, i.e., $\left|\frac{d\chi}{ds}\right| = 1$. Assume that the region Y enclosed by χ is *convex*, i.e., for any $s \in \mathbb{R}$, the tangent line $\{\chi(s) + t\frac{d\chi}{ds} \mid t \in \mathbb{R}\}$ intersects $X := \partial Y$ (= the image of χ) at only the point $\chi(s)$.

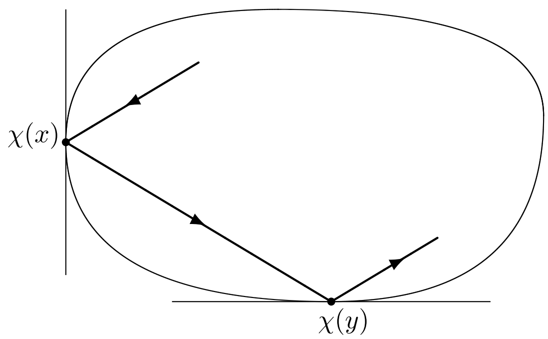


Suppose that we throw a ball into Y rolling with constant velocity and bouncing off the boundary with the usual law of reflection. This determines a map

$$\begin{array}{rccc} \varphi : & \mathbb{R}/\mathbb{Z} \times (-1,1) & \longrightarrow & \mathbb{R}/\mathbb{Z} \times (-1,1) \\ & (x,v) & \longmapsto & (y,w) \end{array}$$

by the rule

when the ball bounces off $\chi(x)$ with angle $\theta = \arccos v$, it will next collide with $\chi(y)$ and bounce off with angle $\nu = \arccos w$.



Let $f : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be defined by $f(x,y) = -|\chi(x) - \chi(y)|$; f is smooth off the diagonal. Use χ to identify $\mathbb{R}/\mathbb{Z}$ with the image curve X.

Suppose that $\varphi(x,v) = (y,w)$, i.e., (x,v) and (y,w) are successive points on the orbit described by the ball. Then

$$\begin{cases} \dfrac{df}{dx} = -\dfrac{x-y}{|x-y|} \text{ projected onto } T_xX = v \\[2ex] \dfrac{df}{dy} = -\dfrac{y-x}{|x-y|} \text{ projected onto } T_yX = -w \end{cases}$$

or, equivalently,

$$\begin{cases} \dfrac{d}{ds}f(\chi(s),y) = \dfrac{y-x}{|x-y|} \cdot \dfrac{d\chi}{ds} = \cos\theta = v \\[2ex] \dfrac{d}{ds}f(x,\chi(s)) = \dfrac{x-y}{|x-y|} \cdot \dfrac{d\chi}{ds} = -\cos\nu = -w\ . \end{cases}$$

We conclude that f is a generating function for φ. Similar approaches work for higher dimensional billiards problems.

Periodic points are obtained by finding critical points of

$$\begin{array}{ccl} \underbrace{X \times \ldots \times X}_{N} & \longrightarrow & \mathbb{R}\ , \qquad\qquad N > 1 \\ (x_1, \ldots, x_N) & \longmapsto & f(x_1, x_2) + \ldots + f(x_{N-1}, x_N) + f(x_N, x_1) \\ & & = |x_1 - x_2| + \ldots + |x_{N-1} - x_N| + |x_N - x_1|\ , \end{array}$$

that is, by finding the N-sided (generalized) polygons inscribed in X of critical perimeter.

Notice that

$$\mathbb{R}/\mathbb{Z} \times (-1, 1) \simeq \{(x, v) \mid x \in X, v \in T_x X, |v| < 1\} \simeq A$$

is the open unit tangent ball bundle of a circle X, that is, an open annulus A. The map $\varphi : A \to A$ is area-preserving.

3.7 Poincaré Recurrence

Theorem 3.6 (Poincaré Recurrence) *Suppose that $\varphi : A \to A$ is an area-preserving diffeomorphism of a finite-area manifold A. Let $p \in A$, and let $\mathcal{U}$ be a neighborhood of p. Then there is $q \in \mathcal{U}$ and a positive integer N such that $\varphi^{(N)}(q) \in \mathcal{U}$.*

Proof. Let $\mathcal{U}_0 = \mathcal{U}, \mathcal{U}_1 = \varphi(\mathcal{U}), \mathcal{U}_2 = \varphi^{(2)}(\mathcal{U}), \ldots$. If all of these sets were disjoint, then, since Area $(\mathcal{U}_i) =$ Area $(\mathcal{U}) > 0$ for all i, we would have

$$\text{Area } A \geq \text{ Area } (\mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2 \cup \ldots) = \sum_i \text{ Area } (\mathcal{U}_i) = \infty\ .$$

To avoid this contradiction we must have $\varphi^{(k)}(\mathcal{U}) \cap \varphi^{(l)}(\mathcal{U}) \neq \emptyset$ for some $k > l$, which implies $\varphi^{(k-l)}(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. □

Hence, eternal return applies to billiards...

Remark. Theorem 3.6 clearly generalizes to volume-preserving diffeomorphisms in higher dimensions. ♢

Theorem 3.7 (Poincaré's Last Geometric Theorem) *Suppose $\varphi : A \to A$ is an area-preserving diffeomorphism of the closed annulus $A = \mathbb{R}/\mathbb{Z} \times [-1,1]$ which preserves the two components of the boundary, and twists them in opposite directions. Then φ has at least two fixed points.*

This theorem was proved in 1913 by Birkhoff, and hence is also called the **Poincaré-Birkhoff theorem**. It has important applications to dynamical systems and celestial mechanics. The Arnold conjecture (1966) on the existence of fixed points for symplectomorphisms of compact manifolds (see Section 3.9) may be regarded as a generalization of the Poincaré-Birkhoff theorem. This conjecture has motivated a significant amount of recent research involving a more general notion of generating function; see, for instance, [18, 20].

3.8 Group of Symplectomorphisms

The symplectomorphisms of a symplectic manifold (M, ω) form the group

$$\mathrm{Sympl}(M, \omega) = \{f : M \xrightarrow{\simeq} M \mid f^*\omega = \omega\} .$$

– What is $T_{\mathrm{id}}(\mathrm{Sympl}(M, \omega))$?
(What is the "Lie algebra" of the group of symplectomorphisms?)
– What does a neighborhood of id in $\mathrm{Sympl}(M, \omega)$ look like?

We will use notions from the $\boldsymbol{C^1}$**-topology**. Let X and Y be manifolds.

Definition 3.8 *A sequence of maps $f_i : X \to Y$* **converges in the $\boldsymbol{C^0}$-topology** *to $f : X \to Y$ if and only if f_i converges uniformly on compact sets.*

Definition 3.9 *A sequence of C^1 maps $f_i : X \to Y$* **converges in the $\boldsymbol{C^1}$-topology** *to $f : X \to Y$ if and only if it and the sequence of derivatives $df_i : TX \to TY$ converge uniformly on compact sets.*

Let (M, ω) be a compact symplectic manifold and $f \in \mathrm{Sympl}(M, \omega)$. Then both Graph f and the diagonal $\Delta =$ Graph id are lagrangian submanifolds of $(M \times M, \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega)$, where $\mathrm{pr}_i : M \times M \to M$, $i = 1, 2$, are the projections to each factor.

By the Weinstein tubular neighborhood theorem, there exists a neighborhood $\mathcal{U}$ of Δ $(\simeq M)$ in $(M \times M, \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega)$ which is symplectomorphic to a neighborhood $\mathcal{U}_0$ of M in (T^*M, ω_0). Let $\varphi : \mathcal{U} \to \mathcal{U}_0$ be that symplectomorphism satisfying $\varphi(p,p) = (p,0)$, $\forall p \in M$.

Suppose that f is sufficiently **C^1-close** to id, i.e., f is in some sufficiently small neighborhood of id in the C^1-topology. Then:

1. We can assume that Graph $f \subseteq \mathcal{U}$.

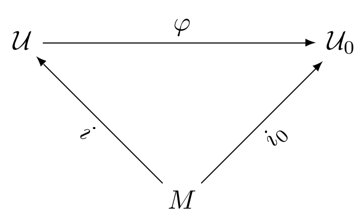
   Let $j : M \hookrightarrow \mathcal{U}$ be the embedding as Graph f ,
   $i : M \hookrightarrow \mathcal{U}$ be the embedding as Graph id $= \Delta$.

2. The map j is sufficiently C^1-close to i.

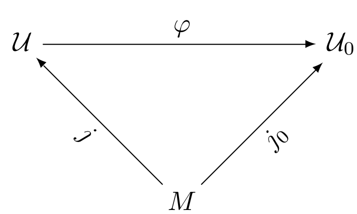
3. By the Weinstein theorem, $\mathcal{U} \simeq \mathcal{U}_0 \subseteq T^*M$, so the above j and i induce

   $j_0 : M \hookrightarrow \mathcal{U}_0$ embedding, where $j_0 = \varphi \circ j$,
   $i_0 : M \hookrightarrow \mathcal{U}_0$ embedding as 0-section .

   Hence, we have



   and



   where $i(p) = (p,p)$, $i_0(p) = (p,0)$, $j(p) = (p, f(p))$ and $j_0(p) = \varphi(p, f(p))$ for $p \in M$.

4. The map j_0 is sufficiently C^1-close to i_0. Therefore, the image set $j_0(M)$ intersects each T_p^*M at one point μ_p depending smoothly on p.

5. The image of j_0 is the image of a smooth section $\mu : M \to T^*M$, that is, a 1-form $\mu = j_0 \circ (\pi \circ j_0)^{-1}$.

We conclude that Graph $f \simeq \{(p, \mu_p) \mid p \in M,\ \mu_p \in T_p^*M\}$.

Exercise 14
Vice-versa: show that, if μ is a 1-form sufficiently C^1-close to the zero 1-form, then there is a diffeomorphism $f : M \to M$ such that

$$\{(p, \mu_p) \mid p \in M,\ \mu_p \in T_p^*M\} \ \simeq\ \text{Graph } f \ .$$

By Section 2.4, we have

$$\text{Graph } f \text{ is lagrangian} \quad \Longleftrightarrow \quad \mu \text{ is closed} \ .$$

Conclusion. A small C^1-neighborhood of id in $\text{Sympl}(M, \omega)$ is homeomorphic to a C^1-neighborhood of zero in the vector space of closed 1-forms on M. So:

$$T_{\text{id}}(\text{Sympl}(M, \omega)) \simeq \{\mu \in \Omega^1(M) \mid d\mu = 0\} \ .$$

In particular, $T_{\text{id}}(\text{Sympl}(M, \omega))$ contains the space of exact 1-forms

$$\{\mu = dh \mid h \in C^\infty(M)\} \ \simeq\ C^\infty(M)/ \text{ locally constant functions} \ .$$

3.9 Fixed Points of Symplectomorphisms

Theorem 3.10 *Let (M, ω) be a compact symplectic manifold with $H^1_{\text{deRham}}(M) = 0$. Then any symplectomorphism of M which is sufficiently C^1-close to the identity has at least two fixed points.*

Proof. Suppose that $f \in \text{Sympl}(M, \omega)$ is sufficiently C^1-close to id. Then the graph of f corresponds to a closed 1-form μ on M.

$$\left.\begin{array}{l} d\mu = 0 \\ H^1_{\text{deRham}}(M) = 0 \end{array}\right\} \Longrightarrow \mu = dh \text{ for some } h \in C^\infty(M) \ .$$

If M is compact, then h has at least 2 critical points.

$$\begin{array}{ccc} \text{Fixed points of f} & = & \text{critical points of } h \\ \| & & \| \\ \text{Graph } f \cap \Delta & = & \{p : \mu_p = dh_p = 0\} \ . \end{array}$$

□

Lagrangian intersection problem:

A submanifold Y of M is **C^1-close** to X when there is a diffeomorphism $X \to Y$ which is, as a map into M, C^1-close to the inclusion $X \hookrightarrow M$.

Theorem 3.11 *Let (M,ω) be a symplectic manifold. Suppose that X is a compact lagrangian submanifold of M with $H^1_{\text{deRham}}(X) = 0$. Then every lagrangian submanifold of M which is C^1-close to X intersects X in at least two points.*

Proof. Exercise. □

Arnold conjecture:

Let (M,ω) be a compact symplectic manifold, and $f : M \to M$ a symplectomorphism which is "exactly homotopic to the identity" (see below). Then

$$\#\{\text{fixed points of } f\} \ \geq \ \begin{array}{l}\text{minimal \# of critical points} \\ \text{a smooth function on } M \text{ can have} \ .\end{array}$$

Together with Morse theory,[1] we obtain[2]

$$\begin{array}{rcl} \#\{\text{nondegenerate fixed points of } f\} & \geq & \begin{array}{l}\text{minimal \# of critical} \\ \text{points a Morse function} \\ \text{on } M \text{ can have}\end{array} \\ & \geq & \displaystyle\sum_{i=0}^{2n} \dim H^i(M) \ . \end{array}$$

[1]A **Morse function** on M is a function $h : M \to \mathbb{R}$ whose critical points (i.e., points p where the differential vanishes: $dh_p = 0$) are all nondegenerate (i.e., the hessian at those points is nonsingular: $\det\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)_p \neq 0$).

[2]A fixed point p of $f : M \to M$ is **nondegenerate** if $df_p : T_pM \to T_pM$ is nonsingular.

The Arnold conjecture was proved by Conley-Zehnder, Floer, Hofer-Salamon, Ono, Fukaya-Ono, Liu-Tian using Floer homology (which is an ∞-dimensional analogue of Morse theory). There are open conjectures for sharper bounds on the number of fixed points.

Meaning of "f is exactly homotopic to the identity:"

Suppose that $h_t : M \to \mathbb{R}$ is a smooth family of functions which is 1-periodic, i.e., $h_t = h_{t+1}$. Let $\rho : M \times \mathbb{R} \to M$ be the isotopy generated by the time-dependent vector field v_t defined by $\omega(v_t, \cdot) = dh_t$. Then "$f$ being exactly homotopic to the identity" means $f = \rho_1$ for some such h_t.

In other words, f is **exactly homotopic to the identity** when f is the time-1 map of an isotopy generated by some smooth time-dependent 1-periodic hamiltonian function.

There is a one-to-one correspondence

$$\{ \text{ fixed points of } f \} \quad \longleftrightarrow \quad \{ \text{ period-1 orbits of } \rho : M \times \mathbb{R} \to M \}$$

because $f(p) = p$ if and only if $\{\rho(t,p) \ , \ t \in [0,1]\}$ is a closed orbit.

Proof of the Arnold conjecture in the case when $h : M \to \mathbb{R}$ is independent of t:

$$\begin{array}{lcl} p \text{ is a critical point of } h & \Longleftrightarrow & dh_p = 0 \\ & \Longleftrightarrow & v_p = 0 \\ & \Longrightarrow & \rho(t,p) = p \ , \ \forall t \in \mathbb{R} \\ & \Longrightarrow & p \text{ is a fixed point of } \rho_1 \ . \end{array}$$

$\square$

Exercise 15
Compute these estimates for the number of fixed points on some compact symplectic manifolds (for instance, S^2, $S^2 \times S^2$ and $T^2 = S^1 \times S^1$).

Lecture 4

Hamiltonian Fields

To any real function on a symplectic manifold, a symplectic geometer associates a vector field whose flow preserves the symplectic form and the given function. The vector field is called the *hamiltonian vector field* of that *(hamiltonian) function.*

The concept of a *moment map* is a generalization of that of a hamiltonian function, and was introduced by Souriau [40] under the french name *application moment* (besides the more standard english translation to *moment map*, the alternative *momentum map* is also used). The notion of a moment map associated to a group action on a symplectic manifold formalizes the Noether principle, which states that to every symmetry (such as a group action) in a mechanical system, there corresponds a conserved quantity.

4.1 Hamiltonian and Symplectic Fields

Let (M, ω) be a symplectic manifold and let $H : M \to \mathbb{R}$ be a smooth function. Its differential dH is a 1-form. By nondegeneracy, there is a unique vector field X_H on M such that $\imath_{X_H}\omega = dH$. Integrate X_H. Supposing that M is compact, or at least that X_H is complete, let $\rho_t : M \to M$, $t \in \mathbb{R}$, be the one-parameter family of diffeomorphisms generated by X_H:

$$\begin{cases} \rho_0 = \mathrm{id}_M \\ \dfrac{d\rho_t}{dt} \circ \rho_t^{-1} = X_H \ . \end{cases}$$

Claim. Each diffeomorphism ρ_t preserves ω, i.e., $\rho_t^*\omega = \omega$, $\forall t$.

Proof. We have $\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{X_H}\omega = \rho_t^*(d\underbrace{\imath_{X_H}\omega}_{dH} + \imath_{X_H}\underbrace{d\omega}_{0}) = 0$. □

Therefore, every function on (M,ω) gives a family of symplectomorphisms. Notice how the proof involved both the *nondegeneracy* and the *closedness* of ω.

Definition 4.1 *A vector field X_H as above is called the* **hamiltonian vector field** *with* **hamiltonian function** *H.*

Example. The height function $H(\theta, h) = h$ on the sphere $(M,\omega) = (S^2, d\theta \wedge dh)$ has

$$\imath_{X_H}(d\theta \wedge dh) = dh \quad \Longleftrightarrow \quad X_H = \frac{\partial}{\partial\theta} \ .$$

Thus, $\rho_t(\theta, h) = (\theta + t, h)$, which is rotation about the vertical axis; the height function H is preserved by this motion. ♢

Exercise 16
Let X be a vector field on an abstract manifold W. There is a unique vector field $X_\sharp$ on the cotangent bundle T^*W, whose flow is the lift of the flow of X. Let α be the tautological 1-form on T^*W and let $\omega = -d\alpha$ be the canonical symplectic form on T^*W. Show that $X_\sharp$ is a hamiltonian vector field with hamiltonian function $H := \imath_{X_\sharp}\alpha$.

Remark. If X_H is hamiltonian, then

$$\mathcal{L}_{X_H}H = \imath_{X_H}dH = \imath_{X_H}\imath_{X_H}\omega = 0 \ .$$

Therefore, hamiltonian vector fields preserve their hamiltonian functions, and each integral curve $\{\rho_t(x) \mid t \in \mathbb{R}\}$ of X_H must be contained in a level set of H:

$$H(x) = (\rho_t^* H)(x) = H(\rho_t(x)) \ , \quad \forall t \ .$$

$\diamondsuit$

Definition 4.2 *A vector field X on M preserving ω (i.e., such that $\mathcal{L}_X\omega = 0$) is called a* **symplectic vector field***.*

$$\begin{cases} X \text{ is symplectic} & \iff \quad \imath_X\omega \text{ is closed ,} \\ X \text{ is hamiltonian} & \iff \quad \imath_X\omega \text{ is exact .} \end{cases}$$

Locally, on every contractible open set, every symplectic vector field is hamiltonian. If $H^1_{\text{deRham}}(M) = 0$, then globally every symplectic vector field is hamiltonian. In general, $H^1_{\text{deRham}}(M)$ measures the obstruction for symplectic vector fields to be hamiltonian.

Example. On the 2-torus $(M, \omega) = (\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$, the vector fields $X_1 = \frac{\partial}{\partial\theta_1}$ and $X_2 = \frac{\partial}{\partial\theta_2}$ are symplectic but not hamiltonian. $\diamondsuit$

To summarize, vector fields on a symplectic manifold (M, ω) which preserve ω are called **symplectic**. The following are equivalent:

- X is a symplectic vector field;
- the flow ρ_t of X preserves ω, i.e., $\rho_t^*\omega = \omega$, for all t;
- $\mathcal{L}_X\omega = 0$;
- $\imath_X\omega$ is closed.

A **hamiltonian** vector field is a vector field X for which

- $\imath_X\omega$ is exact,

i.e., $\imath_X\omega = dH$ for some $H \in C^\infty(M)$. A primitive H of $\imath_X\omega$ is then called a **hamiltonian function** of X.

4.2 Hamilton Equations

Consider euclidean space $\mathbb{R}^{2n}$ with coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ and $\omega_0 = \sum dq_j \wedge dp_j$. The curve $\rho_t = (q(t),p(t))$ is an integral curve for X_H exactly if

$$\begin{cases} \dfrac{dq_i}{dt}(t) = \dfrac{\partial H}{\partial p_i} \\[2ex] \dfrac{dp_i}{dt}(t) = -\dfrac{\partial H}{\partial q_i} \end{cases} \qquad \textbf{(Hamilton equations)}$$

Indeed, let $X_H = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\right)$. Then,

$$\begin{aligned} \imath_{X_H}\omega &= \sum_{j=1}^{n} \imath_{X_H}(dq_j \wedge dp_j) \\ &= \sum_{j=1}^{n} [(\imath_{X_H} dq_j)\wedge dp_j - dq_j \wedge (\imath_{X_H} dp_j)] \\ &= \sum_{j=1}^{n}\left(\tfrac{\partial H}{\partial p_j} dp_j + \tfrac{\partial H}{\partial q_j} dq_j\right) = dH\ . \end{aligned}$$

Remark. The gradient vector field of H relative to the euclidean metric is

$$\nabla H := \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\frac{\partial}{\partial q_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial p_i}\right)\ .$$

If J is the standard (almost) complex structure[1] so that $J(\frac{\partial}{\partial q_i}) = \frac{\partial}{\partial p_i}$ and $J(\frac{\partial}{\partial p_i}) = -\frac{\partial}{\partial q_i}$, we have $JX_H = \nabla H$. ♢

The case where $n = 3$ has a simple physical illustration. Newton's second law states that a particle of mass m moving in **configuration space** $\mathbb{R}^3$ with coordinates $q = (q_1,q_2,q_3)$ under a potential $V(q)$ moves along a curve $q(t)$ such that

$$m\frac{d^2q}{dt^2} = -\nabla V(q)\ .$$

[1] An **almost complex structure** on a manifold M is a vector bundle morphism $J: TM \to TM$ such that $J^2 = -\mathrm{Id}$.

Introduce the **momenta** $p_i = m\frac{dq_i}{dt}$ for $i = 1, 2, 3$, and **energy** function $H(p, q) = \frac{1}{2m}|p|^2 + V(q)$. Let $\mathbb{R}^6 = T^*\mathbb{R}^3$ be the corresponding **phase space**, with coordinates $(q_1, q_2, q_3, p_1, p_2, p_3)$. Newton's second law in $\mathbb{R}^3$ is equivalent to the Hamilton equations in $\mathbb{R}^6$:

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{1}{m}p_i = \dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_i}{dt} = m\dfrac{d^2 q_i}{dt^2} = -\dfrac{\partial V}{\partial q_i} = -\dfrac{\partial H}{\partial q_i} \ . \end{cases}$$

The energy H is conserved by the physical motion.

4.3 Brackets

Vector fields are differential operators on functions: if X is a vector field and $f \in C^\infty(M)$, df being the corresponding 1-form, then

$$X \cdot f := df(X) = \mathcal{L}_X f \ .$$

Given two vector fields X, Y, there is a unique vector field W such that

$$\mathcal{L}_W f = \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f) \ .$$

The vector field W is called the **Lie bracket** of the vector fields X and Y and denoted $W = [X, Y]$, since $\mathcal{L}_W = [\mathcal{L}_X, \mathcal{L}_Y]$ is the commutator.

Exercise 17
Check that, for any form α,

$$\imath_{[X,Y]}\alpha = \mathcal{L}_X \imath_Y \alpha - \imath_Y \mathcal{L}_X \alpha = [\mathcal{L}_X, \imath_Y]\alpha \ .$$

Since each side is an anti-derivation with respect to the wedge product, it suffices to check this formula on local generators of the exterior algebra of forms, namely functions and exact 1-forms.

Proposition 4.3 *If X and Y are symplectic vector fields on a symplectic manifold (M, ω), then $[X, Y]$ is hamiltonian with hamiltonian function $\omega(Y, X)$.*

Proof.

$$\begin{array}{rcl}\imath_{[X,Y]}\omega & = & \mathcal{L}_X\imath_Y\omega - \imath_Y\mathcal{L}_X\omega \\ & = & d\imath_X\imath_Y\omega + \imath_X \underbrace{d\imath_Y\omega}_{0} - \imath_Y \underbrace{d\imath_X\omega}_{0} - \imath_Y\imath_X \underbrace{d\omega}_{0} \\ & = & d(\omega(Y,X)) \ .\end{array}$$

□

A (real) **Lie algebra** is a (real) vector space $\mathfrak{g}$ together with a **Lie bracket** $[\cdot,\cdot]$, i.e., a bilinear map $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying:

(a) $[x,y] = -[y,x]$, $\forall x,y\in\mathfrak{g}$, **(antisymmetry)**

(b) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$, $\forall x,y,z\in\mathfrak{g}$. **(Jacobi identity)**

Let

$$\begin{array}{rcl}\chi(M) & = & \{ \text{ vector fields on } M \ \} \\ \chi^{\text{sympl}}(M) & = & \{ \text{ symplectic vector fields on } M \ \} \\ \chi^{\text{ham}}(M) & = & \{ \text{ hamiltonian vector fields on } M \ \} \ .\end{array}$$

The inclusions $(\chi^{\text{ham}}(M),[\cdot,\cdot]) \subseteq (\chi^{\text{sympl}}(M),[\cdot,\cdot]) \subseteq (\chi(M),[\cdot,\cdot])$ are inclusions of Lie algebras.

Definition 4.4 *The* **Poisson bracket** *of two functions f and g in $C^\infty(M;\mathbb{R})$ is*

$$\{f,g\} := \omega(X_f, X_g) \ .$$

We have $X_{\{f,g\}} = -[X_f,X_g]$ because $X_{\omega(X_f,X_g)} = [X_g,X_f]$.

Theorem 4.5 *The bracket $\{\cdot,\cdot\}$ satisfies the Jacobi identity, i.e.,*

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \ .$$

Proof. Exercise. □

Definition 4.6 *A* **Poisson algebra** $(\mathcal{P},\{\cdot,\cdot\})$ *is a commutative associative algebra $\mathcal{P}$ with a Lie bracket $\{\cdot,\cdot\}$ satisfying the* **Leibniz rule***:*

$$\{f,gh\} = \{f,g\}h + g\{f,h\} \ .$$

Exercise 18
Check that the Poisson bracket $\{\cdot,\cdot\}$ defined above satisfies the Leibniz rule.

We conclude that, if (M,ω) is a symplectic manifold, then $(C^\infty(M), \{\cdot,\cdot\})$ is a Poisson algebra. Furthermore, we have a Lie algebra anti-homomorphism

$$\begin{array}{rcl} C^\infty(M) & \longrightarrow & \chi(M) \\ H & \longmapsto & X_H \\ \{\cdot,\cdot\} & \rightsquigarrow & -[\cdot,\cdot] \ . \end{array}$$

Exercise 19
Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$.

(a) Let ${}^{\mathfrak{g}}X^\#$ be the vector field generated by $X \in \mathfrak{g}$ for the adjoint representation of G on $\mathfrak{g}$. Show that

$${}^{\mathfrak{g}}X^\#_Y = [X,Y] \qquad \forall\ Y \in \mathfrak{g} \ .$$

(b) Let $X^\#$ be the vector field generated by $X \in \mathfrak{g}$ for the coadjoint representation of G on $\mathfrak{g}^*$. Show that

$$\langle X^\#_\xi, Y\rangle = \langle \xi, [Y,X]\rangle \qquad \forall\ Y \in \mathfrak{g} \ .$$

(c) For any $\xi \in \mathfrak{g}^*$, define a skew-symmetric bilinear form on $\mathfrak{g}$ by

$$\omega_\xi(X,Y) := \langle \xi, [X,Y]\rangle \ .$$

Show that the kernel of ω_ξ is the Lie algebra $\mathfrak{g}_\xi$ of the stabilizer of ξ for the coadjoint representation.

(d) Show that ω_ξ defines a nondegenerate 2-form on the tangent space at ξ to the coadjoint orbit through ξ.

(e) Show that ω_ξ defines a closed 2-form on the orbit of ξ in $\mathfrak{g}^*$.
Hint: The tangent space to the orbit being generated by the vector fields $X^\#$, this is a consequence of the Jacobi identity in $\mathfrak{g}$.
This **canonical symplectic form** on the coadjoint orbits is also known as the **Lie-Poisson** or **Kostant-Kirillov symplectic structure**.

(f) The Lie algebra structure of $\mathfrak{g}$ defines a canonical Poisson structure on $\mathfrak{g}^*$:

$$\{f,g\}(\xi) := \langle \xi, [df_\xi, dg_\xi]\rangle$$

for $f,g \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$. Notice that $df_\xi : T_\xi\mathfrak{g}^* \simeq \mathfrak{g}^* \to \mathbb{R}$ is identified with an element of $\mathfrak{g} \simeq \mathfrak{g}^{**}$.
Check that $\{\cdot,\cdot\}$ satisfies the Leibniz rule:

$$\{f,gh\} = g\{f,h\} + h\{f,g\} \ .$$

Example. For the prototype $(\mathbb{R}^{2n}, \omega_0)$, where $\omega_0 = \sum dx_i \wedge dy_i$, we have

$$X_{x_i} = -\frac{\partial}{\partial y_i} \qquad \text{and} \qquad X_{y_i} = \frac{\partial}{\partial x_i}$$

so that

$$\{x_i, x_j\} = \{y_i, y_j\} = 0 \quad \text{and} \quad \{x_i, y_j\} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \forall i, j \ .$$

For arbitrary functions $f, g \in C^\infty(M)$ we have hamiltonian vector fields

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial}{\partial y_i} \right) ,$$

and the classical Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right) .$$

◊

4.4 Integrable Systems

Definition 4.7 *A* **hamiltonian system** *is a triple* (M, ω, H)*, where* (M, ω) *is a symplectic manifold and* $H \in C^\infty(M; \mathbb{R})$ *is a function, called the* **hamiltonian function***.*

Theorem 4.8 *We have* $\{f, H\} = 0$ *if and only if* f *is constant along integral curves of* X_H*.*

Proof. Let ρ_t be the flow of X_H. Then

$$\begin{aligned} \tfrac{d}{dt}(f \circ \rho_t) &= \rho_t^* \mathcal{L}_{X_H} f = \rho_t^* \imath_{X_H} df = \rho_t^* \imath_{X_H} \imath_{X_f} \omega \\ &= \rho_t^* \omega(X_f, X_H) = \rho_t^* \{f, H\} \ . \end{aligned}$$

□

A function f as in Theorem 4.8 is called an **integral of motion** (or a **first integral** or a **constant of motion**). In general, hamiltonian systems do not admit integrals of motion which are *independent* of the hamiltonian function. Functions $f_1, \ldots, f_n$ on M are said to be **independent** if their differentials $(df_1)_p, \ldots, (df_n)_p$ are linearly independent at all points p in some open dense subset of M. Loosely speaking, a hamiltonian system is *(completely) integrable* if it has as many commuting integrals of motion as possible. Commutativity is with respect to the Poisson bracket. Notice that, if $f_1, \ldots, f_n$ are commuting integrals of motion for a hamiltonian system (M, ω, H), then, at each $p \in M$, their hamiltonian vector fields generate an isotropic subspace of T_pM:

$$\omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\} = 0 \ .$$

If $f_1, \ldots, f_n$ are independent, then, by symplectic linear algebra, n can be at most half the dimension of M.

Definition 4.9 *A hamiltonian system (M, ω, H) is* **(completely) integrable** *if it possesses $n = \frac{1}{2} \dim M$ independent integrals of motion, $f_1 = H, f_2, \ldots, f_n$, which are pairwise in involution with respect to the Poisson bracket, i.e., $\{f_i, f_j\} = 0$, for all i, j.*

Examples.

1. The simple pendulum (discussed in the next section) and the harmonic oscillator are trivially integrable systems – any 2-dimensional hamiltonian system (where the set of non-fixed points is dense) is integrable.

2. A hamiltonian system (M, ω, H) where M is 4-dimensional is integrable if there is an integral of motion independent of H (the commutativity condition is automatically satisfied). The next section shows that the spherical pendulum is integrable.

$\diamondsuit$

For sophisticated examples of integrable systems, see [9, 28].

Let (M, ω, H) be an integrable system of dimension $2n$ with integrals of motion $f_1 = H, f_2, \ldots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of $f := (f_1, \ldots, f_n)$. The corresponding level set, $f^{-1}(c)$, is a lagrangian submanifold, because it is n-dimensional and its tangent bundle is isotropic.

Proposition 4.10 *If the hamiltonian vector fields* $X_{f_1}, \ldots, X_{f_n}$ *are complete on the level* $f^{-1}(c)$*, then the connected components of* $f^{-1}(c)$ *are homogeneous spaces for* $\mathbb{R}^n$*, i.e., are of the form* $\mathbb{R}^{n-k} \times \mathbb{T}^k$ *for some* k*,* $0 \leq k \leq n$*, where* $\mathbb{T}^k$ *is a* k*-dimensional torus.*

Proof. Exercise (just follow the flows to obtain coordinates). □

Any compact component of $f^{-1}(c)$ must hence be a torus. These components, when they exist, are called **Liouville tori**. (The easiest way to ensure that compact components exist is to have one of the f_i's proper.)

Theorem 4.11 (Arnold-Liouville [3]) *Let* (M, ω, H) *be an integrable system of dimension* $2n$ *with integrals of motion* $f_1 = H$*,* $f_2, \ldots, f_n$*. Let* $c \in \mathbb{R}^n$ *be a regular value of* $f := (f_1, \ldots, f_n)$*. The corresponding level* $f^{-1}(c)$ *is a lagrangian submanifold of* M*.*

(a) If the flows of $X_{f_1}, \ldots, X_{f_n}$ *starting at a point* $p \in f^{-1}(c)$ *are complete, then the connected component of* $f^{-1}(c)$ *containing* p *is a homogeneous space for* $\mathbb{R}^n$*. With respect to this affine structure, that component has coordinates* $\varphi_1, \ldots, \varphi_n$*, known as* **angle coordinates***, in which the flows of the vector fields* $X_{f_1}, \ldots, X_{f_n}$ *are linear.*

(b) There are coordinates $\psi_1, \ldots, \psi_n$*, known as* **action coordinates***, complementary to the angle coordinates such that the* ψ_i*'s are integrals of motion and* $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$ *form a Darboux chart.*

Therefore, the dynamics of an integrable system is extremely simple and the system has an explicit solution in action-angle coordinates. The proof of part (a) – the easy part – of the Arnold-Liouville theorem is sketched above. For the proof of part (b), see [3, 17].

Geometrically, in a neighborhood of the regular value c, the map $f : M \to \mathbb{R}^n$ collecting the given integrals of motion is a **lagrangian fibration**, i.e., it is locally trivial and its fibers are lagrangian submanifolds. Part (a) of the Arnold-Liouville theorem says that there are coordinates along the fibers, the angle coordinates[2] φ_i, in which

[2]The name "angle coordinates" is used even if the fibers are not tori.

the flows of $X_{f_1}, \ldots, X_{f_n}$ are linear. Part (b) of the theorem guarantees the existence of coordinates on $\mathbb{R}^n$, the action coordinates ψ_i, which (Poisson) commute among themselves and satisfy $\{\varphi_i, \psi_j\} = \delta_{ij}$ with respect to the angle coordinates. Notice that, in general, the action coordinates are not the given integrals of motion because $\varphi_1, \ldots, \varphi_n, f_1, \ldots, f_n$ do not form a Darboux chart.

4.5 Pendula

The **simple pendulum** is a mechanical system consisting of a massless rigid rod of length ℓ, fixed at one end, whereas the other end has a plumb bob of mass m, which may oscillate in the vertical plane. We assume that the force of gravity is constant pointing vertically downwards, and that this is the only external force acting on this system.

Let θ be the oriented angle between the rod (regarded as a line segment) and the vertical direction. Let ξ be the coordinate along the fibers of T^*S^1 induced by the standard angle coordinate on S^1. Then the function $H : T^*S^1 \to \mathbb{R}$ given by

$$H(\theta, \xi) = \underbrace{\frac{\xi^2}{2m\ell^2}}_{K} + \underbrace{m\ell(1 - \cos\theta)}_{V} ,$$

is an appropriate hamiltonian function to describe the simple pendulum. More precisely, gravity corresponds to the potential energy $V(\theta) = m\ell(1 - \cos\theta)$ (we omit universal constants), and the kinetic energy is given by $K(\theta, \xi) = \frac{1}{2m\ell^2}\xi^2$.

For simplicity, we assume that $m = \ell = 1$.

Exercise 20
Show that there exists a number c such that for $0 < h < c$ the level curve $H = h$ in the (θ, ξ) plane is a disjoint union of closed curves. Show that the projection of each of these curves onto the θ-axis is an interval of length less than π.
Show that neither of these assertions is true if $h > c$.
What types of motion are described by these two types of curves?
What about the case $H = c$?

Modulo 2π in θ, the function H has exactly two critical points: a critical point s where H vanishes, and a critical point u where H equals c. These points are called the **stable** and **unstable** points of H, respectively. This terminology is justified by the fact that a trajectory of the hamiltonian vector field of H whose initial point is close to s stays close to s forever, whereas this is not the case for u. (What is happening physically?)

The **spherical pendulum** is a mechanical system consisting of a massless rigid rod of length ℓ, fixed at one end, whereas the other end has a plumb bob of mass m, which may oscillate freely in all directions. We assume that the force of gravity is constant pointing vertically downwards, and that this is the only external force acting on this system.

Let φ, θ ($0 < \varphi < \pi$, $0 < \theta < 2\pi$) be spherical coordinates for the bob. For simplicity we take $m = \ell = 1$.

Let η, ξ be the coordinates along the fibers of T^*S^2 induced by the spherical coordinates φ, θ on S^2. An appropriate hamiltonian function to describe the spherical pendulum is $H : T^*S^2 \to \mathbb{R}$ given by

$$H(\varphi, \theta, \eta, \xi) = \frac{1}{2}\left(\eta^2 + \frac{\xi^2}{(\sin\varphi)^2}\right) + \cos\varphi \ .$$

On S^2, the function H has exactly two critical points: s (where H has a minimum) and u. These points are called the **stable** and **unstable** points of H, respectively. A trajectory whose initial point is close to s stays close to s forever, whereas this is not the case for u.

The group of rotations about the vertical axis is a group of symmetries of the spherical pendulum. In the coordinates above, the integral of motion associated with these symmetries is the function

$$J(\varphi, \theta, \eta, \xi) = \xi \ .$$

Exercise 21
Give a more coordinate-independent description of J, one that makes sense also on the cotangent fibers above the North and South poles.

We will locate all points $p \in T^*S^2$ where dH_p and dJ_p are linearly dependent:

- Clearly, the two critical points s and u belong to this set. These are the only two points where $dH_p = dJ_p = 0$.

- If $x \in S^2$ is in the southern hemisphere ($x_3 < 0$), then there exist exactly two points, $p_+ = (x, \eta, \xi)$ and $p_- = (x, -\eta, -\xi)$, in the cotangent fiber above x where dH_p and dJ_p are linearly dependent.

- Since dH_p and dJ_p are linearly dependent along the trajectory of the hamiltonian vector field of H through p_+, this trajectory is also a trajectory of the hamiltonian vector field of J, and, hence, that its projection onto S^2 is a latitudinal circle (of the form $x_3 =$ constant). The projection of the trajectory through p_- is the same latitudinal circle traced in the opposite direction.

One can check that any nonzero value j is a regular value of J, and that S^1 acts freely on the level set $J = j$.

Exercise 22
What happens on the cotangent fibers above the North and South poles?

The integral curves of the original system on the level set $J = j$ can be obtained from those of the reduced system by "quadrature", in other words, by a simple integration.

The reduced system for $j \neq 0$ has exactly one equilibrium point. The corresponding relative equilibrium for the original system is one of the horizontal curves from above.

The **energy-momentum map** is the map $(H, J) : T^*S^2 \to \mathbb{R}^2$. If $j \neq 0$, the level set $(H, J) = (h, j)$ of the energy-momentum map is either a circle (in which case it is one of the horizontal curves above), or a two-torus. The projection onto the configuration space of the two-torus is an annular region on S^2.

4.6 Symplectic and Hamiltonian Actions

Let (M, ω) be a symplectic manifold, and G a Lie group. Let $\psi : G \longrightarrow \text{Diff}(M)$ be a (smooth) action.

Definition 4.12 *The action ψ is a* **symplectic action** *if*

$$\psi : G \longrightarrow \text{Sympl}(M, \omega) \subset \text{Diff}(M) ,$$

i.e., G acts by symplectomorphisms.

In particular, symplectic actions of $\mathbb{R}$ on (M, ω) are in one-to-one correspondence with complete symplectic vector fields on M.

Examples.

1. On $\mathbb{R}^{2n}$ with $\omega = \sum dx_i \wedge dy_i$, let $X = -\frac{\partial}{\partial y_1}$. The orbits of the action generated by X are lines parallel to the y_1-axis,

$$\{(x_1, y_1 - t, x_2, y_2, \ldots, x_n, y_n) \mid t \in \mathbb{R}\} .$$

   Since $X = X_{x_1}$ is hamiltonian (with hamiltonian function $H = x_1$), this is actually an example of a *hamiltonian action* of $\mathbb{R}$.

2. On the symplectic 2-torus $(\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$, the one-parameter groups of diffeomorphisms given by rotation around each circle, $\psi_{1,t}(\theta_1, \theta_2) = (\theta_1 + t, \theta_2)$ $(t \in \mathbb{R})$ and $\psi_{2,t}$ similarly defined, are symplectic actions of S^1.

3. On the symplectic 2-sphere $(S^2, d\theta \wedge dh)$ in cylindrical coordinates, the one-parameter group of diffeomorphisms given by rotation around the vertical axis, $\psi_t(\theta, h) = (\theta + t, h)$ $(t \in \mathbb{R})$ is a symplectic action of the group $S^1 \simeq \mathbb{R}/\langle 2\pi \rangle$, as it preserves the area form $d\theta \wedge dh$. Since the vector field corresponding to ψ is hamiltonian (with hamiltonian function $H = h$), this is an example of a *hamiltonian action* of S^1.

$\diamondsuit$

Definition 4.13 *A symplectic action ψ of S^1 or $\mathbb{R}$ on (M, ω) is* **hamiltonian** *if the vector field generated by ψ is hamiltonian. Equivalently, an action ψ of S^1 or $\mathbb{R}$ on (M, ω) is* **hamiltonian** *if there is $H : M \to \mathbb{R}$ with $dH = \imath_X \omega$, where X is the vector field generated by ψ.*

What is a "hamiltonian action" of an arbitrary Lie group?

For the case where $G = \mathbb{T}^n = S^1 \times \ldots \times S^1$ is an n-torus, an action $\psi : G \to \mathrm{Sympl}(M, \omega)$ should be called *hamiltonian* when each restriction

$$\psi^i := \psi|_{i\text{th } S^1 \text{ factor}} : S^1 \longrightarrow \mathrm{Sympl}(M, \omega)$$

is hamiltonian in the previous sense with hamiltonian function preserved by the action of the rest of G.

When G is not a product of S^1's or $\mathbb{R}$'s, the solution is to use an upgraded hamiltonian function, known as a *moment map*. Up to an additive constant, a moment map μ is determined by coordinate functions μ_i satisfying $d\mu_i = \imath_{X_i}\omega$ for a basis X_i of the Lie algebra of G. There are various ways to fix that constant, and we can always choose μ *equivariant*, i.e., intertwining the action of G on M with the coadjoint action of G on the dual of the Lie algebra (see Appendix B), as defined in the next section.

4.7 Moment Maps

Let

(M, ω)	be a symplectic manifold,
G	a Lie group, and
$\psi : G \to \mathrm{Sympl}(M, \omega)$	a symplectic action, i.e., a group homomorphism with smooth evaluation map $\mathrm{ev}_\psi(g, p) := \psi_g(p)$.

Case $G = \mathbb{R}$:

We have the following bijective correspondence:

$$\left\{ \begin{array}{c} \text{symplectic actions} \\ \text{of } \mathbb{R} \text{ on } M \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{complete symplectic} \\ \text{vector fields on } M \end{array} \right\}$$

$$\psi \quad \longmapsto \quad X_p = \frac{d\psi_t(p)}{dt}$$

$$\psi = \exp tX \quad \longleftarrow \quad X$$

"flow of X" "vector field generated by ψ"

The action ψ is **hamiltonian** if there is a function $H : M \to \mathbb{R}$ such that $dH = \imath_X \omega$ where X is the vector field on M generated by ψ.

Case $G = S^1$:

An action of S^1 is an action of $\mathbb{R}$ which is 2π-periodic: $\psi_{2\pi} = \psi_0$. The S^1-action is called **hamiltonian** if the underlying $\mathbb{R}$-action is hamiltonian.

General case:

Let

(M, ω)	be a symplectic manifold,
G	a Lie group,
$\mathfrak{g}$	the Lie algebra of G,
$\mathfrak{g}^*$	the dual vector space of $\mathfrak{g}$, and
$\psi : G \longrightarrow \text{Sympl}(M, \omega)$	a symplectic action.

Definition 4.14 *The action ψ is a* **hamiltonian action** *if there exists a map*

$$\mu : M \longrightarrow \mathfrak{g}^*$$

satisfying:

1. *For each $X \in \mathfrak{g}$, let*
   - $\mu^X : M \to \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X \rangle$, *be the component of μ along X,*
   - $X^\#$ *be the vector field on M generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\} \subseteq G$.*

   Then

   $$d\mu^X = \imath_{X^\#} \omega$$

   i.e., μ^X is a hamiltonian function for the vector field $X^\#$.

2. *μ is* equivariant *with respect to the given action ψ of G on M and the coadjoint action* Ad^* *of G on $\mathfrak{g}^*$:*

   $$\mu \circ \psi_g = \text{Ad}_g^* \circ \mu \ , \qquad \text{for all } g \in G \ .$$

The vector (M, ω, G, μ) is then called a **hamiltonian G-space** *and μ is a* **moment map***.*

This definition matches the previous ones for the cases $G = \mathbb{R}$, S^1, torus, where equivariance becomes invariance since the coadjoint action is trivial.

Case $G = S^1$ (or $\mathbb{R}$):

Here $\mathfrak{g} \simeq \mathbb{R}$, $\mathfrak{g}^* \simeq \mathbb{R}$. A moment map $\mu : M \longrightarrow \mathbb{R}$ satisfies:

1. For the generator $X = 1$ of $\mathfrak{g}$, we have $\mu^X(p) = \mu(p) \cdot 1$, i.e., $\mu^X = \mu$, and $X^\#$ is the standard vector field on M generated by S^1. Then $d\mu = \imath_{X^\#}\omega$.

2. μ is invariant: $\mathcal{L}_{X^\#}\mu = \imath_{X^\#} d\mu = 0$.

Case $G = \mathbb{T}^n = n$-torus:

Here $\mathfrak{g} \simeq \mathbb{R}^n$, $\mathfrak{g}^* \simeq \mathbb{R}^n$. A moment map $\mu : M \longrightarrow \mathbb{R}^n$ satisfies:

1. For each basis vector X_i of $\mathbb{R}^n$, μ^{X_i} is a hamiltonian function for $X_i^\#$.

2. μ is invariant.

Atiyah, Guillemin and Sternberg [6, 26] showed that the image of the moment map for a hamiltonian torus action on a compact connected symplectic manifold is always a polytope.[3]

Theorem 4.15 (Atiyah [6], Guillemin-Sternberg [26]) *Let (M, ω) be a compact connected symplectic manifold, and let $\mathbb{T}^m$ be an m-torus. Suppose that $\psi : \mathbb{T}^m \to \mathrm{Sympl}(M, \omega)$ is a hamiltonian action with moment map $\mu : M \to \mathbb{R}^m$. Then:*

(a) the levels of μ are connected;

(b) the image of μ is convex;

(c) the image of μ is the convex hull of the images of the fixed points of the action.

[3]A **polytope** in $\mathbb{R}^n$ is the convex hull of a finite number of points in $\mathbb{R}^n$. A **convex polyhedron** is a subset of $\mathbb{R}^n$ which is the intersection of a finite number of affine half-spaces. Hence, polytopes coincide with bounded convex polyhedra.

The image $\mu(M)$ of the moment map is called the **moment polytope**. A proof of Theorem 4.15 can be found in [35].

An action of a group G on a manifold M is called **effective** if each group element $g \neq e$ moves at least one $p \in M$, that is, $\cap_{p\in M} G_p = \{e\}$, where $G_p = \{g \in G \mid g \cdot p = p\}$ is the stabilizer of p.

Exercise 23
Suppose that $\mathbb{T}^m$ acts linearly on $(\mathbb{C}^n, \omega_0)$. Let $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^m$ be the weights appearing in the corresponding weight space decomposition, that is,

$$\mathbb{C}^n \simeq \bigoplus_{k=1}^{n} V_{\lambda^{(k)}} \ ,$$

where, for $\lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_m^{(k)})$, $\mathbb{T}^m$ acts on the complex line $V_{\lambda^{(k)}}$ by

$$(e^{it_1}, \ldots, e^{it_m}) \cdot v = e^{i \sum_j \lambda_j^{(k)} t_j} v \ , \qquad \forall v \in V_{\lambda^{(k)}} \ , \ \forall k = 1, \ldots, n \ .$$

(a) Show that, if the action is effective, then $m \leq n$ and the weights $\lambda^{(1)}, \ldots, \lambda^{(n)}$ are part of a $\mathbb{Z}$-basis of $\mathbb{Z}^m$.

(b) Show that, if the action is symplectic (hence, hamiltonian), then the weight spaces $V_{\lambda^{(k)}}$ are symplectic subspaces.

(c) Show that, if the action is hamiltonian, then a moment map is given by

$$\mu(v) = -\tfrac{1}{2} \sum_{k=1}^{n} \lambda^{(k)} ||v_{\lambda^{(k)}}||^2 \ (+ \text{ constant }) \ ,$$

where $|| \cdot ||$ is the standard norm[a] and $v = v_{\lambda^{(1)}} + \ldots + v_{\lambda^{(n)}}$ is the weight space decomposition. Cf. Example 1 below.

(d) Conclude that, if $\mathbb{T}^n$ acts on $\mathbb{C}^n$ in a linear, effective and hamiltonian way, then any moment map μ is a submersion, i.e., each differential $d\mu_v : \mathbb{C}^n \to \mathbb{R}^n$ $(v \in \mathbb{C}^n)$ is surjective.

[a]Notice that the standard inner product satisfies $(v, w) = \omega_0(v, Jv)$ where $J\frac{\partial}{\partial z} = i\frac{\partial}{\partial z}$ and $J\frac{\partial}{\partial \bar{z}} = -i\frac{\partial}{\partial \bar{z}}$. In particular, the standard norm is invariant for a symplectic complex-linear action.

The following two results use the crucial fact that any effective action $\mathbb{T}^m \to \text{Diff}(M)$ has orbits of dimension m; a proof may be found in [11].

Corollary 4.16 *Under the conditions of the convexity theorem, if the $\mathbb{T}^m$-action is effective, then there must be at least $m + 1$ fixed points.*

Proof. At a point p of an m-dimensional orbit the moment map is a submersion, i.e., $(d\mu_1)_p, \ldots, (d\mu_m)_p$ are linearly independent. Hence, $\mu(p)$ is an interior point of $\mu(M)$, and $\mu(M)$ is a nondegenerate polytope. Any nondegenerate polytope in $\mathbb{R}^m$ must have at least $m+1$ vertices. The vertices of $\mu(M)$ are images of fixed points. $\square$

Proposition 4.17 *Let $(M, \omega, \mathbb{T}^m, \mu)$ be a hamiltonian $\mathbb{T}^m$-space. If the $\mathbb{T}^m$-action is effective, then* $\dim M \geq 2m$.

Proof. Since the moment map is constant on an orbit $\mathcal{O}$, for $p \in \mathcal{O}$ the exterior derivative

$$d\mu_p : T_pM \longrightarrow \mathfrak{g}^*$$

maps $T_p\mathcal{O}$ to 0. Thus

$$T_p\mathcal{O} \subseteq \ker d\mu_p = (T_p\mathcal{O})^\omega \ ,$$

where $(T_p\mathcal{O})^\omega$ is the symplectic orthogonal of $T_p\mathcal{O}$. This shows that orbits $\mathcal{O}$ of a hamiltonian torus action are always isotropic submanifolds of M. In particular, by symplectic linear algebra we have that $\dim \mathcal{O} \leq \frac{1}{2} \dim M$. Now consider an m-dimensional orbit (which always exists for an effective $\mathbb{T}^m$-action; see for instance [11]). $\square$

Examples.

1. Let $\mathbb{T}^n = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \ : \ |t_j| = 1, \text{ for all } j\}$ be a torus acting on $\mathbb{C}^n$ by

   $$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1^{k_1} z_1, \ldots, t_n^{k_n} z_n) \ ,$$

   where $k_1, \ldots, k_n \in \mathbb{Z}$ are fixed. This action is hamiltonian with moment map $\mu : \mathbb{C}^n \to (\mathfrak{t}^n)^* \simeq \mathbb{R}^n$ given by

   $$\mu(z_1, \ldots, z_n) = -\tfrac{1}{2}(k_1|z_1|^2, \ldots, k_n|z_n|^2) \ (+ \text{ constant }) \ \ .$$

2. Suppose that a Lie group G acts in a hamiltonian way on two symplectic manifolds (M_j, ω_j), $j = 1, 2$, with moment maps $\mu_j : M_j \to \mathfrak{g}^*$. Then the diagonal action of G on $M_1 \times M_2$ is hamiltonian with moment map $\mu : M_1 \times M_2 \to \mathfrak{g}^*$ given by

   $$\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2) \ , \quad \text{for } p_j \in M_j \ .$$

3. The vector field $X^\#$ generated by $X \in \mathfrak{g}$ for the coadjoint representation of a Lie group G on $\mathfrak{g}^*$ satisfies $\langle X^\#_\xi, Y\rangle = \langle \xi, [Y,X]\rangle$, for any $Y \in \mathfrak{g}$. Equip the coadjoint orbits with the canonical symplectic forms (Section 4.3). Then, for each $\xi \in \mathfrak{g}^*$, the coadjoint action on the orbit $G \cdot \xi$ is hamiltonian with moment map the inclusion map:
$$\mu : G \cdot \xi \hookrightarrow \mathfrak{g}^* \ .$$

$\diamondsuit$

Exercises 24

(a) Consider the natural action of $\mathrm{U}(n)$ on $(\mathbb{C}^n, \omega_0)$. Show that this action is hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathfrak{u}(n)$ given by
$$\mu(z) = \tfrac{i}{2} zz^* \ ,$$
where we identify the Lie algebra $\mathfrak{u}(n)$ with its dual via the inner product $(A,B) = \mathrm{trace}(A^*B)$.

Hint: Denote the elements of $\mathrm{U}(n)$ in terms of real and imaginary parts $g = h + i\,k$. Then g acts on $\mathbb{R}^{2n}$ by the linear symplectomorphism $\begin{pmatrix} h & -k \\ k & h \end{pmatrix}$. The Lie algebra $\mathfrak{u}(n)$ is the set of skew-hermitian matrices $X = V + i\,W$ where $V = -V^t \in \mathbb{R}^{n\times n}$ and $W = W^t \in \mathbb{R}^{n \times n}$. Show that the infinitesimal action is generated by the hamiltonian functions $\mu^X(z) = -\frac{1}{2}(x, Wx) + (y, Vx) - \frac{1}{2}(y, Wy)$ where $z = x + i\,y$, $x, y \in \mathbb{R}^n$ and $(\cdot,\cdot)$ is the standard inner product. Show that
$$\mu^X(z) = \tfrac{1}{2} i\, z^* X z = \tfrac{1}{2} i\, \mathrm{trace}(zz^*X) \ .$$
Check that μ is equivariant.

(b) Consider the natural action of $\mathrm{U}(k)$ on the space $(\mathbb{C}^{k\times n}, \omega_0)$ of complex $(k \times n)$-matrices. Identify the Lie algebra $\mathfrak{u}(k)$ with its dual via the inner product $(A,B) = \mathrm{trace}(A^*B)$. Prove that a moment map for this action is given by
$$\mu(A) = \tfrac{i}{2} AA^* + \tfrac{\mathrm{Id}}{2i} \ , \quad \text{for } A \in \mathbb{C}^{k \times n} \ .$$
(The constant $\frac{\mathrm{Id}}{2i}$ is just a choice.)

Hint: Example 2 and Exercise (a).

(c) Consider the $\mathrm{U}(n)$-action by conjugation on the space $(\mathbb{C}^{n^2}, \omega_0)$ of complex $(n \times n)$-matrices. Show that a moment map for this action is given by
$$\mu(A) = \tfrac{i}{2}[A, A^*] \ .$$

Hint: Previous exercise and its "transpose" version.

4.8 Language for Mechanics

Example.

Let $G = \mathrm{SO}(3) = \{A \in \mathrm{GL}(3;\mathbb{R}) \mid A\ {}^tA = \mathrm{Id} \text{ and } \det A = 1\}$. Then $\mathfrak{g} = \{A \in \mathfrak{gl}(3;\mathbb{R}) \mid A + A^t = 0\}$ is the space of 3×3 skew-symmetric matrices and can be identified with $\mathbb{R}^3$. The Lie bracket on $\mathfrak{g}$ can be identified with the exterior product via

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad \longmapsto \quad \vec{a} = (a_1, a_2, a_3)$$

$$[A, B] = AB - BA \quad \longmapsto \quad \vec{a} \times \vec{b}\,.$$

Exercise 25
Under the identifications $\mathfrak{g}, \mathfrak{g}^* \simeq \mathbb{R}^3$, the adjoint and coadjoint actions are the usual SO(3)-action on $\mathbb{R}^3$ by rotations.

Therefore, the coadjoint orbits are the spheres in $\mathbb{R}^3$ centered at the origin. Section 4.3 shows how general coadjoint orbits are symplectic. $\diamondsuit$

The name "moment map" comes from being the generalization of linear and angular momenta in classical mechanics.

Translation: Consider $\mathbb{R}^6$ with coordinates $x_1, x_2, x_3, y_1, y_2, y_3$ and symplectic form $\omega = \sum dx_i \wedge dy_i$. Let $\mathbb{R}^3$ act on $\mathbb{R}^6$ by translations:

$$\vec{a} \in \mathbb{R}^3 \longmapsto \quad \psi_{\vec{a}} \in \mathrm{Sympl}(\mathbb{R}^6, \omega)$$

$$\psi_{\vec{a}}(\vec{x}, \vec{y}) = (\vec{x} + \vec{a}, \vec{y})\ .$$

Then $X^\# = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3}$ for $X = \vec{a}$, and

$$\mu : \mathbb{R}^6 \longrightarrow \mathbb{R}^3\ , \quad \mu(\vec{x}, \vec{y}) = \vec{y}$$

is a moment map, with

$$\mu^{\vec{a}}(\vec{x}, \vec{y}) = \langle \mu(\vec{x}, \vec{y}), \vec{a} \rangle = \vec{y} \cdot \vec{a}\ .$$

Classically, $\overrightarrow{y}$ is called the **momentum vector** corresponding to the **position vector** $\overrightarrow{x}$, and the map μ is called the **linear momentum**.

Rotation: The SO(3)-action on $\mathbb{R}^3$ by rotations lifts to a symplectic action ψ on the cotangent bundle $\mathbb{R}^6$. The infinitesimal version of this action is

$$\overrightarrow{a} \in \mathbb{R}^3 \quad \longmapsto \quad d\psi(\overrightarrow{a}) \in \chi^{\mathrm{sympl}}(\mathbb{R}^6)$$

$$d\psi(\overrightarrow{a})(\overrightarrow{x}, \overrightarrow{y}) = (\overrightarrow{a} \times \overrightarrow{x}, \overrightarrow{a} \times \overrightarrow{y}) \ .$$

Then

$$\mu : \mathbb{R}^6 \longrightarrow \mathbb{R}^3 \ , \quad \mu(\overrightarrow{x}, \overrightarrow{y}) = \overrightarrow{x} \times \overrightarrow{y}$$

is a moment map, with

$$\mu^{\overrightarrow{a}}(\overrightarrow{x}, \overrightarrow{y}) = \langle \mu(\overrightarrow{x}, \overrightarrow{y}), \overrightarrow{a} \rangle = (\overrightarrow{x} \times \overrightarrow{y}) \cdot \overrightarrow{a}.$$

The map μ is called the **angular momentum**.

Let (M, ω, G, μ) be a hamiltonian G-space.

Theorem 4.18 (Noether) *A function $f : M \to \mathbb{R}$ is G-invariant if and only if μ is constant on the trajectories of the hamiltonian vector field of f.*

Proof. Let v_f be the hamiltonian vector field of f. Let $X \in \mathfrak{g}$ and $\mu^X = \langle \mu, X \rangle : M \to \mathbb{R}$. We have

$$\begin{aligned} \mathcal{L}_{v_f} \mu^X &= \imath_{v_f} d\mu^X = \imath_{v_f} \imath_{X^\#} \omega \\ &= -\imath_{X^\#} \imath_{v_f} \omega = -\imath_{X^\#} df \\ &= -\mathcal{L}_{X^\#} f = 0 \end{aligned}$$

because f is G-invariant. ☐

Definition 4.19 *A G-invariant function $f : M \to \mathbb{R}$ is called an* **integral of motion** *of (M, ω, G, μ). If μ is constant on the trajectories of a hamiltonian vector field v_f, then the corresponding one-parameter group of diffeomorphisms $\{\exp t v_f \mid t \in \mathbb{R}\}$ is called a* **symmetry** *of (M, ω, G, μ).*

The **Noether principle** asserts that there is a one-to-one correspondence between symmetries and integrals of motion.

4.9 Obstructions to Moment Maps

Let $\mathfrak{g}$ be a Lie algebra, and

$$\begin{aligned} C^k &:= \Lambda^k \mathfrak{g}^* = k\text{-cochains on } \mathfrak{g} \\ &= \text{alternating } k\text{-linear maps } \underbrace{\mathfrak{g} \times \ldots \times \mathfrak{g}}_{k} \longrightarrow \mathbb{R} \ . \end{aligned}$$

Define a linear operator $\delta : C^k \to C^{k+1}$ by

$$\delta c(X_0, \ldots, X_k) = \sum_{i<j} (-1)^{i+j} c([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k) \ .$$

Exercise 26
Check that $\delta^2 = 0$.

The **Lie algebra cohomology groups** (or **Chevalley cohomology groups**) of $\mathfrak{g}$ are the cohomology groups of the complex $0 \xrightarrow{\delta} C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} \ldots$:

$$H^k(\mathfrak{g}; \mathbb{R}) := \frac{\ker \delta : C^k \longrightarrow C^{k+1}}{\text{im } \delta : C^{k-1} \longrightarrow C^k} \ .$$

Theorem 4.20 *If $\mathfrak{g}$ is the Lie algebra of a compact connected Lie group G, then*

$$H^k(\mathfrak{g}; \mathbb{R}) = H^k_{\text{deRham}}(G) \ .$$

Proof. Exercise. Hint: by averaging show that the de Rham cohomology can be computed from the subcomplex of G-invariant forms.
□

Meaning of $H^1(\mathfrak{g}; \mathbb{R})$ and $H^2(\mathfrak{g}; \mathbb{R})$:

- An element of $C^1 = \mathfrak{g}^*$ is a linear functional on $\mathfrak{g}$. If $c \in \mathfrak{g}^*$, then $\delta c(X_0, X_1) = -c([X_0, X_1])$. The **commutator ideal** of $\mathfrak{g}$ is

  $$[\mathfrak{g}, \mathfrak{g}] := \{\text{linear combinations of } [X, Y] \text{ for any } X, Y \in \mathfrak{g}\} \ .$$

  Since $\delta c = 0$ if and only if c vanishes on $[\mathfrak{g}, \mathfrak{g}]$, we conclude that

  $$H^1(\mathfrak{g}; \mathbb{R}) = [\mathfrak{g}, \mathfrak{g}]^0$$

  where $[\mathfrak{g}, \mathfrak{g}]^0 \subseteq \mathfrak{g}^*$ is the annihilator of $[\mathfrak{g}, \mathfrak{g}]$.

- An element of C^2 is an alternating bilinear map $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

$$\delta c(X_0, X_1, X_2) =$$
$$-c([X_0, X_1], X_2) + c([X_0, X_2], X_1) - c([X_1, X_2], X_0) \ .$$

If $c = \delta b$ for some $b \in C^1$, then

$$c(X_0, X_1) = (\delta b)(X_0, X_1) = -b([X_0, X_1]\ ).$$

Theorem 4.21 *If $H^1(\mathfrak{g}; \mathbb{R}) = H^2(\mathfrak{g}, \mathbb{R}) = 0$, then any symplectic G-action is hamiltonian.*

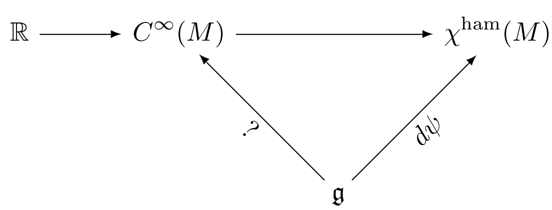
Proof. Let $\psi : G \to \mathrm{Sympl}(M, \omega)$ be a symplectic action of G on a symplectic manifold (M, ω). Since

$$H^1(\mathfrak{g}; \mathbb{R}) = 0 \iff [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$$

and since commutators of symplectic vector fields are hamiltonian, we have

$$d\psi : \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \longrightarrow \chi^{\mathrm{ham}}(M).$$

The action ψ is hamiltonian if and only if there is a Lie algebra homomorphism $\mu^* : \mathfrak{g} \to C^\infty(M)$ such that the following diagram commutes.



We first take an arbitrary vector space lift $\tau : \mathfrak{g} \to C^\infty(M)$ making the diagram commute, i.e., for each basis vector $X \in \mathfrak{g}$, we choose

$$\tau(X) = \tau^X \in C^\infty(M) \qquad \text{such that} \qquad v_{(\tau^X)} = d\psi(X) \ .$$

The map $X \mapsto \tau^X$ may not be a Lie algebra homomorphism. By construction, $\tau^{[X,Y]}$ is a hamiltonian function for $[X, Y]^\#$, and (as computed in Section 4.3) $\{\tau^X, \tau^Y\}$ is a hamiltonian function for

$-[X^\#, Y^\#]$. Since $[X,Y]^\# = -[X^\#, Y^\#]$, the corresponding hamiltonian functions must differ by a constant:

$$\tau^{[X,Y]} - \{\tau^X, \tau^Y\} = c(X,Y) \in \mathbb{R} \; .$$

By the Jacobi identity, $\delta c = 0$. Since $H^2(\mathfrak{g};\mathbb{R}) = 0$, there is $b \in \mathfrak{g}^*$ satisfying $c = \delta b$, $c(X,Y) = -b([X,Y])$. We define

$$\begin{array}{rccl} \mu^* : & \mathfrak{g} & \longrightarrow & C^\infty(M) \\ & X & \longmapsto & \mu^*(X) = \tau^X + b(X) = \mu^X \; . \end{array}$$

Now μ^* is a Lie algebra homomorphism:

$$\mu^*([X,Y]) = \tau^{[X,Y]} + b([X,Y]) = \{\tau^X, \tau^Y\} = \{\mu^X, \mu^Y\} \; .$$

□

So when is $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$?

A compact Lie group G is **semisimple** if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Examples. The unitary group $\mathrm{U}(n)$ is not semisimple because the multiples of the identity, $S^1 \cdot \mathrm{Id}$, form a nontrivial center; at the level of the Lie algebra, this corresponds to the 1-dimensional subspace $\mathbb{R} \cdot \mathrm{Id}$ of scalar matrices which are not commutators since they are not traceless.

Any direct product of the other compact classical groups $\mathrm{SU}(n)$, $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$ is semisimple ($n > 1$). Any commutative Lie group is not semisimple. ♢

Theorem 4.22 (Whitehead Lemmas) *Let G be a compact Lie group.*

$$G \textit{ is semisimple} \quad \Longleftrightarrow \quad H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0 \; .$$

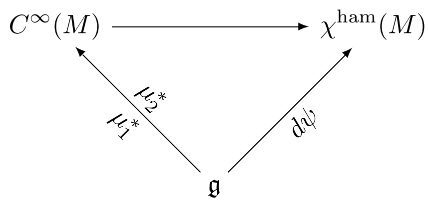
A proof can be found in [30, pages 93-95].

Corollary 4.23 *If G is semisimple, then any symplectic G-action is hamiltonian.*

As for the question of uniqueness, let G be a compact connected Lie group.

Theorem 4.24 *If $H^1(\mathfrak{g};\mathbb{R}) = 0$, then moment maps for hamiltonian G-actions are unique.*

Proof. Suppose that μ_1^* and μ_2^* are two comoment maps for an action ψ:



For each $X \in \mathfrak{g}$, μ_1^X and μ_2^X are both hamiltonian functions for $X^\#$, thus $\mu_1^X - \mu_2^X = c(X)$ is locally constant. This defines $c \in \mathfrak{g}^*$, $X \mapsto c(X)$.

Since μ_1^*, μ_2^* are Lie algebra homomorphisms, we have $c([X,Y]) = 0$, $\forall X, Y \in \mathfrak{g}$, i.e., $c \in [\mathfrak{g},\mathfrak{g}]^0 = \{0\}$. Hence, $\mu_1^* = \mu_2^*$. ☐

Corollary of this proof. *In general, if $\mu : M \to \mathfrak{g}^*$ is a moment map, then given any $c \in [\mathfrak{g},\mathfrak{g}]^0$, $\mu_1 = \mu + c$ is another moment map.*

In other words, moment maps are unique up to elements of the dual of the Lie algebra which annihilate the commutator ideal.

The two extreme cases are:

G semisimple:	any symplectic action is hamiltonian , moment maps are unique .
G commutative:	symplectic actions may not be hamiltonian , moment maps are unique up to constants $c \in \mathfrak{g}^*$.

Example. The circle action on $(\mathbb{T}^2, \omega = d\theta_1 \wedge d\theta_2)$ by rotations in the θ_1 direction has vector field $X^\# = \frac{\partial}{\partial\theta_1}$; this is a symplectic action but is not hamiltonian. ♢

Lecture 5

Symplectic Reduction

The phase space of a system of n particles is the space parametrizing the position and momenta of the particles. The mathematical model for the phase space is a symplectic manifold. Classical physicists realized that, whenever there is a symmetry group of dimension k acting on a mechanical system, then the number of degrees of freedom for the position and momenta of the particles may be reduced by $2k$. Symplectic reduction formulates this feature mathematically.

5.1 Marsden-Weinstein-Meyer Theorem

Let $\omega = \frac{i}{2}\sum dz_i \wedge d\bar{z}_i = \sum dx_i \wedge dy_i = \sum r_i dr_i \wedge d\theta_i$ be the standard symplectic form on $\mathbb{C}^n$. Consider the following S^1-action on $(\mathbb{C}^n, \omega)$:

$$t \in S^1 \longmapsto \psi_t = \text{multiplication by } t \ .$$

The action ψ is hamiltonian with moment map

$$\begin{array}{rccl} \mu : & \mathbb{C}^n & \longrightarrow & \mathbb{R} \\ & z & \longmapsto & -\frac{|z|^2}{2} + \text{constant} \end{array}$$

since

$$\begin{aligned} d\mu &= -\tfrac{1}{2} d(\sum r_i^2) \\ X^\# &= \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2} + \ldots + \frac{\partial}{\partial\theta_n} \\ \imath_{X^\#}\omega &= -\sum r_i dr_i = -\tfrac{1}{2}\sum dr_i^2 \ . \end{aligned}$$

If we choose the constant to be $\frac{1}{2}$, then $\mu^{-1}(0) = S^{2n-1}$ is the unit sphere. The orbit space of the zero level of the moment map is

$$\mu^{-1}(0)/S^1 = S^{2n-1}/S^1 = \mathbb{CP}^{n-1} \ ,$$

which is thus called a **reduced space**. This is a particular observation of the major theorem Marsden-Weinstein-Meyer which shows that reduced spaces are symplectic manifolds.

Theorem 5.1 (Marsden-Weinstein-Meyer [34, 36]) *Let (M, ω, G, μ) be a hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*

- *the orbit space $M_{\rm red} = \mu^{-1}(0)/G$ is a manifold,*
- *$\pi : \mu^{-1}(0) \to M_{\rm red}$ is a principal G-bundle, and*
- *there is a symplectic form $\omega_{\rm red}$ on $M_{\rm red}$ satisfying $i^*\omega = \pi^*\omega_{\rm red}$.*

Definition 5.2 *The pair $(M_{\rm red}, \omega_{\rm red})$ is the* **reduction** *of (M, ω) with respect to G, μ, or the* **reduced space**, *or the* **symplectic quotient**, *or the* **Marsden-Weinstein-Meyer quotient**, *etc.*

Low-brow proof for the case $G = S^1$ and $\dim M = 4$.

In this case the moment map is $\mu : M \to \mathbb{R}$. Let $p \in \mu^{-1}(0)$. Choose local coordinates:

- θ along the orbit through p,
- μ given by the moment map, and
- η_1, η_2 pullback of coordinates on $\mu^{-1}(0)/S^1$.

Then the symplectic form can be written

$$\omega = A\ d\theta \wedge d\mu + B_j\ d\theta \wedge d\eta_j + C_j\ d\mu \wedge d\eta_j + D\ d\eta_1 \wedge d\eta_2\ .$$

Since $d\mu = \imath\left(\frac{\partial}{\partial\theta}\right)\omega$, we must have $A = 1$, $B_j = 0$. Hence,

$$\omega = d\theta \wedge d\mu + C_j\ d\mu \wedge d\eta_j + D\ d\eta_1 \wedge d\eta_2\ .$$

Since ω is symplectic, we must have $D \neq 0$. Therefore, $i^*\omega = D\ d\eta_1 \wedge d\eta_2$ is the pullback of a symplectic form on M_{red}. □

Examples.

1. For the natural action of $\mathrm{U}(k)$ on $\mathbb{C}^{k\times n}$ with moment map computed in Section 4.7, we have $\mu^{-1}(0) = \{A \in \mathbb{C}^{k\times n} \mid AA^* = \mathrm{Id}\}$. Then the quotient manifold

$$\mu^{-1}(0)/\mathrm{U}(k) = \mathbb{G}(k,n)$$

is the grassmannian of k-planes in $\mathbb{C}^n$.

2. Consider the S^1-action on $(\mathbb{R}^{2n+2}, \omega_0)$ which, under the usual identification of $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$, corresponds to multiplication by e^{it}. This action is hamiltonian with a moment map $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$ given by

$$\mu(z) = -\tfrac{1}{2}|z|^2 + \tfrac{1}{2}\ .$$

Then the reduction $\mu^{-1}(0)/S^1$ is $\mathbb{CP}^n$ with the Fubini-Study symplectic form $\omega_{\text{red}} = \omega_{\text{FS}}$. To prove this assertion, let $\mathrm{pr} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ denote the standard projection, and check that

$$\mathrm{pr}^*\omega_{\text{FS}} = \tfrac{i}{2}\partial\bar\partial \log(|z|^2)\ .$$

This form has the same restriction to S^{2n+1} as ω_0.

♢

Exercise 27
The natural actions of $\mathbb{T}^{n+1}$ and $\mathrm{U}(n+1)$ on $(\mathbb{CP}^n, \omega_{\text{FS}})$ are hamiltonian, and find formulas for their moment maps.

Hint: Previous example and Section 4.7.

5.2 Ingredients

The actual proof of the Marsden-Weinstein-Meyer theorem requires the following ingredients.

1. Let $\mathfrak{g}_p$ be the Lie algebra of the stabilizer of $p \in M$. Then $d\mu_p : T_pM \to \mathfrak{g}^*$ has

$$\begin{array}{rcl} \ker\, d\mu_p & = & (T_p\mathcal{O}_p)^{\omega_p} \\ \mathrm{im}\, d\mu_p & = & \mathfrak{g}_p^0 \end{array}$$

where $\mathcal{O}_p$ is the G-orbit through p, and $\mathfrak{g}_p^0 = \{\xi \in \mathfrak{g}^* \mid \langle \xi, X\rangle = 0,\ \forall X \in \mathfrak{g}_p\}$ is the annihilator of $\mathfrak{g}_p$.

Proof. Stare at the expression $\omega_p(X_p^{\#}, v) = \langle d\mu_p(v), X\rangle$, for all $v \in T_pM$ and all $X \in \mathfrak{g}$, and count dimensions. □

Consequences:

- The action is locally free at p
  $\iff$ $\mathfrak{g}_p = \{0\}$
  $\iff$ $d\mu_p$ is surjective
  $\iff$ p is a regular point of μ.
- G acts freely on $\mu^{-1}(0)$
  $\Longrightarrow$ 0 is a regular value of μ
  $\Longrightarrow$ $\mu^{-1}(0)$ is a closed submanifold of M of codimension equal to $\dim G$.
- G acts freely on $\mu^{-1}(0)$
  $\Longrightarrow$ $T_p\mu^{-1}(0) = \ker d\mu_p$ (for $p \in \mu^{-1}(0)$)
  $\Longrightarrow$ $T_p\mu^{-1}(0)$ and $T_p\mathcal{O}_p$ are symplectic orthocomplements in T_pM.

  In particular, the tangent space to the orbit through $p \in \mu^{-1}(0)$ is an isotropic subspace of T_pM. Hence, orbits in $\mu^{-1}(0)$ are isotropic.

Since any tangent vector to the orbit is the value of a vector field generated by the group, we can confirm that orbits are isotropic directly by computing, for any $X, Y \in \mathfrak{g}$ and any $p \in \mu^{-1}(0)$,

$$\begin{aligned} \omega_p(X_p^\#, Y_p^\#) &= \text{hamiltonian function for } [Y^\#, X^\#] \text{ at } p \\ &= \text{hamiltonian function for } [Y, X]^\# \text{ at } p \\ &= \mu^{[Y,X]}(p) = 0 \ . \end{aligned}$$

2. **Lemma 5.3** *Let (V, ω) be a symplectic vector space. Suppose that I is an isotropic subspace, that is, $\omega|_I \equiv 0$. Then ω induces a canonical symplectic form Ω on I^ω / I.*

   Proof. Let $u, v \in I^\omega$, and $[u], [v] \in I^\omega/I$. Define $\Omega([u],[v]) = \omega(u, v)$.

   - Ω is well-defined:

     $$\omega(u+i, v+j) = \omega(u,v) + \underbrace{\omega(u,j)}_{0} + \underbrace{\omega(i,v)}_{0} + \underbrace{\omega(i,j)}_{0}\ , \forall i, j \in I\ .$$

   - Ω is nondegenerate:
     Suppose that $u \in I^\omega$ has $\omega(u, v) = 0$, for all $v \in I^\omega$.
     Then $u \in (I^\omega)^\omega = I$, i.e., $[u] = 0$.

   □

3. **Proposition 5.4** *If a compact Lie group G acts freely on a manifold M, then M/G is a manifold and the map $\pi : M \to M/G$ is a principal G-bundle.*

   Proof.

   We will first show that, for any $p \in M$, the G-orbit through p is a compact embedded submanifold of M diffeomorphic to G.

   Since the action is smooth, the evaluation map $\mathrm{ev} : G \times M \to M$, $\mathrm{ev}(g, p) = g \cdot p$, is smooth. Let $\mathrm{ev}_p : G \to M$ be defined by $\mathrm{ev}_p(g) = g \cdot p$. The map ev_p provides the embedding we seek:

   The image of ev_p is the G-orbit through p. Injectivity of ev_p follows from the action of G being free. The map ev_p is proper because, if A is a compact, hence closed, subset of M, then its inverse image $(\mathrm{ev}_p)^{-1}(A)$, being a closed subset of the compact Lie group G, is also compact. It remains to show that ev_p is an immersion. For $X \in \mathfrak{g} \simeq T_eG$, we have

$$d(\mathrm{ev}_p)_e(X) = 0 \iff X_p^\# = 0 \iff X = 0 \ ,$$

as the action is free. We conclude that $d(\mathrm{ev}_p)_e$ is injective. At any other point $g \in G$, for $X \in T_gG$, we have

$$d(\mathrm{ev}_p)_g(X) = 0 \iff d(\mathrm{ev}_p \circ R_g)_e \circ (dR_{g^{-1}})_g(X) = 0 \ ,$$

where $R_g : G \to G$ is right multiplication by g. But $\mathrm{ev}_p \circ R_g = \mathrm{ev}_{g\cdot p}$ has an injective differential at e, and $(dR_{g^{-1}})_g$ is an isomorphism. It follows that $d(\mathrm{ev}_p)_g$ is always injective.

Exercise 28
Show that, even if the action is not free, the G-orbit through p is a compact embedded submanifold of M. In that case, the orbit is diffeomorphic to the quotient of G by the isotropy of p: $\mathcal{O}_p \simeq G/G_p$.

Let S be a transverse section to $\mathcal{O}_p$ at p; this is called a **slice**. Choose a coordinate system $x_1, \dots, x_n$ centered at p such that

$$\begin{array}{lcccccccc} \mathcal{O}_p \simeq G & : & x_1 & = & \dots & = & x_k & = & 0 \\ S & : & x_{k+1} & = & \dots & = & x_n & = & 0 \ . \end{array}$$

Let $S_\varepsilon = S \cap B_\varepsilon(0, \mathbb{R}^n)$ where $B_\varepsilon(0, \mathbb{R}^n)$ is the ball of radius ε centered at 0 in $\mathbb{R}^n$. Let $\eta : G \times S \to M$, $\eta(g, s) = g \cdot s$. Apply the following equivariant tubular neighborhood theorem.

Theorem 5.5 (Slice Theorem) *Let G be a compact Lie group acting on a manifold M such that G acts freely at $p \in M$. For sufficiently small ε, $\eta : G \times S_\varepsilon \to M$ maps $G \times S_\varepsilon$ diffeomorphically onto a G-invariant neighborhood $\mathcal{U}$ of the G-orbit through p.*

The proof of this slice theorem is sketched further below.

Corollary 5.6 *If the action of G is free at p, then the action is free on $\mathcal{U}$.*

Corollary 5.7 *The set of points where G acts freely is open.*

Corollary 5.8 *The set $G \times S_\varepsilon \simeq \mathcal{U}$ is G-invariant. Hence, the quotient $\mathcal{U}/G \simeq S_\varepsilon$ is smooth.*

Conclusion of the proof that M/G is a manifold and $\pi : M \to M/G$ is a smooth fiber map.

For $p \in M$, let $q = \pi(p) \in M/G$. Choose a G-invariant neighborhood $\mathcal{U}$ of p as in the slice theorem: $\mathcal{U} \simeq G \times S$ (where $S = S_\varepsilon$ for an appropriate ε). Then $\pi(\mathcal{U}) = \mathcal{U}/G =: \mathcal{V}$ is an open neighborhood of q in M/G. By the slice theorem, $S \stackrel{\simeq}{\to} \mathcal{V}$ is a homeomorphism. We will use such neighborhoods $\mathcal{V}$ as charts on M/G. To show that the transition functions associated with these charts are smooth, consider two G-invariant open sets $\mathcal{U}_1, \mathcal{U}_2$ in M and corresponding slices S_1, S_2 of the G-action. Then $S_{12} = S_1 \cap \mathcal{U}_2$, $S_{21} = S_2 \cap \mathcal{U}_1$ are both slices for the G-action on $\mathcal{U}_1 \cap \mathcal{U}_2$. To compute the transition map $S_{12} \to S_{21}$, consider the diagram

$$\begin{array}{ccccccc}
S_{12} & \stackrel{\simeq}{\longrightarrow} & \mathrm{id} \times S_{12} & \hookrightarrow & G \times S_{12} & & \\
 & & & & & \searrow^{\simeq} & \\
 & & & & & & \mathcal{U}_1 \cap \mathcal{U}_2\ . \\
 & & & & & \nearrow_{\simeq} & \\
S_{21} & \stackrel{\simeq}{\longrightarrow} & \mathrm{id} \times S_{21} & \hookrightarrow & G \times S_{21} & &
\end{array}$$

Then the composition

$$S_{12} \hookrightarrow \mathcal{U}_1 \cap \mathcal{U}_2 \stackrel{\simeq}{\longrightarrow} G \times S_{21} \stackrel{pr}{\longrightarrow} S_{21}$$

is smooth.

Finally, we need to show that $\pi : M \to M/G$ is a smooth fiber map. For $p \in M$, $q = \pi(p)$, choose a G-invariant neighborhood $\mathcal{U}$ of the G-orbit through p of the form $\eta : G \times S \stackrel{\simeq}{\to} \mathcal{U}$. Then

$\mathcal{V} = \mathcal{U}/G \simeq S$ is the corresponding neighborhood of q in M/G:

$$\begin{array}{ccccccc} M \supseteq & \mathcal{U} & \overset{\eta}{\simeq} & G \times S & \simeq & G \times \mathcal{V} \\ & \downarrow \pi & & & & \downarrow \\ M/G \supseteq & \mathcal{V} & & = & & \mathcal{V} \end{array}$$

Since the projection on the right is smooth, π is smooth.

Exercise 29
Check that the transition functions for the bundle defined by π are smooth.

□

Sketch for the proof of the slice theorem. We need to show that, for ε sufficiently small, $\eta : G \times S_\varepsilon \to \mathcal{U}$ is a diffeomorphism where $\mathcal{U} \subseteq M$ is a G-invariant neighborhood of the G-orbit through p. Show that:

(a) $d\eta_{(\text{id},p)}$ is bijective.

(b) Let G act on $G \times S$ by the product of its left action on G and trivial action on S. Then $\eta : G \times S \to M$ is G-equivariant.

(c) $d\eta$ is bijective at all points of $G \times \{p\}$. This follows from (a) and (b).

(d) The set $G \times \{p\}$ is compact, and $\eta : G \times S \to M$ is injective on $G \times \{p\}$ with $d\eta$ bijective at all these points. By the implicit function theorem, there is a neighborhood $\mathcal{U}_0$ of $G \times \{p\}$ in $G \times S$ such that η maps $\mathcal{U}_0$ diffeomorphically onto a neighborhood $\mathcal{U}$ of the G-orbit through p.

(e) The sets $G \times S_\varepsilon$, varying ε, form a neighborhood base for $G \times \{p\}$ in $G \times S$. So in (d) we may take $\mathcal{U}_0 = G \times S_\varepsilon$.

□

5.3 Proof of the Reduction Theorem

Since

$$\begin{array}{rcl} G \text{ acts freely on } \mu^{-1}(0) & \Longrightarrow & d\mu_p \text{ is surjective for all } p \in \mu^{-1}(0) \\ & \Longrightarrow & 0 \text{ is a regular value} \\ & \Longrightarrow & \mu^{-1}(0) \text{ is a submanifold} \\ & & \text{of codimension} = \dim G \end{array}$$

for the first two parts of the Marsden-Weinstein-Meyer theorem it is enough to apply the third ingredient from Section 5.2 to the free action of G on $\mu^{-1}(0)$.

At $p \in \mu^{-1}(0)$ the tangent space to the orbit $T_p\mathcal{O}_p$ is an isotropic subspace of the symplectic vector space (T_pM, ω_p), i.e., $T_p\mathcal{O}_p \subseteq (T_p\mathcal{O}_p)^\omega$.

$$(T_p\mathcal{O}_p)^\omega = \ker d\mu_p = T_p\mu^{-1}(0) \ .$$

The lemma (second ingredient) gives a canonical symplectic structure on the quotient $T_p\mu^{-1}(0)/T_p\mathcal{O}_p$. The point $[p] \in M_{\text{red}} = \mu^{-1}(0)/G$ has tangent space $T_{[p]}M_{\text{red}} \simeq T_p\mu^{-1}(0)/T_p\mathcal{O}_p$. Thus the lemma defines a nondegenerate 2-form ω_{red} on M_{red}. This is well-defined because ω is G-invariant.

By construction $i^*\omega = \pi^*\omega_{\text{red}}$ where

$$\begin{array}{ccc} \mu^{-1}(0) & \stackrel{i}{\hookrightarrow} & M \\ \downarrow \pi & & \\ M_{\text{red}} & & \end{array}$$

Hence, $\pi^* d\omega_{\text{red}} = d\pi^*\omega_{\text{red}} = d\imath^*\omega = \imath^* d\omega = 0$. The closedness of ω_{red} follows from the injectivity of π^*. $\square$

Remark. Suppose that another Lie group H acts on (M, ω) in a hamiltonian way with moment map $\phi : M \to \mathfrak{h}^*$. If the H-action commutes with the G-action, and if ϕ is G-invariant, then M_{red} inherits a hamiltonian action of H, with moment map $\phi_{\text{red}} : M_{\text{red}} \to \mathfrak{h}^*$ satisfying $\phi_{\text{red}} \circ \pi = \phi \circ i$. $\diamondsuit$

5.4 Elementary Theory of Reduction

Finding a symmetry for a $2n$-dimensional mechanical problem may reduce it to a $(2n-2)$-dimensional problem as follows: an integral of motion f for a $2n$-dimensional hamiltonian system (M, ω, H) may enable us to understand the trajectories of this system in terms of the trajectories of a $(2n-2)$-dimensional hamiltonian system $(M_{\text{red}}, \omega_{\text{red}}, H_{\text{red}})$. To make this precise, we will describe this process locally. Suppose that $\mathcal{U}$ is an open set in M with Darboux coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ such that $f = \xi_n$ for this chart, and write H in these coordinates: $H = H(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$. Then

$$\xi_n \text{ is integral of motion } \Longrightarrow \begin{cases} \text{the trajectories of } v_H \text{ lie on the} \\ \text{ hyperplane } \xi_n = \text{constant} \\ \{\xi_n, H\} = 0 = -\frac{\partial H}{\partial x_n} \\ \Rightarrow H = H(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_n). \end{cases}$$

If we set $\xi_n = c$, the motion of the system on this hyperplane is described by the following Hamilton equations:

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{\partial H}{\partial \xi_1}(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c) \\ \vdots \\ \dfrac{dx_{n-1}}{dt} = \dfrac{\partial H}{\partial \xi_{n-1}}(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c) \\ \dfrac{d\xi_1}{dt} = -\dfrac{\partial H}{\partial x_1}(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c) \\ \vdots \\ \dfrac{d\xi_{n-1}}{dt} = -\dfrac{\partial H}{\partial x_{n-1}}(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c) \end{cases}$$

$$\frac{dx_n}{dt} = \frac{\partial H}{\partial \xi_n}$$

$$\frac{d\xi_n}{dt} = -\frac{\partial H}{\partial x_n} = 0 \, .$$

The **reduced phase space** is

$$\begin{array}{rl}\mathcal{U}_{\mathrm{red}} = & \{(x_1,\ldots,x_{n-1},\xi_1,\ldots,\xi_{n-1}) \in \mathbb{R}^{2n-2} \mid \\ & (x_1,\ldots,x_{n-1},a,\xi_1,\ldots,\xi_{n-1},c) \in \mathcal{U} \text{ for some } a\} \ .\end{array}$$

The **reduced hamiltonian** is

$$\begin{array}{l} H_{\mathrm{red}} : \mathcal{U}_{\mathrm{red}} \longrightarrow \mathbb{R} \ , \\ H_{\mathrm{red}}(x_1,\ldots,x_{n-1},\xi_1,\ldots,\xi_{n-1}) = H(x_1,\ldots,x_{n-1},\xi_1,\ldots,\xi_{n-1},c) \ .\end{array}$$

In order to find the trajectories of the original system on the hypersurface $\xi_n = c$, we look for the trajectories

$$x_1(t),\ldots,x_{n-1}(t),\xi_1(t),\ldots,\xi_{n-1}(t)$$

of the reduced system on $\mathcal{U}_{\mathrm{red}}$. We integrate the equation

$$\frac{dx_n}{dt}(t) = \frac{\partial H}{\partial \xi_n}(x_1(t),\ldots,x_{n-1}(t),\xi_1(t),\ldots,\xi_{n-1}(t),c)$$

to obtain the original trajectories

$$\left\{ \begin{array}{rcl} x_n(t) & = & x_n(0) + \int_0^t \frac{\partial H}{\partial \xi_n}(\ldots)dt \\ \xi_n(t) & = & c \ . \end{array} \right.$$

5.5 Reduction for Product Groups

Let G_1 and G_2 be compact connected Lie groups and let $G = G_1 \times G_2$. Then

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \qquad \text{and} \qquad \mathfrak{g}^* = \mathfrak{g}_1^* \oplus \mathfrak{g}_2^* \ .$$

Suppose that (M,ω,G,ψ) is a hamiltonian G-space with moment map

$$\psi : M \longrightarrow \mathfrak{g}_1^* \oplus \mathfrak{g}_2^* \ .$$

Write $\psi = (\psi_1,\psi_2)$ where $\psi_i : M \to \mathfrak{g}_i^*$ for $i = 1, 2$. The fact that ψ is equivariant implies that ψ_1 is invariant under G_2 and ψ_2 is invariant under G_1. Now reduce (M,ω) with respect to the G_1-action. Let

$$Z_1 = \psi_1^{-1}(0) \ .$$

Assume that G_1 acts freely on Z_1. Let $M_1 = Z_1/G_1$ be the reduced space and let ω_1 be the corresponding reduced symplectic form. The action of G_2 on Z_1 commutes with the G_1-action. Since G_2 preserves ω, it follows that G_2 acts symplectically on (M_1, ω_1). Since G_1 preserves ψ_2, G_1 also preserves $\psi_2 \circ \iota_1 : Z_1 \to \mathfrak{g}_2^*$, where $\iota_1 : Z_1 \hookrightarrow M$ is inclusion. Thus $\psi_2 \circ \iota$ is constant on fibers of $Z_1 \stackrel{p_1}{\to} M_1$. We conclude that there exists a smooth map $\mu_2 : M_1 \to \mathfrak{g}_2^*$ such that $\mu_2 \circ p_1 = \psi_2 \circ \iota_1$.

Exercise 30
Show that:

(a) the map μ_2 is a moment map for the action of G_2 on (M_1, ω_1), and

(b) if G acts freely on $\psi^{-1}(0,0)$, then G_2 acts freely on $\mu_2^{-1}(0)$, and there is a natural symplectomorphism

$$\mu_2^{-1}(0)/G_2 \simeq \psi^{-1}(0,0)/G .$$

This technique of performing reduction with respect to one factor of a product group at a time is called **reduction in stages**. It may be extended to reduction by a normal subgroup $H \subset G$ and by the corresponding quotient group G/H.

5.6 Reduction at Other Levels

Suppose that a compact Lie group G acts on a symplectic manifold (M, ω) in a hamiltonian way with moment map $\mu : M \to \mathfrak{g}^*$. Let $\xi \in \mathfrak{g}^*$.

To reduce at the level ξ of μ, we need $\mu^{-1}(\xi)$ to be preserved by G, or else take the G-orbit of $\mu^{-1}(\xi)$, or else take the quotient by the maximal subgroup of G which preserves $\mu^{-1}(\xi)$.

Since μ is equivariant,

$$\begin{aligned} G \text{ preserves } \mu^{-1}(\xi) \quad &\iff \quad G \text{ preserves } \xi \\ &\iff \quad \mathrm{Ad}_g^* \xi = \xi, \ \forall g \in G . \end{aligned}$$

Of course the level 0 is always preserved. Also, when G is a torus, any level is preserved and reduction at ξ for the moment map μ, is equivalent to reduction at 0 for a shifted moment map $\phi : M \to \mathfrak{g}^*$, $\phi(p) := \mu(p) - \xi$.

Let $\mathcal{O}$ be a coadjoint orbit in $\mathfrak{g}^*$ equipped with the **canonical symplectic form** (also know as the **Kostant-Kirillov symplectic form** or the **Lie-Poisson symplectic form**) $\omega_{\mathcal{O}}$ defined in Section 4.3. Let $\mathcal{O}^-$ be the orbit $\mathcal{O}$ equipped with $-\omega_{\mathcal{O}}$. The natural product action of G on $M \times \mathcal{O}^-$ is hamiltonian with moment map $\mu_{\mathcal{O}}(p, \xi) = \mu(p) - \xi$. If the Marsden-Weinstein-Meyer hypothesis is satisfied for $M \times \mathcal{O}^-$, then one obtains a **reduced space with respect to the coadjoint orbit** $\mathcal{O}$.

5.7 Orbifolds

Example. Let $G = \mathbb{T}^n$ be an n-torus. For any $\xi \in (\mathfrak{t}^n)^*$, $\mu^{-1}(\xi)$ is preserved by the $\mathbb{T}^n$-action. Suppose that ξ is a regular value of μ. (By Sard's theorem, the singular values of μ form a set of measure zero.) Then $\mu^{-1}(\xi)$ is a submanifold of codimension n. Note that

$$\begin{aligned}
\xi \text{ regular} &\implies d\mu_p \text{ is surjective at all } p \in \mu^{-1}(\xi) \\
&\implies \mathfrak{g}_p = 0 \;\text{ for all } p \in \mu^{-1}(\xi) \\
&\implies \text{the stabilizers on } \mu^{-1}(\xi) \text{ are finite} \\
&\implies \mu^{-1}(\xi)/G \text{ is an } \textbf{orbifold} \text{ [38, 39]} .
\end{aligned}$$

Let G_p be the stabilizer of p. By the slice theorem (Theorem 5.5), $\mu^{-1}(\xi)/G$ is modeled by S/G_p, where S is a G_p-invariant disk in $\mu^{-1}(\xi)$ through p and transverse to $\mathcal{O}_p$. Hence, locally $\mu^{-1}(\xi)/G$ looks indeed like $\mathbb{R}^n$ divided by a finite group action. $\diamondsuit$

Example. Consider the S^1-action on $\mathbb{C}^2$ given by $e^{i\theta} \cdot (z_1, z_2) = (e^{ik\theta} z_1, e^{i\theta} z_2)$ for some fixed integer $k \geq 2$. This is hamiltonian with moment map

$$\begin{aligned}
\mu : \quad \mathbb{C}^2 &\longrightarrow \mathbb{R} \\
(z_1, z_2) &\longmapsto -\tfrac{1}{2}(k|z_1|^2 + |z_2|^2) .
\end{aligned}$$

Any $\xi < 0$ is a regular value and $\mu^{-1}(\xi)$ is a 3-dimensional ellipsoid. The stabilizer of $(z_1, z_2) \in \mu^{-1}(\xi)$ is $\{1\}$ if $z_2 \neq 0$, and is $\mathbb{Z}_k =$

$\left\{e^{i\frac{2\pi\ell}{k}} \mid \ell = 0,1,\dots,k-1\right\}$ if $z_2 = 0$. The reduced space $\mu^{-1}(\xi)/S^1$ is called a **teardrop** orbifold or **conehead**; it has one **cone** (also known as a **dunce cap**) singularity of type k (with cone angle $\frac{2\pi}{k}$). ♢

Example. Let S^1 act on $\mathbb{C}^2$ by $e^{i\theta}\cdot(z_1,z_2) = (e^{ik\theta}z_1, e^{i\ell\theta}z_2)$ for some integers $k,\ell \geq 2$. Suppose that k and ℓ are relatively prime. Then

$$\begin{array}{rll} (z_1,0) & \text{has stabilizer } \mathbb{Z}_k & (\text{for } z_1 \neq 0)\ , \\ (0,z_2) & \text{has stabilizer } \mathbb{Z}_\ell & (\text{for } z_2 \neq 0)\ , \\ (z_1,z_2) & \text{has stabilizer } \{1\} & (\text{for } z_1,z_2 \neq 0)\ . \end{array}$$

The quotient $\mu^{-1}(\xi)/S^1$ is called a **football** orbifold. It has two cone singularities, one of type k and another of type ℓ. ♢

Example. More generally, the reduced spaces of S^1 acting on $\mathbb{C}^n$ by

$$e^{i\theta}\cdot(z_1,\dots,z_n) = (e^{ik_1\theta}z_1,\dots,e^{ik_n\theta}z_n)\ ,$$

are called **weighted** (or **twisted**) **projective spaces**. ♢

5.8 Symplectic Toric Manifolds

Definition 5.9 *A* **symplectic toric manifold** *is a compact connected symplectic manifold (M,ω) equipped with an effective hamiltonian action of a torus $\mathbb{T}$ of dimension equal to half the dimension of the manifold,*

$$\dim \mathbb{T} = \frac{1}{2}\dim M\ ,$$

and with a choice of a corresponding moment map μ.

Exercise 31
Show that an effective hamiltonian action of a torus $\mathbb{T}^n$ on a $2n$-dimensional symplectic manifold gives rise to an integrable system.

Hint: The coordinates of the moment map are commuting integrals of motion.

Definition 5.10 *Two symplectic toric manifolds,* $(M_i, \omega_i, \mathbb{T}_i, \mu_i)$*,* $i = 1, 2$*, are* **equivalent** *if there exists an isomorphism* $\lambda : \mathbb{T}_1 \to \mathbb{T}_2$ *and a* λ*-equivariant symplectomorphism* $\varphi : M_1 \to M_2$ *such that* $\mu_1 = \mu_2 \circ \varphi$*.*

Equivalent symplectic toric manifolds are often undistinguished.

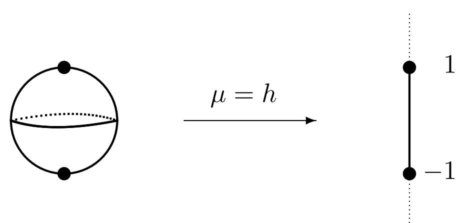
Examples of symplectic toric manifolds.

1. The circle S^1 acts on the 2-sphere $(S^2, \omega_{\text{standard}} = d\theta \wedge dh)$ by rotations

   $$e^{i\nu} \cdot (\theta, h) = (\theta + \nu, h)$$

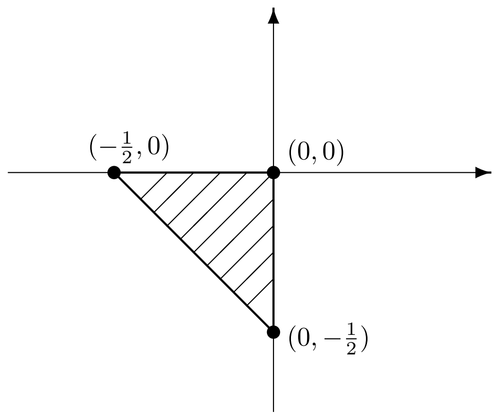
   with moment map $\mu = h$ equal to the height function and moment polytope $[-1, 1]$.

   Equivalently, the circle S^1 acts on $\mathbb{P}^1 = \mathbb{C}^2 - 0/\sim$ with the Fubini-Study form $\omega_{\text{FS}} = \frac{1}{4}\omega_{\text{standard}}$, by $e^{i\theta} \cdot [z_0 : z_1] = [z_0 : e^{i\theta} z_1]$. This is hamiltonian with moment map $\mu[z_0 : z_1] = -\frac{1}{2} \cdot \frac{|z_1|^2}{|z_0|^2 + |z_1|^2}$, and moment polytope $\left[-\frac{1}{2}, 0\right]$.



2. Let $(\mathbb{P}^2, \omega_{\text{FS}})$ be 2-(complex-)dimensional complex projective space equipped with the Fubini-Study form defined in Section 5.1. The $\mathbb{T}^2$-action on $\mathbb{P}^2$ by $(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$ has moment map

   $$\mu[z_0 : z_1 : z_2] = -\frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right) .$$



The fixed points get mapped as

$$\begin{array}{lcl} [1:0:0] & \longmapsto & (0,0) \\ [0:1:0] & \longmapsto & \left(-\frac{1}{2},0\right) \\ [0:0:1] & \longmapsto & \left(0,-\frac{1}{2}\right) \end{array} .$$

Notice that the stabilizer of a preimage of the edges is S^1, while the action is free at preimages of interior points of the moment polytope.

Exercise 32
Compute a moment polytope for the $\mathbb{T}^3$-action on $\mathbb{P}^3$ as

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2 : e^{i\theta_3} z_3] .$$

Exercise 33
Compute a moment polytope for the $\mathbb{T}^2$-action on $\mathbb{P}^1 \times \mathbb{P}^1$ as

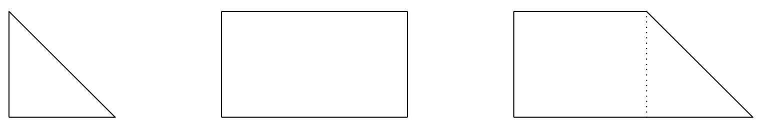
$$(e^{i\theta}, e^{i\eta}) \cdot ([z_0 : z_1], [w_0 : w_1]) = ([z_0 : e^{i\theta} z_1], [w_0 : e^{i\eta} w_1]) .$$

By Proposition 4.17, symplectic toric manifolds are optimal hamiltonian torus-spaces. By Theorem 4.15, they have an associated polytope. It turns out that the moment polytope contains enough information to sort all symplectic toric manifolds. We now define the class of polytopes which arise in the classification.

Definition 5.11 *A* **Delzant polytope** Δ *in* $\mathbb{R}^n$ *is a polytope satisfying:*

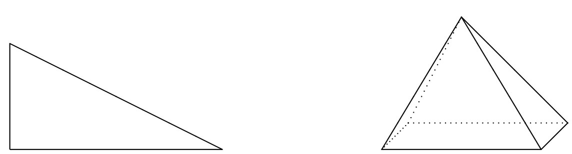
- **simplicity**, *i.e., there are* n *edges meeting at each vertex;*
- **rationality**, *i.e., the edges meeting at the vertex* p *are rational in the sense that each edge is of the form* $p + tu_i$, $t \geq 0$, *where* $u_i \in \mathbb{Z}^n$*;*
- **smoothness**, *i.e., for each vertex, the corresponding* $u_1, \ldots, u_n$ *can be chosen to be a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^n$*.*

Examples of Delzant polytopes in $\mathbb{R}^2$:



The dotted vertical line in the trapezoidal example is there just to stress that it is a picture of a rectangle plus an isosceles triangle. For "taller" triangles, smoothness would be violated. "Wider" triangles (with integral slope) may still be Delzant. The family of the Delzant trapezoids of this type, starting with the rectangle, correspond, under the Delzant construction, to the so-called *Hirzebruch surfaces.* ♢

Examples of polytopes which are not Delzant:



The picture on the left fails the smoothness condition, since the triangle is not isosceles, whereas the one on the right fails the simplicity condition. ♢

Delzant's theorem classifies (equivalence classes of) symplectic toric manifolds in terms of the combinatorial data encoded by a Delzant polytope.

Theorem 5.12 (Delzant [14]) *Toric manifolds are classified by Delzant polytopes. More specifically, the bijective correspondence between these two sets is given by the moment map:*

$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \longleftrightarrow & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) & \longmapsto & \mu(M) \ . \end{array}$$

In Section 5.9 we describe the construction which proves the (easier) existence part, or surjectivity, in Delzant's theorem. In order to prepare that, we will next give an algebraic description of Delzant polytopes.

Let Δ be a Delzant polytope in $(\mathbb{R}^n)^*$[1] and with d facets.[2] Let $v_i \in \mathbb{Z}^n$, $i = 1, \ldots, d$, be the primitive[3] outward-pointing normal vectors to the facets of Δ. Then we can describe Δ as an intersection of halfspaces

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \leq \lambda_i, \ i = 1, \ldots, d\} \quad \text{for some } \lambda_i \in \mathbb{R} \ .$$

Example. For the picture below, we have

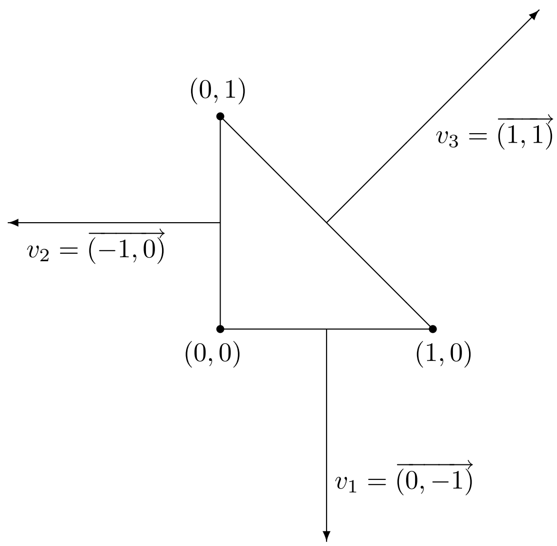
$$\begin{aligned} \Delta &= \{x \in (\mathbb{R}^2)^* \mid x_1 \geq 0, \ x_2 \geq 0, \ x_1 + x_2 \leq 1\} \\ &= \{x \in (\mathbb{R}^2)^* \mid \langle x, (-1,0) \rangle \leq 0, \ \langle x, (0,-1) \rangle \leq 0, \ \langle x, (1,1) \rangle \leq 1\}. \end{aligned}$$

$\diamondsuit$

[1] Although we identify $\mathbb{R}^n$ with its dual via the euclidean inner product, it may be more clear to see Δ in $(\mathbb{R}^n)^*$ for Delzant's construction.

[2] A **face** of a polytope Δ is a set of the form $F = P \cap \{x \in \mathbb{R}^n \mid f(x) = c\}$ where $c \in \mathbb{R}$ and $f \in (\mathbb{R}^n)^*$ satisfies $f(x) \geq c$, $\forall x \in P$. A **facet** of an n-dimensional polytope is an $(n-1)$-dimensional face.

[3] A lattice vector $v \in \mathbb{Z}^n$ is **primitive** if it cannot be written as $v = ku$ with $u \in \mathbb{Z}^n$, $k \in \mathbb{Z}$ and $|k| > 1$; for instance, $(1,1)$, $(4,3)$, $(1,0)$ are primitive, but $(2,2)$, $(3,6)$ are not.



5.9 Delzant's Construction

Following [14, 24], we prove the existence part (or surjectivity) in Delzant's theorem, by using symplectic reduction to associate to an n-dimensional Delzant polytope Δ a symplectic toric manifold $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$.

Let Δ be a Delzant polytope with d facets. Let $v_i \in \mathbb{Z}^n$, $i = 1, \ldots, d$, be the primitive outward-pointing normal vectors to the facets. For some $\lambda_i \in \mathbb{R}$, we can write

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \leq \lambda_i,\ i = 1, \ldots, d\} .$$

Let $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ be the standard basis of $\mathbb{R}^d$. Consider

$$\begin{array}{rccc} \pi : & \mathbb{R}^d & \longrightarrow & \mathbb{R}^n \\ & e_i & \longmapsto & v_i \ . \end{array}$$

Lemma 5.13 *The map π is onto and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$.*

Proof. The set $\{e_1, \dots, e_d\}$ is a basis of $\mathbb{Z}^d$. The set $\{v_1, \dots, v_d\}$ spans $\mathbb{Z}^n$ for the following reason. At a vertex p, the edge vectors $u_1, \dots, u_n \in (\mathbb{R}^n)^*$, form a basis for $(\mathbb{Z}^n)^*$ which, by a change of basis if necessary, we may assume is the standard basis. Then the corresponding primitive normal vectors to the facets meeting at p are symmetric (in the sense of multiplication by -1) to the u_i's, hence form a basis of $\mathbb{Z}^n$. $\square$

Therefore, π induces a surjective map, still called π, between tori:

$$\begin{array}{ccccc} \mathbb{R}^d/(2\pi\mathbb{Z}^d) & \stackrel{\pi}{\longrightarrow} & \mathbb{R}^n/(2\pi\mathbb{Z}^n) & & \\ \| & & \| & & \\ \mathbb{T}^d & \longrightarrow & \mathbb{T}^n & \longrightarrow & 1 \ . \end{array}$$

The kernel N of π is a $(d-n)$-dimensional Lie subgroup of $\mathbb{T}^d$ with inclusion $i : N \hookrightarrow \mathbb{T}^d$. Let $\mathfrak{n}$ be the Lie algebra of N. The exact sequence of tori

$$1 \longrightarrow N \stackrel{i}{\longrightarrow} \mathbb{T}^d \stackrel{\pi}{\longrightarrow} \mathbb{T}^n \longrightarrow 1$$

induces an exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{n} \stackrel{i}{\longrightarrow} \mathbb{R}^d \stackrel{\pi}{\longrightarrow} \mathbb{R}^n \longrightarrow 0$$

with dual exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \stackrel{\pi^*}{\longrightarrow} (\mathbb{R}^d)^* \stackrel{i^*}{\longrightarrow} \mathfrak{n}^* \longrightarrow 0 \ .$$

Now consider $\mathbb{C}^d$ with symplectic form $\omega_0 = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k$, and standard hamiltonian action of $\mathbb{T}^d$ given by

$$(e^{it_1}, \dots, e^{it_d}) \cdot (z_1, \dots, z_d) = (e^{it_1} z_1, \dots, e^{it_d} z_d) \ .$$

The moment map is $\phi : \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^*$ defined by

$$\phi(z_1, \dots, z_d) = -\frac{1}{2}(|z_1|^2, \dots, |z_d|^2) + \text{ constant } ,$$

where we choose the constant to be $(\lambda_1, \ldots, \lambda_d)$. The subtorus N acts on $\mathbb{C}^d$ in a hamiltonian way with moment map

$$i^* \circ \phi : \mathbb{C}^d \longrightarrow \mathfrak{n}^* \ .$$

Let $Z = (i^* \circ \phi)^{-1}(0)$ be the zero-level set.

Claim 1. The set Z is compact and N acts freely on Z.

We postpone the proof of this claim until further down.

Since i^* is surjective, $0 \in \mathfrak{n}^*$ is a regular value of $i^* \circ \phi$. Hence, Z is a compact submanifold of $\mathbb{C}^d$ of (real) dimension $2d-(d-n) = d+n$. The orbit space $M_\Delta = Z/N$ is a compact manifold of (real) dimension $\dim Z - \dim N = (d+n) - (d-n) = 2n$. The point-orbit map $p : Z \to M_\Delta$ is a principal N-bundle over M_Δ. Consider the diagram

$$\begin{array}{ccc} Z & \stackrel{j}{\hookrightarrow} & \mathbb{C}^d \\ {\scriptstyle p} \downarrow & & \\ M_\Delta & & \end{array}$$

where $j : Z \hookrightarrow \mathbb{C}^d$ is inclusion. The Marsden-Weinstein-Meyer theorem guarantees the existence of a symplectic form ω_Δ on M_Δ satisfying

$$p^* \omega_\Delta = j^* \omega_0 \ .$$

Since Z is connected, the compact symplectic $2n$-dimensional manifold $(M_\Delta, \omega_\Delta)$ is also connected.

Proof of Claim 1. The set Z is clearly closed, hence in order to show that it is compact it suffices (by the Heine-Borel theorem) to show that Z is bounded. Let Δ' be the image of Δ by π^*. We will show that $\phi(Z) = \Delta'$.

Lemma 5.14 *Let $y \in (\mathbb{R}^d)^*$. Then:*

$$y \in \Delta' \iff y \textit{ is in the image of } Z \textit{ by } \phi \ .$$

Proof. The value y is in the image of Z by ϕ if and only if both of the following conditions hold:

1. y is in the image of ϕ;

2. $i^* y = 0$.

Using the expression for ϕ and the third exact sequence, we see that these conditions are equivalent to:

1. $\langle y, e_i \rangle \leq \lambda_i$ for $i = 1, \dots, d$;

2. $y = \pi^*(x)$ for some $x \in (\mathbb{R}^n)^*$.

Suppose that the second condition holds, so that $y = \pi^*(x)$. Then

$$\begin{aligned} \langle y, e_i \rangle \leq \lambda_i, \forall i \quad &\iff \quad \langle \pi^*(x), e_i \rangle \leq \lambda_i, \forall i \\ &\iff \quad \langle x, \pi(e_i) \rangle \leq \lambda_i, \forall i \\ &\iff \quad \langle x, v_i \rangle \leq \lambda_i, \forall i \\ &\iff \quad x \in \Delta \ . \end{aligned}$$

Thus, $y \in \phi(Z) \iff y \in \pi^*(\Delta) = \Delta'$. □

Since we have that Δ' is compact, that ϕ is a proper map and that $\phi(Z) = \Delta'$, we conclude that Z must be bounded, and hence compact.

It remains to show that N acts freely on Z.

Pick a vertex p of Δ, and let $I = \{i_1, \dots, i_n\}$ be the set of indices for the n facets meeting at p. Pick $z \in Z$ such that $\phi(z) = \pi^*(p)$. Then p is characterized by n equations $\langle p, v_i \rangle = \lambda_i$ where i ranges in I:

$$\begin{aligned} \langle p, v_i \rangle = \lambda_i \quad &\iff \quad \langle p, \pi(e_i) \rangle = \lambda_i \\ &\iff \quad \langle \pi^*(p), e_i \rangle = \lambda_i \\ &\iff \quad \langle \phi(z), e_i \rangle = \lambda_i \\ &\iff \quad i\text{-th coordinate of } \phi(z) \text{ is equal to } \lambda_i \\ &\iff \quad -\tfrac{1}{2}|z_i|^2 + \lambda_i = \lambda_i \\ &\iff \quad z_i = 0 \ . \end{aligned}$$

Hence, those z's are points whose coordinates in the set I are zero, and whose other coordinates are nonzero. Without loss of generality, we may assume that $I = \{1, \dots, n\}$. The stabilizer of z is

$$(\mathbb{T}^d)_z = \{(t_1, \dots, t_n, 1, \dots, 1) \in \mathbb{T}^d\} \ .$$

As the restriction $\pi : (\mathbb{R}^d)_z \to \mathbb{R}^n$ maps the vectors $e_1, \ldots, e_n$ to a $\mathbb{Z}$-basis $v_1, \ldots, v_n$ of $\mathbb{Z}^n$ (respectively), at the level of groups, $\pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$ must be bijective. Since $N = \ker(\pi : \mathbb{T}^d \to \mathbb{T}^n)$, we conclude that $N \cap (\mathbb{T}^d)_z = \{e\}$, i.e., $N_z = \{e\}$. Hence all N-stabilizers at points mapping to vertices are trivial. But this was the worst case, since other stabilizers $N_{z'}$ ($z' \in Z$) are contained in stabilizers for points z which map to vertices. This concludes the proof of Claim 1. □

Given a Delzant polytope Δ, we have constructed a symplectic manifold $(M_\Delta, \omega_\Delta)$ where $M_\Delta = Z/N$ is a compact $2n$-dimensional manifold and ω_Δ is the reduced symplectic form.

Claim 2. The manifold $(M_\Delta, \omega_\Delta)$ is a hamiltonian $\mathbb{T}^n$-space with a moment map μ_Δ having image $\mu_\Delta(M_\Delta) = \Delta$.

Proof of Claim 2. Let z be such that $\phi(z) = \pi^*(p)$ where p is a vertex of Δ, as in the proof of Claim 1. Let $\sigma : \mathbb{T}^n \to (\mathbb{T}^d)_z$ be the inverse for the earlier bijection $\pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$. Since we have found a *section*, i.e., a right inverse for π, in the exact sequence

$$1 \longrightarrow N \xrightarrow{i} \mathbb{T}^d \underset{\overset{\longleftarrow}{\sigma}}{\xrightarrow{\pi}} \mathbb{T}^n \longrightarrow 1 ,$$

the exact sequence *splits*, i.e., becomes like a sequence for a product, as we obtain an isomorphism

$$(i, \sigma) : N \times \mathbb{T}^n \xrightarrow{\simeq} \mathbb{T}^d .$$

The action of the $\mathbb{T}^n$ factor (or, more rigorously, $\sigma(\mathbb{T}^n) \subset \mathbb{T}^d$) descends to the quotient $M_\Delta = Z/N$.

It remains to show that the $\mathbb{T}^n$-action on M_Δ is hamiltonian with appropriate moment map.

Consider the diagram

$$\begin{array}{l} Z \overset{j}{\hookrightarrow} \mathbb{C}^d \xrightarrow{\phi} (\mathbb{R}^d)^* \simeq \eta^* \oplus (\mathbb{R}^n)^* \xrightarrow{\sigma^*} (\mathbb{R}^n)^* \\ p \downarrow \\ M_\Delta \end{array}$$

where the last horizontal map is simply projection onto the second factor. Since the composition of the horizontal maps is constant along N-orbits, it descends to a map

$$\mu_\Delta : M_\Delta \longrightarrow (\mathbb{R}^n)^*$$

which satisfies

$$\mu_\Delta \circ p = \sigma^* \circ \phi \circ j \ .$$

By Section 5.5 on reduction for product groups, this is a moment map for the action of $\mathbb{T}^n$ on $(M_\Delta, \omega_\Delta)$. Finally, the image of μ_Δ is:

$$\mu_\Delta(M_\Delta) = (\mu_\Delta \circ p)(Z) = (\sigma^* \circ \phi \circ j)(Z) = (\sigma^* \circ \pi^*)(\Delta) = \Delta \ ,$$

because $\phi(Z) = \pi^*(\Delta)$ and $\sigma^* \circ \pi^* = (\pi \circ \sigma)^* = \mathrm{id}$.

We conclude that $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ is the required toric manifold corresponding to Δ. □

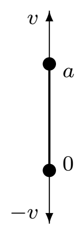
Exercise 34
Let Δ be an n-dimensional Delzant polytope, and let $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ be the associated symplectic toric manifold. Show that μ_Δ maps the fixed points of $\mathbb{T}^n$ bijectively onto the vertices of Δ.

Exercise 35

Follow through the details of Delzant's construction for the case of $\Delta = [0, a] \subset \mathbb{R}^*$ ($n = 1, d = 2$). Let $v(= 1)$ be the standard basis vector in $\mathbb{R}$. Then Δ is described by

$$\langle x, -v\rangle \leq 0 \qquad \text{and} \qquad \langle x, v\rangle \leq a \ ,$$

where $v_1 = -v$, $v_2 = v$, $\lambda_1 = 0$ and $\lambda_2 = a$.



The projection

$$\begin{array}{ccc} \mathbb{R}^2 & \stackrel{\pi}{\longrightarrow} & \mathbb{R} \\ e_1 & \longmapsto & -v \\ e_2 & \longmapsto & v \end{array}$$

has kernel equal to the span of $(e_1 + e_2)$, so that N is the diagonal subgroup of $\mathbb{T}^2 = S^1 \times S^1$. The exact sequences become

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \stackrel{i}{\longrightarrow} & \mathbb{T}^2 & \stackrel{\pi}{\longrightarrow} & S^1 & \longrightarrow & 1 \\ & & t & \longmapsto & (t, t) & & & & \\ & & & & (t_1, t_2) & \longmapsto & t_1^{-1} t_2 & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n} & \stackrel{i}{\longrightarrow} & \mathbb{R}^2 & \stackrel{\pi}{\longrightarrow} & \mathbb{R} & \longrightarrow & 0 \\ & & x & \longmapsto & (x, x) & & & & \\ & & & & (x_1, x_2) & \longmapsto & x_2 - x_1 & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}^* & \stackrel{\pi^*}{\longrightarrow} & (\mathbb{R}^2)^* & \stackrel{i^*}{\longrightarrow} & \mathfrak{n}^* & \longrightarrow & 0 \\ & & x & \longmapsto & (-x, x) & & & & \\ & & & & (x_1, x_2) & \longmapsto & x_1 + x_2 \ . & & \end{array}$$

The action of the diagonal subgroup $N = \{(e^{it}, e^{it}) \in S^1 \times S^1\}$ on $\mathbb{C}^2$,

$$(e^{it}, e^{it}) \cdot (z_1, z_2) = (e^{it} z_1, e^{it} z_2) \ ,$$

has moment map

$$(i^* \circ \phi)(z_1, z_2) = -\tfrac{1}{2}(|z_1|^2 + |z_2|^2) + a \ ,$$

with zero-level set

$$(i^* \circ \phi)^{-1}(0) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 2a\} \ .$$

Hence, the reduced space is a projective space:

$$(i^* \circ \phi)^{-1}(0)/N = \mathbb{P}^1 \ .$$

Example. Consider

$$(S^2, \omega = d\theta \wedge dh, S^1, \mu = h) \ ,$$

where S^1 acts on S^2 by rotation. The image of μ is the line segment $I = [-1, 1]$. The product $S^1 \times I$ is an open-ended cylinder. By collapsing each end of the cylinder to a point, we recover the 2-sphere. ♢

Exercise 36
Build $\mathbb{P}^2$ from $\mathbb{T}^2 \times \Delta$ where Δ is a right-angled isosceles triangle.

Exercise 37
Consider the standard $(S^1)^3$-action on $\mathbb{P}^3$:

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2 : e^{i\theta_3} z_3] \ .$$

Exhibit explicitly the subsets of $\mathbb{P}^3$ for which the stabilizer under this action is $\{1\}$, S^1, $(S^1)^2$ and $(S^1)^3$. Show that the images of these subsets under the moment map are the interior, the facets, the edges and the vertices, respectively.

Exercise 38
What would be the classification of symplectic toric manifolds if, instead of the equivalence relation defined in Section 5.8, one considered to be equivalent those $(M_i, \omega_i, \mathbb{T}_i, \mu_i)$, $i = 1, 2$, related by an isomorphism $\lambda : \mathbb{T}_1 \to \mathbb{T}_2$ and a λ-equivariant symplectomorphism $\varphi : M_1 \to M_2$ such that:

(a) the maps μ_1 and $\mu_2 \circ \varphi$ are equal up to a constant?

(b) we have $\mu_1 = \ell \circ \mu_2 \circ \varphi$ for some $\ell \in \mathrm{SL}(n; \mathbb{Z})$?

Exercise 39

(a) Classify all 2-dimensional Delzant polytopes with 3 vertices, i.e., triangles, up to translation, change of scale and the action of $\mathrm{SL}(2;\mathbb{Z})$.

Hint: By a linear transformation in $\mathrm{SL}(2;\mathbb{Z})$, we can make one of the angles in the polytope into a square angle. How are the lengths of the two edges forming that angle related?

(b) Classify all 2-dimensional Delzant polytopes with 4 vertices, up to translation and the action of $\mathrm{SL}(2;\mathbb{Z})$.

Hint: By a linear transformation in $\mathrm{SL}(2;\mathbb{Z})$, we can make one of the angles in the polytope into a square angle. Check that automatically another angle also becomes 90^o.

(c) What are all the 4-dimensional symplectic toric manifolds that have four fixed points?

Exercise 40
Let Δ be the n-simplex in $\mathbb{R}^n$ spanned by the origin and the standard basis vectors $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. Show that the corresponding symplectic toric manifold is projective space, $M_\Delta = \mathbb{P}^n$.

Exercise 41
Which $2n$-dimensional toric manifolds have exactly $n + 1$ fixed points?

Appendix A

Prerequisites from Differential Geometry

A.1 Isotopies and Vector Fields

Let M be a manifold, and $\rho : M \times \mathbb{R} \to M$ a map, where we set $\rho_t(p) := \rho(p,t)$.

Definition A.1 *The map ρ is an* **isotopy** *if each $\rho_t : M \to M$ is a diffeomorphism, and $\rho_0 = \mathrm{id}_M$.*

Given an isotopy ρ, we obtain a **time-dependent vector field**, that is, a family of vector fields v_t, $t \in \mathbb{R}$, which at $p \in M$ satisfy

$$v_t(p) = \left.\frac{d}{ds}\rho_s(q)\right|_{s=t} \qquad \text{where} \qquad q = \rho_t^{-1}(p) \ ,$$

i.e.,

$$\frac{d\rho_t}{dt} = v_t \circ \rho_t \ .$$

Conversely, given a time-dependent vector field v_t, if M is compact or if the v_t's are compactly supported, there exists an isotopy ρ satisfying the previous ordinary differential equation.

Suppose that M is compact. Then we have a one-to-one correspondence

$$\begin{array}{rcl} \{\text{isotopies of } M\} & \longleftrightarrow & \{\text{time-dependent vector fields on } M\} \\ \rho_t,\ t \in \mathbb{R} & \longleftrightarrow & v_t,\ t \in \mathbb{R} \end{array}$$

Definition A.2 *When $v_t = v$ is independent of t, the associated isotopy is called the* **exponential map** *or the* **flow** *of v and is denoted $\exp tv$; i.e., $\{\exp tv : M \to M \mid t \in \mathbb{R}\}$ is the unique smooth family of diffeomorphisms satisfying*

$$\exp tv|_{t=0} = \mathrm{id}_M \qquad \textit{and} \qquad \frac{d}{dt}(\exp tv)(p) = v(\exp tv(p)) \ .$$

Definition A.3 *The* **Lie derivative** *is the operator*

$$\mathcal{L}_v : \Omega^k(M) \longrightarrow \Omega^k(M) \qquad \textit{defined by} \qquad \mathcal{L}_v\omega := \frac{d}{dt}(\exp tv)^*\omega|_{t=0} \ .$$

When a vector field v_t is time-dependent, its flow, that is, the corresponding isotopy ρ, still locally exists by Picard's theorem. More precisely, in the neighborhood of any point p and for sufficiently small time t, there is a one-parameter family of local diffeomorphisms ρ_t satisfying

$$\frac{d\rho_t}{dt} = v_t \circ \rho_t \qquad \text{and} \qquad \rho_0 = \mathrm{id} \ .$$

Hence, we say that the **Lie derivative** by v_t is

$$\mathcal{L}_{v_t} : \Omega^k(M) \longrightarrow \Omega^k(M) \qquad \text{defined by} \qquad \mathcal{L}_{v_t}\omega := \frac{d}{dt}(\rho_t)^*\omega|_{t=0} \ .$$

Exercise 42
Prove the **Cartan magic formula**,

$$\mathcal{L}_v \omega = \imath_v d\omega + d\imath_v \omega \ ,$$

and the formula

$$\frac{d}{dt}\rho_t^* \omega = \rho_t^* \mathcal{L}_{v_t} \omega \ , \qquad (\star)$$

where ρ is the (local) isotopy generated by v_t. A good strategy for each formula is to follow the steps:

(a) Check the formula for 0-forms $\omega \in \Omega^0(M) = C^\infty(M)$.

(b) Check that both sides commute with d.

(c) Check that both sides are derivations of the algebra $(\Omega^*(M), \wedge)$. For instance, check that

$$\mathcal{L}_v(\omega \wedge \alpha) = (\mathcal{L}_v \omega) \wedge \alpha + \omega \wedge (\mathcal{L}_v \alpha) \ .$$

(d) Notice that, if $\mathcal{U}$ is the domain of a coordinate system, then $\Omega^\bullet(\mathcal{U})$ is generated as an algebra by $\Omega^0(\mathcal{U})$ and $d\Omega^0(\mathcal{U})$, i.e., every element in $\Omega^\bullet(\mathcal{U})$ is a linear combination of wedge products of elements in $\Omega^0(\mathcal{U})$ and elements in $d\Omega^0(\mathcal{U})$.

We will need the following improved version of formula $(\star)$.

Proposition A.4 *For a smooth family ω_t, $t \in \mathbb{R}$, of d-forms, we have*

$$\frac{d}{dt}\rho_t^* \omega_t = \rho_t^* \left(\mathcal{L}_{v_t} \omega_t + \frac{d\omega_t}{dt} \right) \ .$$

Proof. If $f(x,y)$ is a real function of two variables, by the chain rule we have

$$\frac{d}{dt} f(t,t) = \left. \frac{d}{dx} f(x,t) \right|_{x=t} + \left. \frac{d}{dy} f(t,y) \right|_{y=t} \ .$$

Therefore,

$$\frac{d}{dt}\rho_t^* \omega_t = \underbrace{\left. \frac{d}{dx}\rho_x^* \omega_t \right|_{x=t}}_{\left. \rho_x^* \mathcal{L}_{v_x} \omega_t \right|_{x=t} \text{ by } (\star)} + \underbrace{\left. \frac{d}{dy}\rho_t^* \omega_y \right|_{y=t}}_{\left. \rho_t^* \frac{d\omega_y}{dy} \right|_{y=t}} = \rho_t^* \left(\mathcal{L}_{v_t} \omega_t + \frac{d\omega_t}{dt} \right) \ .$$

$\square$

A.2 Submanifolds

Let M and X be manifolds with $\dim X < \dim M$.

Definition A.5 *A map* $i : X \to M$ *is an* **immersion** *if* $di_p : T_pX \to T_{i(p)}M$ *is injective for any point* $p \in X$.

An **embedding** *is an immersion which is a homeomorphism onto its image.*[1] *A* **closed embedding** *is a proper*[2] *injective immersion.*

Exercise 43
Show that a map $i : X \to M$ is a closed embedding if and only if i is an embedding and its image $i(X)$ is closed in M.

Hint:

- If i is injective and proper, then for any neighborhood $\mathcal{U}$ of $p \in X$, there is a neighborhood $\mathcal{V}$ of $i(p)$ such that $f^{-1}(\mathcal{V}) \subseteq \mathcal{U}$.
- On a Hausdorff space, any compact set is closed. On any topological space, a closed subset of a compact set is compact.
- An embedding is proper if and only if its image is closed.

Definition A.6 *A* **submanifold** *of* M *is a manifold* X *with a closed embedding* $i : X \hookrightarrow M$.[3]

Notation. Given a submanifold, we regard the embedding $i : X \hookrightarrow M$ as an inclusion, in order to identify points and tangent vectors:

$$p = i(p) \quad \text{and} \quad T_pX = di_p(T_pX) \subset T_pM \ .$$

A.3 Tubular Neighborhood Theorem

Let M be an n-dimensional manifold, and let X be a k-dimensional submanifold where $k < n$ and with inclusion map

$$i : X \hookrightarrow M \ .$$

[1]The image has the topology induced by the target manifold.

[2]A map is **proper** if the preimage of any compact set is compact.

[3]When X is an open subset of a manifold M, we refer to it as an *open* submanifold.

At each $x \in X$, the tangent space to X is viewed as a subspace of the tangent space to M via the linear inclusion $di_x : T_xX \hookrightarrow T_xM$, where we denote $x = i(x)$. The quotient $N_xX := T_xM/T_xX$ is an $(n-k)$-dimensional vector space, known as the **normal space** to X at x. The **normal bundle** of X is

$$NX = \{(x, v) \mid x \in X \ , \ v \in N_xX\} \ .$$

The set NX has the structure of a vector bundle over X of rank $n-k$ under the natural projection, hence as a manifold NX is n-dimensional.

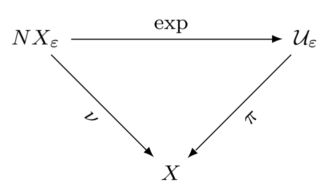
Exercises 44
Let M be $\mathbb{R}^n$ and let X be a k-dimensional compact submanifold of $\mathbb{R}^n$.

(a) Show that in this case N_xX can be identified with the usual "normal space" to X in $\mathbb{R}^n$, that is, the orthogonal complement in $\mathbb{R}^n$ of the tangent space to X at x.

(b) Given $\varepsilon > 0$ let $\mathcal{U}_\varepsilon$ be the set of all points in $\mathbb{R}^n$ which are at a distance less than ε from X. Show that, for ε sufficiently small, every point $p \in \mathcal{U}_\varepsilon$ has a *unique* nearest point $\pi(p) \in X$.

(c) Let $\pi : \mathcal{U}_\varepsilon \to X$ be the map defined in the previous exercise for ε sufficiently small. Show that, if $p \in \mathcal{U}_\varepsilon$, then the line segment $(1-t) \cdot p + t \cdot \pi(p)$, $0 \leq t \leq 1$, joining p to $\pi(p)$ lies in $\mathcal{U}_\varepsilon$.

(d) Let $NX_\varepsilon = \{(x, v) \in NX$ such that $|v| < \varepsilon\}$. Let $\exp : NX \to \mathbb{R}^n$ be the map $(x, v) \mapsto x + v$, and let $\nu : NX_\varepsilon \to X$ be the map $(x, v) \mapsto x$. Show that, for ε sufficiently small, exp maps NX_ε diffeomorphically onto $\mathcal{U}_\varepsilon$, and show also that the following diagram commutes:



(e) Suppose now that the manifold X is not compact. Prove that the assertion about exp is still true provided we replace ε by a continuous function

$$\varepsilon : X \to \mathbb{R}^+$$

which tends to zero fast enough as x tends to infinity. You have thus proved the *tubular neighborhood theorem in* $\mathbb{R}^n$.

In general, the zero section of NX,

$$i_0 : X \hookrightarrow NX \ , \qquad x \mapsto (x, 0) \ ,$$

embeds X as a closed submanifold of NX. A neighborhood $\mathcal{U}_0$ of the zero section X in NX is called **convex** if the intersection $\mathcal{U}_0 \cap N_x X$ with each fiber is convex.

Theorem A.7 (Tubular Neighborhood Theorem) *Let M be an n-dimensional manifold, X a k-dimensional submanifold, NX the normal bundle of X in M, $i_0 : X \hookrightarrow NX$ the zero section, and $i : X \hookrightarrow M$ inclusion. Then there exist a convex neighborhood $\mathcal{U}_0$ of X in NX, a neighborhood $\mathcal{U}$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}$ such that*

$$\begin{array}{ccc} NX \supseteq \mathcal{U}_0 & \xrightarrow[\simeq]{\varphi} & \mathcal{U} \subseteq M \\ {\scriptstyle i_0}\nwarrow & & \nearrow{\scriptstyle i} \\ & X & \end{array}$$

commutes.

Outline of the proof.

- *Case of $M = \mathbb{R}^n$, and X is a compact submanifold of $\mathbb{R}^n$.*

  Theorem A.8 (ε-Neighborhood Theorem)
  Let $\mathcal{U}^\varepsilon = \{p \in \mathbb{R}^n : \ |p - q| < \varepsilon$ for some $q \in X\}$ be the set of points at a distance less than ε from X. Then, for ε sufficiently small, each $p \in \mathcal{U}^\varepsilon$ has a unique nearest point $q \in X$ (i.e., a unique $q \in X$ minimizing $|q - x|$).

  Moreover, setting $q = \pi(p)$, the map $\mathcal{U}^\varepsilon \stackrel{\pi}{\to} X$ is a (smooth) submersion with the property that, for all $p \in \mathcal{U}^\varepsilon$, the line segment $(1-t)p + tq$, $0 \le t \le 1$, is in $\mathcal{U}^\varepsilon$.

  Here is a sketch. At any $x \in X$, the *normal* space $N_x X$ may be regarded as an $(n-k)$-dimensional subspace of $\mathbb{R}^n$, namely the orthogonal complement in $\mathbb{R}^n$ of the tangent space to X at x:

  $$N_x X \simeq \{v \in \mathbb{R}^n : \ v \perp w \ , \text{ for all } w \in T_x X\} \ .$$

  We define the following open neighborhood of X in NX:

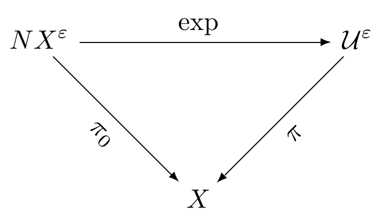
  $$NX^\varepsilon = \{(x, v) \in NX : \ |v| < \varepsilon\} \ .$$

Let

$$\begin{array}{rccc} \exp : & NX & \longrightarrow & \mathbb{R}^n \\ & (x,v) & \longmapsto & x+v \ . \end{array}$$

Restricted to the zero section, exp is the identity map on X.

Prove that, for ε sufficiently small, exp maps NX^ε diffeomorphically onto $\mathcal{U}^\varepsilon$, and show also that the diagram



commutes.

- *Case where X is a compact submanifold of an arbitrary manifold M.*

  Put a riemannian metric g on M, and let $d(p,q)$ be the riemannian distance between $p, q \in M$. The ε-neighborhood of a compact submanifold X is

  $$\mathcal{U}^\varepsilon = \{p \in M \mid d(p,q) < \varepsilon \text{ for some } q \in X\} \ .$$

  Prove the ε-neighborhood theorem in this setting: for ε small enough, the following assertions hold.

  – Any $p \in \mathcal{U}^\varepsilon$ has a unique point $q \in X$ with minimal $d(p,q)$. Set $q = \pi(p)$.

  – The map $\mathcal{U}^\varepsilon \xrightarrow{\pi} X$ is a submersion and, for all $p \in \mathcal{U}^\varepsilon$, there is a unique geodesic curve γ joining p to $q = \pi(p)$.

  – The normal space to X at $x \in X$ is naturally identified with a subspace of T_xM:

  $$N_xX \simeq \{v \in T_xM \mid g_x(v,w) = 0 \ , \text{ for any } w \in T_xX\} \ .$$

  Let $NX^\varepsilon = \{(x,v) \in NX \mid \sqrt{g_x(v,v)} < \varepsilon\}$.

  – Define $\exp : NX^\varepsilon \to M$ by $\exp(x,v) = \gamma(1)$, where $\gamma : [0,1] \to M$ is the geodesic with $\gamma(0) = x$ and $\frac{d\gamma}{dt}(0) = v$. Then exp maps NX^ε diffeomorphically to $\mathcal{U}^\varepsilon$.

- *General case.*

  When X is not compact, adapt the previous argument by replacing ε by an appropriate continuous function $\varepsilon : X \to \mathbb{R}^+$ which tends to zero fast enough as x tends to infinity.

$\square$

Restricting to the subset $\mathcal{U}^0 \subseteq NX$ from the tubular neighborhood theorem, we obtain a submersion $\mathcal{U}_0 \xrightarrow{\pi_0} X$ with all fibers $\pi_0^{-1}(x)$ convex. We can carry this fibration to $\mathcal{U}$ by setting $\pi = \pi_0 \circ \varphi^{-1}$:

$$\begin{array}{ccc} \mathcal{U}_0 & \subseteq NX & \text{is a fibration} \\ \pi_0 \downarrow & & \\ X & & \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \mathcal{U} & \subseteq M & \text{is a fibration} \\ \pi \downarrow & & \\ X & & \end{array}$$

This is called the **tubular neighborhood fibration**.

A.4 Homotopy Formula

Let $\mathcal{U}$ be a tubular neighborhood of a submanifold X in M. The restriction $i^* : H^d_{\text{deRham}}(\mathcal{U}) \to H^d_{\text{deRham}}(X)$ by the inclusion map is surjective. As a corollary of the tubular neighborhood fibration, i^* is also injective: this follows from the homotopy-invariance of de Rham cohomology.

Corollary A.9 *For any degree ℓ, $H^\ell_{\text{deRham}}(\mathcal{U}) \simeq H^\ell_{\text{deRham}}(X)$.*

At the level of forms, this means that, if ω is a closed ℓ-form on $\mathcal{U}$ and $i^*\omega$ is exact on X, then ω is exact. We will need the following related result.

Proposition A.10 *If a closed ℓ-form ω on $\mathcal{U}$ has restriction $i^*\omega = 0$, then ω is exact, i.e., $\omega = d\mu$ for some $\mu \in \Omega^{d-1}(\mathcal{U})$. Moreover, we can choose μ such that $\mu_x = 0$ at all $x \in X$.*

Proof. Via $\varphi : \mathcal{U}_0 \xrightarrow{\simeq} \mathcal{U}$, it is equivalent to work over $\mathcal{U}_0$. Define for every $0 \leq t \leq 1$ a map

$$\begin{array}{rccc} \rho_t : & \mathcal{U}_0 & \longrightarrow & \mathcal{U}_0 \\ & (x,v) & \longmapsto & (x,tv) \ . \end{array}$$

This is well-defined since $\mathcal{U}_0$ is convex. The map ρ_1 is the identity, $\rho_0 = i_0 \circ \pi_0$, and each ρ_t fixes X, that is, $\rho_t \circ i_0 = i_0$. We hence say that the family $\{\rho_t \mid 0 \leq t \leq 1\}$ is a **homotopy** from $i_0 \circ \pi_0$ to the identity fixing X. The map $\pi_0 : \mathcal{U}_0 \to X$ is called a **retraction** because $\pi_0 \circ i_0$ is the identity. The submanifold X is then called a **deformation retract** of $\mathcal{U}$.

A (de Rham) **homotopy operator** between $\rho_0 = i_0 \circ \pi_0$ and $\rho_1 = \mathrm{id}$ is a linear map

$$Q : \Omega^d(\mathcal{U}_0) \longrightarrow \Omega^{d-1}(\mathcal{U}_0)$$

satisfying the **homotopy formula**

$$\mathrm{Id} - (i_0 \circ \pi_0)^* = dQ + Qd \ .$$

When $d\omega = 0$ and $i_0^*\omega = 0$, the operator Q gives $\omega = dQ\omega$, so that we can take $\mu = Q\omega$. A concrete operator Q is given by the formula:

$$Q\omega = \int_0^1 \rho_t^*(\imath_{v_t}\omega) \ dt \ ,$$

where v_t, at the point $q = \rho_t(p)$, is the vector tangent to the curve $\rho_s(p)$ at $s = t$. The proof that Q satisfies the homotopy formula is below.

In our case, for $x \in X$, $\rho_t(x) = x$ (all t) is the constant curve, so v_t vanishes at all x for all t, hence $\mu_x = 0$. $\square$

To check that Q above satisfies the homotopy formula, we compute

$$\begin{aligned} Qd\omega + dQ\omega &= \int_0^1 \rho_t^*(\imath_{v_t} d\omega)dt + d\int_0^1 \rho_t^*(\imath_{v_t}\omega)dt \\ &= \int_0^1 \rho_t^*(\underbrace{\imath_{v_t} d\omega + d\imath_{v_t}\omega}_{\mathcal{L}_{v_t}\omega})dt \ , \end{aligned}$$

where $\mathcal{L}_v$ denotes the Lie derivative along v (reviewed in the next section), and we used the Cartan magic formula: $\mathcal{L}_v\omega = \imath_v d\omega + d\imath_v\omega$. The result now follows from

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{v_t}\omega$$

and from the fundamental theorem of calculus:

$$Qd\omega + dQ\omega = \int_0^1 \frac{d}{dt}\rho_t^*\omega \ dt = \rho_1^*\omega - \rho_0^*\omega \ .$$

A.5 Whitney Extension Theorem

Theorem A.11 (Whitney Extension Theorem) *Let M be an n-dimensional manifold and X a k-dimensional submanifold with $k < n$. Suppose that at each $p \in X$ we are given a linear isomorphism $L_p : T_pM \stackrel{\simeq}{\to} T_pM$ such that $L_p|_{T_pX} = \mathrm{Id}_{T_pX}$ and L_p depends smoothly on p. Then there exists an embedding $h : \mathcal{N} \to M$ of some neighborhood $\mathcal{N}$ of X in M such that $h|_X = \mathrm{id}_X$ and $dh_p = L_p$ for all $p \in X$.*

The linear maps L serve as "germs" for the embedding.

Sketch of proof for the Whitney theorem.

Case $M = \mathbb{R}^n$: For a compact k-dimensional submanifold X, take a neighborhood of the form

$$\mathcal{U}^\varepsilon = \{p \in M \mid \text{ distance } (p, X) \leq \varepsilon\} \ .$$

For ε sufficiently small so that any $p \in \mathcal{U}^\varepsilon$ has a unique nearest point in X, define a projection $\pi : \mathcal{U}^\varepsilon \to X$, $p \mapsto$ point on X closest to p. If $\pi(p) = q$, then $p = q + v$ for some $v \in N_qX$ where $N_qX = (T_qX)^\perp$ is the normal space at q; see Appendix A. Let

$$\begin{array}{rccl} h : & \mathcal{U}^\varepsilon & \longrightarrow & \mathbb{R}^n \\ & p & \longmapsto & q + L_q v \ , \end{array}$$

where $q = \pi(p)$ and $v = p - \pi(p) \in N_qX$. Then $h_X = \mathrm{id}_X$ and $dh_p = L_p$ for $p \in X$. If X is not compact, replace ε by a continuous function $\varepsilon : X \to \mathbb{R}^+$ which tends to zero fast enough as x tends to infinity.

General case: Choose a riemannian metric on M. Replace distance by riemannian distance, replace straight lines $q + tv$ by geodesics $\exp(q, v)(t)$ and replace $q + L_qv$ by the value at $t = 1$ of the geodesic with initial value q and initial velocity L_qv. $\square$

Appendix B

Prerequisites from Lie Group Actions

B.1 One-Parameter Groups of Diffeos

Let M be a manifold and X a complete vector field on M. Let $\rho_t : M \to M$, $t \in \mathbb{R}$, be the family of diffeomorphisms generated by X. For each $p \in M$, $\rho_t(p)$, $t \in \mathbb{R}$, is by definition the unique integral curve of X passing through p at time 0, i.e., $\rho_t(p)$ satisfies

$$\begin{cases} \rho_0(p) & = & p \\ \dfrac{d\rho_t(p)}{dt} & = & X(\rho_t(p)) \ . \end{cases}$$

Claim. We have that $\rho_t \circ \rho_s = \rho_{t+s}$.

Proof. Let $\rho_s(q) = p$. We need to show that $(\rho_t \circ \rho_s)(q) = \rho_{t+s}(q)$, for all $t \in \mathbb{R}$. Reparametrize as $\tilde{\rho}_t(q) := \rho_{t+s}(q)$. Then

$$\begin{cases} \tilde{\rho}_0(q) & = & \rho_s(q) = p \\ \dfrac{d\tilde{\rho}_t(q)}{dt} & = & \dfrac{d\rho_{t+s}(q)}{dt} = X(\rho_{t+s}(q)) = X(\tilde{\rho}_t(q)) \ , \end{cases}$$

i.e., $\tilde{\rho}_t(q)$ is an integral curve of X through p. By uniqueness we must have $\tilde{\rho}_t(q) = \rho_t(p)$, that is, $\rho_{t+s}(q) = \rho_t(\rho_s(q))$. $\square$

Consequence. We have that $\rho_t^{-1} = \rho_{-t}$.

In terms of the group $(\mathbb{R}, +)$ and the group $(\mathrm{Diff}(M), \circ)$ of all diffeomorphisms of M, these results can be summarized as:

Corollary B.1 *The map $\mathbb{R} \to \mathrm{Diff}(M)$, $t \mapsto \rho_t$, is a group homomorphism.*

The family $\{\rho_t \mid t \in \mathbb{R}\}$ is then called a **one-parameter group of diffeomorphisms** of M and denoted

$$\rho_t = \exp tX \ .$$

B.2 Lie Groups

Definition B.2 *A* **Lie group** *is a manifold G equipped with a group structure for which the group operations are smooth maps,*

$$\begin{array}{rcl} G \times G & \longrightarrow & G \\ (a,b) & \longmapsto & a \cdot b \end{array} \qquad \textit{and} \qquad \begin{array}{rcl} G & \longrightarrow & G \\ a & \longmapsto & a^{-1} \ . \end{array}$$

Examples.

- $\mathbb{R}$ (with addition[1]).
- S^1 regarded as unit complex numbers with multiplication, represents rotations of the plane: $S^1 = \mathrm{U}(1) = \mathrm{SO}(2)$.
- $\mathrm{U}(n)$, unitary linear transformations of $\mathbb{C}^n$.
- $\mathrm{SU}(n)$, unitary linear transformations of $\mathbb{C}^n$ with $\det = 1$.
- $\mathrm{O}(n)$, orthogonal linear transformations of $\mathbb{R}^n$.
- $\mathrm{SO}(n)$, elements of $\mathrm{O}(n)$ with $\det = 1$.
- $\mathrm{GL}(V)$, invertible linear transformations of a vector space V.

$\diamondsuit$

Definition B.3 *A* **representation** *of a Lie group G on a vector space V is a group homomorphism $G \to \mathrm{GL}(V)$.*

[1] The operation will be omitted when it is clear from the context.

B.3 Smooth Actions

Let M be a manifold.

Definition B.4 *An* **action** *of a Lie group G on M is a group homomorphism*

$$\begin{array}{rccl} \psi : & G & \longrightarrow & \mathrm{Diff}(M) \\ & g & \longmapsto & \psi_g \ . \end{array}$$

(We will only consider left actions where ψ is a homomorphism. A **right action** *is defined with ψ being an anti-homomorphism.) The* **evaluation map** *associated with an action $\psi : G \to \mathrm{Diff}(M)$ is*

$$\begin{array}{rccl} \mathrm{ev}_\psi : & M \times G & \longrightarrow & M \\ & (p,g) & \longmapsto & \psi_g(p) \ . \end{array}$$

The action ψ is **smooth** *if ev_ψ is a smooth map.*

Example. If X is a complete vector field on M, then

$$\begin{array}{rccl} \rho : & \mathbb{R} & \longrightarrow & \mathrm{Diff}(M) \\ & t & \longmapsto & \rho_t = \exp tX \end{array}$$

is a smooth action of $\mathbb{R}$ on M. ◊

Every complete vector field gives rise to a smooth action of $\mathbb{R}$ on M. Conversely, every smooth action of $\mathbb{R}$ on M is defined by a complete vector field.

$$\begin{array}{rcl} \{\text{complete vector fields on } M\} & \longleftrightarrow & \{\text{smooth actions of } \mathbb{R} \text{ on } M\} \\ X & \longmapsto & \exp tX \\ X_p = \left.\dfrac{d\psi_t(p)}{dt}\right|_{t=0} & \longleftarrow & \psi \end{array}$$

B.4 Adjoint and Coadjoint Actions

Let G be a Lie group. Given $g \in G$ let

$$\begin{array}{rccl} L_g : & G & \longrightarrow & G \\ & a & \longmapsto & g \cdot a \end{array}$$

be **left multiplication** by g. A vector field X on G is called **left-invariant** if $(L_g)_* X = X$ for every $g \in G$. (There are similar *right* notions.)

Let $\mathfrak{g}$ be the vector space of all left-invariant vector fields on G. Together with the Lie bracket $[\cdot,\cdot]$ of vector fields, $\mathfrak{g}$ forms a Lie algebra, called the **Lie algebra of the Lie group** G.

Exercise 45
Show that the map

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & T_eG \\ X & \longmapsto & X_e \end{array}$$

where e is the identity element in G, is an isomorphism of vector spaces.

Any Lie group G acts on itself by **conjugation**:

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Diff}(G) \\ g & \longmapsto & \psi_g \ , \end{array} \qquad \psi_g(a) = g \cdot a \cdot g^{-1} \ .$$

The derivative at the identity of

$$\begin{array}{cccc} \psi_g : & G & \longrightarrow & G \\ & a & \longmapsto & g \cdot a \cdot g^{-1} \end{array}$$

is an invertible linear map $\mathrm{Ad}_g : \mathfrak{g} \longrightarrow \mathfrak{g}$. Here we identify the Lie algebra $\mathfrak{g}$ with the tangent space T_eG. Letting g vary, we obtain the **adjoint representation** (or **adjoint action**) of G on $\mathfrak{g}$:

$$\begin{array}{cccc} \mathrm{Ad} : & G & \longrightarrow & \mathrm{GL}(\mathfrak{g}) \\ & g & \longmapsto & \mathrm{Ad}_g \ . \end{array}$$

Exercise 46
Check for matrix groups that

$$\left.\frac{d}{dt}\mathrm{Ad}_{\exp tX} Y\right|_{t=0} = [X,Y] \ , \qquad \forall X, Y \in \mathfrak{g} \ .$$

Hint: For a matrix group G (i.e., a subgroup of $\mathrm{GL}(n;\mathbb{R})$ for some n), we have

$$\mathrm{Ad}_g(Y) = gYg^{-1} \ , \qquad \forall g \in G \ , \ \forall Y \in \mathfrak{g}$$

and

$$[X,Y] = XY - YX \ , \qquad \forall X, Y \in \mathfrak{g} \ .$$

Let $\langle \cdot,\cdot \rangle$ be the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$:

$$\begin{array}{cccc} \langle \cdot,\cdot \rangle : & \mathfrak{g}^* \times \mathfrak{g} & \longrightarrow & \mathbb{R} \\ & (\xi, X) & \longmapsto & \langle \xi, X \rangle = \xi(X) \ . \end{array}$$

Given $\xi \in \mathfrak{g}^*$, we define $\mathrm{Ad}_g^*\xi$ by

$$\langle \mathrm{Ad}_g^*\xi, X\rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X\rangle \ , \qquad \text{for any } X \in \mathfrak{g} \ .$$

The collection of maps Ad_g^* forms the **coadjoint representation** (or **coadjoint action**) of G on $\mathfrak{g}^*$:

$$\begin{array}{rccc} \mathrm{Ad}^* : & G & \longrightarrow & \mathrm{GL}(\mathfrak{g}^*) \\ & g & \longmapsto & \mathrm{Ad}_g^* \ . \end{array}$$

We take g^{-1} in the definition of $\mathrm{Ad}_g^*\xi$ in order to obtain a (left) representation, i.e., a group homomorphism, instead of a "right" representation, i.e., a group anti-homomorphism.

Exercise 47
Show that $\mathrm{Ad}_g \circ \mathrm{Ad}_h = \mathrm{Ad}_{gh}$ and $\mathrm{Ad}_g^* \circ \mathrm{Ad}_h^* = \mathrm{Ad}_{gh}^*$.

B.5 Orbit Spaces

Let $\psi : G \to \mathrm{Diff}(M)$ be any action.

Definition B.5 *The* **orbit** *of G through $p \in M$ is $\{\psi_g(p) \mid g \in G\}$. The* **stabilizer** *(or* **isotropy***) of $p \in M$ is the subgroup $G_p := \{g \in G \mid \psi_g(p) = p\}$.*

Exercise 48
If q is in the orbit of p, then G_q and G_p are conjugate subgroups.

Definition B.6 *We say that the action of G on M is ...*

- **transitive** *if there is just one orbit,*
- **free** *if all stabilizers are trivial $\{e\}$,*
- **locally free** *if all stabilizers are discrete.*

Let $\sim$ be the orbit equivalence relation; for $p, q \in M$,

$$p \sim q \quad \Longleftrightarrow \quad p \text{ and } q \text{ are on the same orbit.}$$

The space of orbits $M/\sim\, = M/G$ is called the **orbit space**. Let

$$\begin{array}{rccl} \pi : & M & \longrightarrow & M/G \\ & p & \longmapsto & \text{orbit through } p \end{array}$$

be the **point-orbit projection**.

Topology of the orbit space:

We equip M/G with the weakest topology for which π is continuous, i.e., $\mathcal{U} \subseteq M/G$ is open if and only if $\pi^{-1}(\mathcal{U})$ is open in M. This is called the **quotient topology**. This topology can be "bad." For instance:

Example. Let $G = \mathbb{R}$ act on $M = \mathbb{R}$ by

$$t \longmapsto \psi_t = \text{multiplication by } e^t.$$

There are three orbits $\mathbb{R}^+$, $\mathbb{R}^-$ and $\{0\}$. The point in the three-point orbit space corresponding to the orbit $\{0\}$ is not open, so the orbit space with the quotient topology is *not* Hausdorff. ♢

Example. Let $G = \mathbb{C}\backslash\{0\}$ act on $M = \mathbb{C}^n$ by

$$\lambda \longmapsto \psi_\lambda = \text{multiplication by } \lambda \ .$$

The orbits are the punctured complex lines (through non-zero vectors $z \in \mathbb{C}^n$), plus one "unstable" orbit through 0, which has a single point. The orbit space is

$$M/G = \mathbb{CP}^{n-1} \sqcup \{\text{point}\} \ .$$

The quotient topology restricts to the usual topology on $\mathbb{CP}^{n-1}$. The only open set containing $\{\text{point}\}$ in the quotient topology is the full space. Again the quotient topology in M/G is *not* Hausdorff.

However, it suffices to remove 0 from $\mathbb{C}^n$ to obtain a Hausdorff orbit space: $\mathbb{CP}^{n-1}$. Then there is also a compact (yet not complex) description of the orbit space by taking only unit vectors:

$$\mathbb{CP}^{n-1} = \Big(\mathbb{C}^n\backslash\{0\}\Big)\Big/\Big(\mathbb{C}\backslash\{0\}\Big) = S^{2n-1}/S^1 \ .$$

♢

Appendix C

Variational Principles

C.1 Principle of Least Action

The equations of motion in classical mechanics arise as solutions of variational problems. For a general mechanical system of n particles in $\mathbb{R}^3$, the physical path satisfies Newton's second law. On the other hand, the physical path minimizes the mean value of kinetic minus potential energy. This quantity is called the action. For a system with constraints, the physical path is the path which minimizes the action among all paths satisfying the constraint.

Example. Suppose that a point-particle of mass m moves in $\mathbb{R}^3$ under a force field F; let $x(t)$, $a \leq t \leq b$, be its path of motion in $\mathbb{R}^3$. Newton's second law states that

$$m\frac{d^2x}{dt^2}(t) = F(x(t)) \ .$$

Define the **work** of a path $\gamma : [a,b] \longrightarrow \mathbb{R}^3$, with $\gamma(a) = p$ and $\gamma(b) = q$, to be

$$W_\gamma = \int_a^b F(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)dt \ .$$

Suppose that F is **conservative**, i.e., W_γ depends only on p and q. Then we can define the **potential energy** $V : \mathbb{R}^3 \longrightarrow \mathbb{R}$ of the system as

$$V(q) := W_\gamma$$

where γ is a path joining a fixed base point $p_0 \in \mathbb{R}^3$ (the "origin") to q. Newton's second law can now be written

$$m\frac{d^2x}{dt^2}(t) = -\frac{\partial V}{\partial x}(x(t)) \ .$$

In Lecture 4 we saw that

Newton's second law in $\mathbb{R}^3 = \{(q_1, q_2, q_3)\}$	$\Longleftrightarrow$	Hamilton equations in $T^*\mathbb{R}^3 = \{(q_1, q_2, q_3, p_1, p_2, p_3)\}$

where $p_i = m\frac{dq_i}{dt}$ and the hamiltonian is $H(p, q) = \frac{1}{2m}|p|^2 + V(q)$. Hence, solving Newton's second law in **configuration space** $\mathbb{R}^3$ is equivalent to solving in **phase space** $T^*\mathbb{R}^3$ for the integral curve of the hamiltonian vector field with hamiltonian function H. $\diamondsuit$

Example. The motion of earth about the sun, both regarded as point-masses and assuming that the sun to be stationary at the origin, obeys the **inverse square law**

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} \ ,$$

where $x(t)$ is the position of earth at time t, and $V(x) = \frac{\text{const.}}{|x|}$ is the **gravitational potential**. $\diamondsuit$

When we need to deal with systems with constraints, such as the simple pendulum, or two point masses attached by a rigid rod, or a rigid body, the language of variational principles becomes more appropriate than the explicit analogues of Newton's second laws. Variational principles are due mostly to D'Alembert, Maupertius, Euler and Lagrange.

Example. (The n-particle system.) Suppose that we have n point-particles of masses $m_1, \ldots, m_n$ moving in 3-space. At any time t, the configuration of this system is described by a vector in configuration space $\mathbb{R}^{3n}$

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^{3n}$$

with $x_i \in \mathbb{R}^3$ describing the position of the ith particle. If $V \in C^\infty(\mathbb{R}^{3n})$ is the potential energy, then a path of motion $x(t)$, $a \leq t \leq b$, satisfies

$$m_i \frac{d^2 x_i}{dt^2}(t) = -\frac{\partial V}{\partial x_i}(x_1(t), \ldots, x_n(t)) \ .$$

Consider this path in configuration space as a map $\gamma_0 : [a,b] \to \mathbb{R}^{3n}$ with $\gamma_0(a) = p$ and $\gamma_0(b) = q$, and let

$$\mathcal{P} = \{\gamma : [a,b] \longrightarrow \mathbb{R}^{3n} \mid \gamma(a) = p \text{ and } \gamma(b) = q\}$$

be the set of all paths going from p to q over time $t \in [a,b]$. ♢

Definition C.1 *The* **action** *of a path $\gamma \in \mathcal{P}$ is*

$$\mathcal{A}_\gamma := \int_a^b \left(\sum_{i=1}^n \frac{m_i}{2} \left| \frac{d\gamma_i}{dt}(t) \right|^2 - V(\gamma(t)) \right) dt \ .$$

Principle of least action.

The physical path γ_0 is the path for which $\mathcal{A}_\gamma$ is minimal.

Newton's second law for a constrained system.

Suppose that the n point-masses are restricted to move on a submanifold M of $\mathbb{R}^{3n}$ called the **constraint set**. We can now single out the actual physical path $\gamma_0 : [a,b] \to M$, with $\gamma_0(a) = p$ and $\gamma_0(b) = q$, as being "the" path which minimizes $\mathcal{A}_\gamma$ among all those hypothetical paths $\gamma : [a,b] \to \mathbb{R}^{3n}$ with $\gamma(a) = p$, $\gamma(b) = q$ and satisfying the rigid constraints $\gamma(t) \in M$ for all t.

C.2 Variational Problems

Let M be an n-dimensional manifold. Its tangent bundle TM is a $2n$-dimensional manifold. Let $F : TM \to \mathbb{R}$ be a smooth function.

If $\gamma : [a,b] \to M$ is a smooth curve on M, define the **lift of γ to TM** to be the smooth curve on TM given by

$$\begin{array}{rcl} \tilde{\gamma} : [a,b] & \longrightarrow & TM \\ t & \longmapsto & \left(\gamma(t), \frac{d\gamma}{dt}(t)\right) \ . \end{array}$$

The **action** of γ is

$$\mathcal{A}_\gamma := \int_a^b (\tilde{\gamma}^* F)(t) dt = \int_a^b F\left(\gamma(t), \frac{d\gamma}{dt}(t)\right) dt \ .$$

For fixed $p, q \in M$, let

$$\mathcal{P}(a, b, p, q) := \{\gamma : [a, b] \longrightarrow M \mid \gamma(a) = p,\ \gamma(b) = q\} \ .$$

Problem.
Find, among all $\gamma \in \mathcal{P}(a, b, p, q)$, the curve γ_0 which "minimizes" $\mathcal{A}_\gamma$.

First observe that minimizing curves are always locally minimizing:

Lemma C.2 *Suppose that $\gamma_0 : [a, b] \to M$ is minimizing. Let $[a_1, b_1]$ be a subinterval of $[a, b]$ and let $p_1 = \gamma_0(a_1)$, $q_1 = \gamma_0(b_1)$. Then $\gamma_0|_{[a_1,b_1]}$ is minimizing among the curves in $\mathcal{P}(a_1, b_1, p_1, q_1)$.*

Proof. Exercise: Argue by contradiction. Suppose that there were $\gamma_1 \in \mathcal{P}(a_1, b_1, p_1, q_1)$ for which $\mathcal{A}_{\gamma_1} < \mathcal{A}_{\gamma_0|_{[a_1,b_1]}}$. Consider a broken path obtained from γ_0 by replacing the segment $\gamma_0|_{[a_1,b_1]}$ by γ_1. Construct a smooth curve $\gamma_2 \in \mathcal{P}(a, b, p, q)$ for which $\mathcal{A}_{\gamma_2} < \mathcal{A}_{\gamma_0}$ by rounding off the corners of the broken path. $\square$

We now assume that p, q and γ_0 lie in a coordinate neighborhood $(\mathcal{U}, x_1, \ldots, x_n)$. On $T\mathcal{U}$ we have coordinates $(x_1, \ldots, x_n, v_1, \ldots, v_n)$ associated with a trivialization of $T\mathcal{U}$ by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. Using this trivialization, the curve

$$\gamma : [a, b] \longrightarrow \mathcal{U} \ , \qquad \gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$$

lifts to

$$\tilde{\gamma} : [a, b] \longrightarrow T\mathcal{U} \ , \qquad \tilde{\gamma}(t) = \left(\gamma_1(t), \ldots, \gamma_n(t), \frac{d\gamma_1}{dt}(t), \ldots, \frac{d\gamma_n}{dt}(t)\right) \ .$$

Necessary condition for γ_0 to minimize the action.

Let $c_1, \ldots, c_n \in C^\infty([a,b])$ be such that $c_i(a) = c_i(b) = 0$. For $\gamma_0 \in \mathcal{P}(a,b,p,q)$ let $\gamma_\varepsilon : [a,b] \longrightarrow \mathcal{U}$ be the curve

$$\gamma_\varepsilon(t) = (\gamma_1(t) + \varepsilon c_1(t), \ldots, \gamma_n(t) + \varepsilon c_n(t)) \ .$$

For ε small, γ_ε is well-defined and in $\mathcal{P}(a,b,p,q)$.

Let $\mathcal{A}_\varepsilon = \mathcal{A}_{\gamma_\varepsilon} = \int_a^b F\left(\gamma_\varepsilon(t), \frac{d\gamma_\varepsilon}{dt}(t)\right) dt$. If γ_0 minimizes $\mathcal{A}$, then

$$\frac{d\mathcal{A}_\varepsilon}{d\varepsilon}(0) = 0 \ .$$

$$\begin{aligned} \frac{d\mathcal{A}_\varepsilon}{d\varepsilon}(0) &= \int_a^b \sum_i \left[\frac{\partial F}{\partial x_i}\left(\gamma_0(t), \frac{d\gamma_0}{dt}(t)\right) c_i(t) \right. \\ &\quad \left. + \frac{\partial F}{\partial v_i}\left(\gamma_0(t), \frac{d\gamma_0}{dt}(t)\right) \frac{dc_i}{dt}(t) \right] dt \\ &= \int_a^b \sum_i \left[\frac{\partial F}{\partial x_i}(\ldots) - \frac{d}{dt}\frac{\partial F}{\partial v_i}(\ldots) \right] c_i(t) dt = 0 \end{aligned}$$

where the first equality follows from the Leibniz rule and the second equality follows from integration by parts. Since this is true for all c_i's satisfying the boundary conditions $c_i(a) = c_i(b) = 0$, we conclude that

$$\frac{\partial F}{\partial x_i}\left(\gamma_0(t), \frac{d\gamma_0}{dt}(t)\right) = \frac{d}{dt}\frac{\partial F}{\partial v_i}\left(\gamma_0(t), \frac{d\gamma_0}{dt}(t)\right) \ . \qquad \textbf{(E-L)}$$

These are the **Euler-Lagrange equations**.

Example. Let (M, g) be a riemannian manifold. From the riemannian metric, we get a function $F : TM \to \mathbb{R}$, whose restriction to each tangent space T_pM is the quadratic form defined by the metric. On a coordinate chart $(\mathcal{U}, x^1, \ldots, x^n)$ on M, we have

$$F(x,v) = \sum g_{ij}(x) v^i v^j \ .$$

Let p and q be points on M, and let $\gamma : [a,b] \to M$ be a smooth curve joining p to q. Let $\tilde{\gamma} : [a,b] \to TM$, $\tilde{\gamma}(t) = (\gamma(t), \frac{d\gamma}{dt}(t))$ be the lift of γ to TM. The *action* of γ is

$$\mathcal{A}(\gamma) = \int_a^b (\tilde{\gamma}^* F) \ dt = \int_a^b \left| \frac{d\gamma}{dt} \right|^2 dt \ .$$

It is not hard to show that the Euler-Lagrange equations associated to the action reduce to the **Christoffel equations** for a geodesic

$$\frac{d^2\gamma^k}{dt^2} + \sum (\Gamma^k_{ij} \circ \gamma) \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0 \ ,$$

where the Γ^k_{ij}'s (called the **Christoffel symbols**) are defined in terms of the coefficients of the riemannian metric by

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell} g^{\ell k} \left(\frac{\partial g_{\ell i}}{\partial x_j} + \frac{\partial g_{\ell j}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_\ell} \right) \ ,$$

(g^{ij}) being the matrix inverse to (g_{ij}). ♢

C.3 Solving the Euler-Lagrange Equations

Case 1: Suppose that $F(x,v)$ does not depend on v.

The Euler-Lagrange equations become

$$\frac{\partial F}{\partial x_i} \left(\gamma_0(t), \frac{d\gamma_0}{dt}(t) \right) = 0$$

which mean that the curve γ_0 sits on the critical set of F. For generic F, the critical points are isolated, hence $\gamma_0(t)$ must be a constant curve.

Case 2: Suppose that $F(x,v)$ depends affinely on v:

$$F(x,v) = F_0(x) + \sum_{j=1}^n F_j(x) v_j \ .$$

LHS of **(E-L)** : $$\frac{\partial F_0}{\partial x_i}(\gamma(t)) + \sum_{j=1}^n \frac{\partial F_j}{\partial x_i}(\gamma(t)) \frac{d\gamma_j}{dt}(t)$$

RHS of **(E-L)** : $$\frac{d}{dt} F_i(\gamma(t)) = \sum_{j=1}^n \frac{\partial F_i}{\partial x_j}(\gamma(t)) \frac{d\gamma_j}{dt}(t)$$

The Euler-Lagrange equations become

$$\frac{\partial F_0}{\partial x_i}(\gamma(t)) = \sum_{j=1}^{n} \underbrace{\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right)}_{n\times n \text{ matrix}}(\gamma(t))\frac{d\gamma_j}{dt}(t) \ .$$

If the $n \times n$ matrix $\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right)$ has an inverse $G_{ij}(x)$, then

$$\frac{d\gamma_j}{dt}(t) = \sum_{i=1}^{n} G_{ji}(\gamma(t))\frac{\partial F_0}{\partial x_i}(\gamma(t))$$

is a system of first order ordinary differential equations. Locally it has a unique solution through each point p. If q is not on this curve, there is no solution at all to the Euler-Lagrange equations belonging to $\mathcal{P}(a,b,p,q)$.

Therefore, we need non-linear dependence of F on the v variables in order to have appropriate solutions. From now on, assume that the

Legendre condition: $$\det\left(\frac{\partial^2 F}{\partial v_i \partial v_j}\right) \neq 0 \ .$$

Letting $G_{ij}(x,v) = \left(\frac{\partial^2 F}{\partial v_i \partial v_j}(x,v)\right)^{-1}$, the Euler-Lagrange equations become

$$\frac{d^2\gamma_j}{dt^2} = \sum_i G_{ji}\frac{\partial F}{\partial x_i}\left(\gamma, \frac{d\gamma}{dt}\right) - \sum_{i,k} G_{ji}\frac{\partial^2 F}{\partial v_i \partial x_k}\left(\gamma, \frac{d\gamma}{dt}\right)\frac{d\gamma_k}{dt} \ .$$

This second order ordinary differential equation has a unique solution given initial conditions

$$\gamma(a) = p \qquad \text{and} \qquad \frac{d\gamma}{dt}(a) = v \ .$$

To check whether the above solution is locally minimizing, assume that $\left(\frac{\partial^2 F}{\partial v_i \partial v_j}(x,v)\right) \gg 0$, $\forall (x,v)$, i.e., with the x variable frozen, the function $v \mapsto F(x,v)$ is **strictly convex**.

Suppose that $\gamma_0 \in \mathcal{P}(a,b,p,q)$ satisfies **(E-L)**. Does γ_0 minimize $\mathcal{A}_\gamma$? Locally, yes, according to the following theorem. (Globally it is only critical.)

Proposition C.3 *For every sufficiently small subinterval* $[a_1, b_1]$ *of* $[a, b]$, $\gamma_0|_{[a_1,b_1]}$ *is locally minimizing in* $\mathcal{P}(a_1, b_1, p_1, q_1)$ *where* $p_1 = \gamma_0(a_1)$, $q_1 = \gamma_0(b_1)$.

Proof. As an exercise in Fourier series, show the **Wirtinger inequality**: for $f \in C^1([a, b])$ with $f(a) = f(b) = 0$, we have

$$\int_a^b \left|\frac{df}{dt}\right|^2 dt \geq \frac{\pi^2}{(b-a)^2} \int_a^b |f|^2 dt \ .$$

Suppose that $\gamma_0 : [a, b] \to \mathcal{U}$ satisfies **(E-L)**. Take $c_i \in C^\infty([a, b])$, $c_i(a) = c_i(b) = 0$. Let $c = (c_1, \ldots, c_n)$. Let $\gamma_\varepsilon = \gamma_0 + \varepsilon c \in \mathcal{P}(a, b, p, q)$, and let $\mathcal{A}_\varepsilon = \mathcal{A}_{\gamma_\varepsilon}$.

(E-L) $\iff \frac{d\mathcal{A}_\varepsilon}{d\varepsilon}(0) = 0$.

$$\begin{aligned} \frac{d^2\mathcal{A}_\varepsilon}{d\varepsilon^2}(0) &= \int_a^b \sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial x_j}\left(\gamma_0, \frac{d\gamma_0}{dt}\right) c_i \ c_j \ dt & \text{(I)} \\ &+ 2\int_a^b \sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial v_j}\left(\gamma_0, \frac{d\gamma_0}{dt}\right) c_i \ \frac{dc_j}{dt} \ dt & \text{(II)} \\ &+ \int_a^b \sum_{i,j} \frac{\partial^2 F}{\partial v_i \partial v_j}\left(\gamma_0, \frac{d\gamma_0}{dt}\right) \frac{dc_i}{dt} \ \frac{dc_j}{dt} \ dt & \text{(III)} \ . \end{aligned}$$

Since $\left(\frac{\partial^2 F}{\partial v_i \partial v_j}(x, v)\right) \gg 0$ at all x, v,

$$\begin{aligned} \text{III} &\geq K_{\text{III}} \left|\frac{dc}{dt}\right|^2_{L^2[a,b]} \\ |\text{I}| &\leq K_{\text{I}} |c|^2_{L^2[a,b]} \\ |\text{II}| &\leq K_{\text{II}} |c|_{L^2[a,b]} \left|\frac{dc}{dt}\right|_{L^2[a,b]} \end{aligned}$$

where $K_{\text{I}}, K_{\text{II}}, K_{\text{III}} > 0$. By the Wirtinger inequality, if $b - a$ is very small, then $\text{III} > |\text{I}| + |\text{II}|$ when $c \neq 0$. Hence, γ_0 is a local minimum.
$\square$

C.4 Legendre Transform

The Legendre transform gives the relation between the variational (Euler-Lagrange) and the symplectic (Hamilton-Jacobi) formulations of the equations of motion.

Let V be an n-dimensional vector space, with $e_1, \ldots, e_n$ a basis of V and $v_1, \ldots, v_n$ the associated coordinates. Let $F : V \to \mathbb{R}$, $F = F(v_1, \ldots, v_n)$, be a smooth function. Let $p \in V$, $u = \sum_{i=1}^{n} u_i e_i \in V$. The **hessian** of F is the quadratic function on V defined by

$$(d^2F)_p(u) := \sum_{i,j} \frac{\partial^2 F}{\partial v_i \partial v_j}(p) u_i u_j \ .$$

Exercise 49
Show that $(d^2F)_p(u) = \frac{d^2}{dt^2} F(p + tu)|_{t=0}$.

Exercise 50
A smooth function $f : \mathbb{R} \to \mathbb{R}$ is called **strictly convex** if $f''(x) > 0$ for all $x \in \mathbb{R}$. Assuming that f is strictly convex, prove that the following four conditions are equivalent:

(a) $f'(x) = 0$ for some point x_0,

(b) f has a local minimum at some point x_0,

(c) f has a unique (global) minimum at some point x_0,

(d) $f(x) \to +\infty$ as $x \to \pm\infty$.

The function f is **stable** if it satisfies one (and hence all) of these conditions.

Definition C.4 *The function F is said to be* **strictly convex** *if for every pair of elements $p, v \in V$, $v \neq 0$, the restriction of F to the line $\{p + xv \mid x \in \mathbb{R}\}$ is strictly convex.*

Exercise 51
Show that F is strictly convex if and only if d^2F_p is positive definite for all $p \in V$.

Proposition C.5 *For a strictly convex function F on V, the following are equivalent:*

(a) F has a critical point, i.e., a point where $dF_p = 0$;

(b) F has a local minimum at some point;

(c) F has a unique critical point (global minimum); and

(d) F is proper, that is, $F(p) \to +\infty$ as $p \to \infty$ in V.

Proof. Exercise. (Hint: exercise above.) □

Definition C.6 *A strictly convex function F is* **stable** *when it satisfies conditions (a)-(d) in Proposition C.5.*

Example. The function $e^x + ax$ is strictly convex for any $a \in \mathbb{R}$, but it is stable only for $a < 0$. (What does the graph look like for the values of $a \geq 0$ for which it is not stable?) The function $x^2 + ax$ is strictly convex and stable for any $a \in \mathbb{R}$. ♢

Since V is a vector space, there is a canonical identification $T_p^*V \simeq V^*$, for every $p \in V$.

Definition C.7 *The* **Legendre transform** *associated to a function $F \in C^\infty(V;\mathbb{R})$ is the map*

$$\begin{array}{rccl} L_F: & V & \longrightarrow & V^* \\ & p & \longmapsto & dF_p \in T_p^*V \simeq V^* \ . \end{array}$$

Exercise 52
Show that, if F is strictly convex, then, for every point $p \in V$, L_F maps a neighborhood of p diffeomorphically onto a neighborhood of $L_F(p)$.

Let F be any strictly convex function on V. Given $\ell \in V^*$, let

$$F_\ell : V \longrightarrow \mathbb{R} \ , \qquad F_\ell(v) = F(v) - \ell(v) \ .$$

Since $(d^2F)_p = (d^2F_\ell)_p$,

$$F \text{ is strictly convex} \quad \Longleftrightarrow \quad F_\ell \text{ is strictly convex.}$$

Definition C.8 *The* **stability set** *of a strictly convex function F is*

$$S_F = \{\ell \in V^* \mid F_\ell \text{ is stable}\} .$$

Exercise 53
Suppose that F is strictly convex. Prove that:

(a) The set S_F is open and convex.

(b) L_F maps V diffeomorphically onto S_F.

(c) If $\ell \in S_F$ and $p_0 = L_F^{-1}(\ell)$, then p_0 is the unique minimum point of the function F_ℓ.

Exercise 54
Let F be a strictly convex function. F is said to have **quadratic growth at infinity** if there exists a positive-definite quadratic form Q on V and a constant K such that $F(p) \geq Q(p) - K$, for all p. Show that, if F has quadratic growth at infinity, then $S_F = V^*$ and hence L_F maps V diffeomorphically onto V^*.

For F strictly convex, the inverse to L_F is the map $L_F^{-1} : S_F \to V$ described as follows: for $\ell \in S_F$, the value $L_F^{-1}(\ell)$ is the unique minimum point $p_\ell \in V$ of $F_\ell = F - \ell$.

Exercise 55
Check that p is the minimum of $F(v) - dF_p(v)$.

Definition C.9 *The* **dual function** F^* *to* F *is*

$$F^* : S_F \longrightarrow \mathbb{R} , \quad F^*(\ell) = -\min_{p \in V} F_\ell(p) .$$

Exercise 56
Show that the function F^* is smooth.

Exercise 57
Let $F : V \to \mathbb{R}$ be strictly convex and let $F^* : S_F \to \mathbb{R}$ be the dual function. Prove that for all $p \in V$ and all $\ell \in S_F$,

$$F(p) + F^*(\ell) \geq \ell(p) \qquad \textbf{(Young inequality)} .$$

On one hand we have $V \times V^* \simeq T^*V$, and on the other hand, since $V = V^{**}$, we have $V \times V^* \simeq V^* \times V \simeq T^*V^*$. Let α_1 be the canonical 1-form on T^*V and α_2 be the canonical 1-form on T^*V^*. Via the identifications above, we can think of both of these forms as living on $V \times V^*$. Since $\alpha_1 = d\beta - \alpha_2$, where $\beta : V \times V^* \to \mathbb{R}$ is the function $\beta(p, \ell) = \ell(p)$, we conclude that the forms $\omega_1 = d\alpha_1$ and $\omega_2 = d\alpha_2$ satisfy $\omega_1 = -\omega_2$.

Theorem C.10 *For a strictly convex $F : V \to \mathbb{R}$, we have that $L_F^{-1} = L_{F^*}$.*

Proof. Assume that F has quadratic growth at infinity so that $S_F = V^*$ (otherwise we just have to restrict to S_F). Let Λ_F be the graph of the Legendre transform L_F, $\mathrm{pr}_1 : \Lambda_F \to V$ and $\mathrm{pr}_2 : \Lambda_F \to V^*$ the restrictions of the projection maps $V \times V^* \to V$ and $V \times V^* \to V^*$, and $i : \Lambda_F \hookrightarrow V \times V^*$ the inclusion map. Then

$$i^*\alpha_1 = d(\mathrm{pr}_1)^*F$$

as both sides have value dF_p at each $(p, dF_p) \in \Lambda_F$ (in particular, the graph Λ_F is a lagrangian submanifold of $V \times V^*$ with respect to the symplectic forms ω_1 and ω_2). We conclude that

$$i^*\alpha_2 = d(i^*\beta - (\mathrm{pr}_1)^*F) = d(\mathrm{pr}_2)^*F^* \ ,$$

where the last equality follows from both $i^*\beta - (\mathrm{pr}_1)^*F$ and $(\mathrm{pr}_2)^*F^*$ having value $dF_p - F(p)$ at each (p, dF_p). Hence, Λ_F is the graph of the inverse of L_{F^*} and the inverse of the Legendre transform associated with F is the Legendre transform associated with F^*. □

C.5 Application to Variational Problems

Let M be a manifold and $F : TM \to \mathbb{R}$ a function on TM.

Problem. Minimize $\mathcal{A}_\gamma = \int \tilde{\gamma}^*F$.

At $p \in M$, let

$$F_p := F|_{T_pM} : T_pM \longrightarrow \mathbb{R} \ .$$

Assume that F_p is strictly convex for all $p \in M$. To simplify notation, assume also that $S_{F_p} = T_p^*M$. The Legendre transform on each tangent space

$$L_{F_p} : T_pM \stackrel{\simeq}{\longrightarrow} T_p^*M$$

is essentially given by the first derivatives of F in the v directions. The dual function to F_p is $F_p^* : T_p^*M \longrightarrow \mathbb{R}$. Collect these fiberwise maps into

$$\begin{array}{llllllll} \mathcal{L}: & TM & \longrightarrow & T^*M , & \mathcal{L}|_{T_pM} & = & L_{F_p} , & \text{and} \\ H: & T^*M & \longrightarrow & \mathbb{R} , & H|_{T_p^*M} & = & F_p^* . & \end{array}$$

Exercise 58
The maps H and $\mathcal{L}$ are smooth, and $\mathcal{L}$ is a diffeomorphism.

Let

$$\begin{array}{lll} \gamma : [a,b] \longrightarrow M & \text{be a curve,} & \text{and} \\ \tilde{\gamma} : [a,b] \longrightarrow TM & \text{its lift.} & \end{array}$$

Theorem C.11 *The curve γ satisfies the Euler-Lagrange equations on every coordinate chart if and only if $\mathcal{L} \circ \tilde{\gamma} : [a,b] \to T^*M$ is an integral curve of the hamiltonian vector field X_H.*

Proof. Let

$$\begin{array}{ll} (\mathcal{U}, x_1, \ldots, x_n) & \text{coordinate neighborhood in } M , \\ (T\mathcal{U}, x_1, \ldots, x_n, v_1, \ldots, v_n) & \text{coordinates in } TM , \\ (T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) & \text{coordinates in } T^*M . \end{array}$$

On $T\mathcal{U}$ we have $F = F(x, v)$. On $T^*\mathcal{U}$ we have $H = H(u, \xi)$.

$$\begin{array}{lllllll} \mathcal{L}: & T\mathcal{U} & \longrightarrow & T^*\mathcal{U} & & \\ & (x,v) & \longmapsto & (x,\xi) & \text{where} & \xi = L_{F_x}(v) = \frac{\partial F}{\partial v}(x,v) . \end{array}$$

This is the definition of **momentum** ξ. Then

$$H(x,\xi) = F_x^*(\xi) = \xi \cdot v - F(x,v) \quad \text{where} \quad \mathcal{L}(x,v) = (x,\xi) .$$

Integral curves $(x(t), \xi(t))$ of X_H satisfy the Hamilton equations:

$$\textbf{(H)} \qquad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial \xi}(x,\xi) \\[2ex] \dfrac{d\xi}{dt} = -\dfrac{\partial H}{\partial x}(x,\xi) \ , \end{cases}$$

whereas the physical path $x(t)$ satisfies the Euler-Lagrange equations:

$$\textbf{(E-L)} \qquad \frac{\partial F}{\partial x}\left(x, \frac{dx}{dt}\right) = \frac{d}{dt}\frac{\partial F}{\partial v}\left(x, \frac{dx}{dt}\right) \ .$$

Let $(x(t), \xi(t)) = \mathcal{L}\left(x(t), \frac{dx}{dt}(t)\right)$. We want to prove:

$$t \mapsto (x(t), \xi(t)) \text{ satisfies } \textbf{(H)} \iff t \mapsto \left(x(t), \frac{dx}{dt}(t)\right) \text{ satisfies } \textbf{(E-L)}.$$

The first line of **(H)** is automatically satisfied:

$$\frac{dx}{dt} = \frac{\partial H}{\partial \xi}(x,\xi) = L_{F_x^*}(\xi) = L_{F_x}^{-1}(\xi) \qquad \iff \qquad \xi = L_{F_x}\left(\frac{dx}{dt}\right)$$

Claim. If $(x, \xi) = \mathcal{L}(x, v)$, then $\frac{\partial F}{\partial x}(x,v) = -\frac{\partial H}{\partial x}(x,\xi)$.

This follows from differentiating both sides of $H(x,\xi) = \xi \cdot v - F(x,v)$ with respect to x, where $\xi = L_{F_x}(v) = \xi(x,v)$.

$$\frac{\partial H}{\partial x} + \underbrace{\frac{\partial H}{\partial \xi}}_{v}\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial x} \cdot v - \frac{\partial F}{\partial x} \ .$$

Now the second line of **(H)** becomes

$$\underbrace{\frac{d}{dt}\frac{\partial F}{\partial v}(x,v) = \frac{d\xi}{dt}}_{\text{since } \xi = L_{F_x}(v)} = \underbrace{-\frac{\partial H}{\partial x}(x,\xi) = \frac{\partial F}{\partial x}(x,v)}_{\text{by the claim}} \qquad \iff \qquad \textbf{(E-L)} \ .$$

□

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