Thompson Group Representations and Euler's Theorem

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The Thompson group F

Definition

Call a rational number dyadic if it can be written in the form $\frac{m}{2^n}$ for some $m, n \in \mathbb{Z}$.

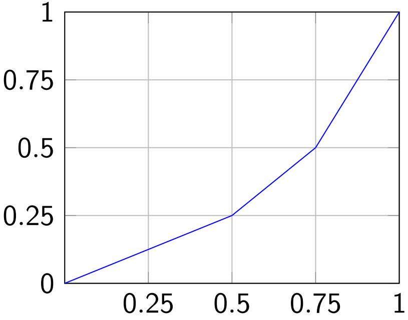
Definition

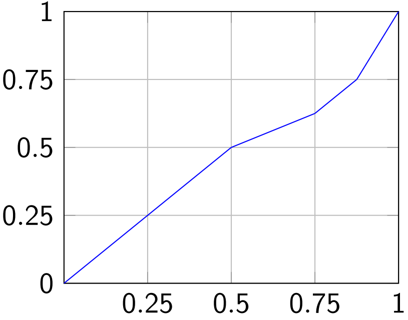
The Thompson group F is the set of piecewise linear homeomorphisms of $[0, 1)$ which satisfy:

- non-differentiability in a finite set of dyadic numbers;
- at the points of differentiability, the derivative is a power of 2;

equipped with the composition operation.

Finite generating set of F





$$A(x) = \begin{cases} \frac{1}{2}x & \text{if } 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{4} & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2x - 1 & \text{if } \frac{3}{4} \leq x < 1 \end{cases}$$

$$B(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2}x + \frac{1}{4} & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{8} & \text{if } \frac{3}{4} \leq x \leq \frac{7}{8} \\ 2x - 1 & \text{if } \frac{7}{8} \leq x < 1 \end{cases}$$

Hilbert space

Definition

A Hilbert space is an inner product space which is complete with regard to the norm induced by the inner product.

Definition

Given a Hilbert space H, define $B(H)$ to be the algebra of bounded linear operators from H to itself.

Theorem

Given an operator $S \in B(H)$, there exists a unique operator $S^* \in B(H)$, such that for every $u, v \in H$, $\langle Su, v \rangle = \langle u, S^* v \rangle$.

Unitary group representations

Definition

A unitary representation of F on an Hilbert Space H is a group homomorphism $\tau : F \to B(H)$ such that $\tau(f^{-1}) = (\tau(f))^*$ for every $f \in F$.

In 2018 Barata & Pinto introduced a family of representations of F.

$$\{\tau_x : F \to H_x | x \in [0,1)\}$$

Every $x \in [0,1)$ induces a representation τ_x of F on H_x.

Unitary equivalence $\tau_x \sim \tau_y$

Definition

Define the equivalence relation $x \sim y \Leftrightarrow \text{frac}(2^m x) = \text{frac}(2^n y)$, for some $m, n \in \mathbb{Z}_0^+$.

Definition

We say that τ_x is unitarily equivalent to τ_y, and write $\tau_x \sim \tau_y$ if there is some unitary operator $U : H_x \to H_y$ such that $U\tau_x(g) = \tau_y(g)U$ for every $g \in F$.

Problem

$$\tau_x \sim \tau_y \Leftrightarrow ?$$

Proposition (Barata & Pinto, 2019)

Given a representation τ_x of F, $\tau_x \sim \tau_{\frac{1}{2}} \Leftrightarrow x \sim \frac{1}{2}$.

The general case

What about the general case? If we find a function which fixes only 0 and some other point, we can use the same idea!

Remark

Linear pieces of functions in F are of the form $f(x) = 2^a x + \frac{b}{2^c}$ with $a, b, c \in \mathbb{Z}$. The only linear piece which may fix an irrational value is $f(x) = x$, which also fixes every value in a neighbourhood of x.

Therefore, this approach doesn't work for $x \in \mathbb{R} \backslash \mathbb{Q}$.

What about $x \in \mathbb{Q}$?

Euler's Theorem

Definition

The function $\phi : \mathbb{N} \to \mathbb{N}$ such that, for every $n \in \mathbb{N}$, $\phi(n)$ is the cardinal of the set $\{x \in \mathbb{N} : x \leq n \text{ and } \gcd(x, n) = 1\}$ is Euler's totient function.

Theorem (Euler)

Given $a, n \in \mathbb{N}$ such that $\gcd(a, n) = 1$, then $a^{\phi(n)} - 1$ is divisible by n.

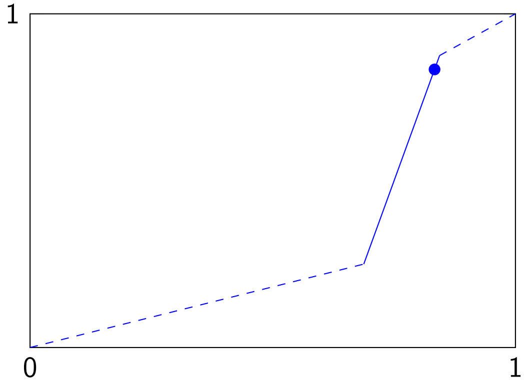
Lemma

Let $z \in \mathbb{Q} \cap [0, 1)$ with $z \nsim \frac{1}{2}$. Write $z = \frac{m}{2^n s}$ for some $m, n, s \in \mathbb{N}$ such that $s \neq 1$ (since $z \nsim \frac{1}{2}$) and $\gcd(s, 2) = \gcd(s, m) = 1$. The linear map

$$f(x) = 2^{\phi(s)}x - \frac{m \cdot \frac{2^{\phi(s)}-1}{s}}{2^n}$$

satisfies $f(x) = x \Leftrightarrow x = z$ and can be a linear piece of an element in F.

The rational case



$$f(x) = 2^{\phi(s)}x - \frac{\frac{2^{\phi(s)}-1}{s}}{2^n}$$

It needs to be checked that some function in F does in fact have this as a linear piece.

The rational case

Theorem

Given a rational $z \in [0,1)$ there exists a function $f \in F$ such that $f(x) = x \Leftrightarrow x \in \{0, z\}$.

Theorem

Let $x \in \mathbb{Q} \cap [0,1)$. Then, $\tau_x \sim \tau_y \Leftrightarrow x \sim y$.

The above result completely solves the problem for the rational case. The irrational case remains to be solved.