Let $N(s,\tilde{s})$ be the number of times $\tilde{s}$ is used in s.

Then
$$A = \max_{\tilde{s}\in\tilde{S}} \left\{ \frac{1}{\nu(\tilde{s})} \sum_{s\in S} \mathrm{length}(s)\, N(s,\tilde{s})\,\mu(s) \right\}$$

Example: Consider S_n with random transpositions (Diaconis, Shah...)

deck of n cards

left + right hand picks cards uniformly at random (maybe the same)

Then transpose the cards.

[This is related to how DNA encodes information]

$S = \{(a,b) \mid a\neq b\} \cup \{id\}$

We have $\mu(id) = \frac{1}{n}$ and $\mu(ab) = \frac{2}{n^2}$

Since again S is a union of conjugacy classes, rep theory can be used to find eigenvalues and eigenvectors.

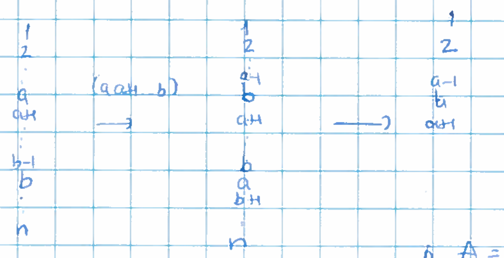
Can see that $\lambda_2 = 1-\frac{2}{n}$

Random to random: deck of n cards. Pick a card uniformly at random, remove it from deck and insert it in a uniformly random position.

The generators of these transformations are $\tilde{S} = \{\text{cycles}\} \cup \{id\}$

Then $\nu(id) = \frac{1}{n}$ $\quad \nu(a\ a{+}1) = \frac{2}{n^2}$ $\quad \nu(a, a{+}1, \dots, a{+}k) = \frac{1}{n^2}$

Given a transposition (a,b) can write it as a product of elements in $\tilde{S}$:



hence $\mathrm{length}(a,b) = 2$

$N((a,b), s) = 1$ for $s = (a, a{+}1, \dots, b)$

2 transpositions at most

We get the following formula for A: $A = n^2 \cdot \frac{2}{n^2} \cdot 2 \cdot 2 \cdot 1 = 8$, $\tilde{s} = (a, a{+}1, \dots, b)$ (this is the maximum you can get)

The inequality says
$$\lambda_2 \leq 1 - \frac{1-\left(1-\frac{2}{n}\right)}{8} = 1 - \frac{1}{4n}$$

Lecture 4: Lower bound techniques

Configuration space $X = (\mathbb{Z}/_{2\mathbb{Z}})^n$ (hypercube)
$= \{(a_1,\dots,a_n) : a_i \in \mathbb{Z}_2\}$

this is an abelian group $\Rightarrow$ can use rep theory to find eigenvalues and eigenfunctions

Want to prove: If we consider the transition matrix
$$P(x,y) = \begin{cases} \frac{1}{2} & \text{if } y = x \\ \frac{1}{2n} & \text{if } y = x + e_i \quad e_i = (0,\dots,0,1,0,\dots,0) \\ 0 & \text{otherwise} \end{cases}$$

If $t = \frac{1}{2} n\log n - cn$ $\quad \|P_0^t - U\|_{TV} \geq 1 - \frac{1}{c^2}$ (quite big)

approx.

The only prerequisite for today is Chebyshov's inequality

Z random variable $\boxed{P(|Z-\mathbb{E}(Z)| \geq \alpha\sqrt{\mathrm{Var}(Z)}) \leq \frac{1}{\alpha^2}}$

Eigenfunctions $f_1(x) = 1$, eigenvalue 1

$f_2(x) = n-2x$ eigenvalue $1-\frac{1}{n}$

$f_3(x) = \binom{n}{2} - 2nx + x^2$ eigenvalue $1-\frac{2}{n}$

here for X_t the process, $f(X_t) = f(\|X_t\|)$ ↑ number of 1s on X_t

For instance $f_2(e_1) = f_2(\|e_1\|) = n-2$

Now $(f_2(x))^2 = n f_1(x) + 2 f_3(x)$

1) Y a random variable on X $Y \sim U$ uniform distribution

Let $U(f_2) \overset{def}{=} \sum_{\bar{x}\in X} \frac{1}{2^n} f_2(\bar{x}) = \sum_x \frac{1}{2^n} f_2(x) f_1(x) = 0$ because $\langle f_1, f_2\rangle = 0$

means $E_U(f)$

check

$\mathrm{Var}_U(f_2) = U(f^2) - \underbrace{U(f)^2}_{0} = \sum_x \frac{1}{2^n}(n-2x)^2 = n$

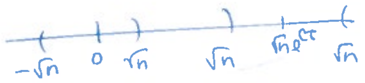
f_2 concentrated here

Chebyshov for uniform measure

2) $P_0^t(f_2) = \underbrace{\sum_{x\in X} P^t(0,x) f_2(x)}_{(P^t f)(0)} = f_2(0)(1-\frac{1}{n})^t = n(1-\frac{1}{n})^t \sim \sqrt{n}\, e^{c/2}$ if $t = \frac{1}{2} n\log n - cn$

$\mathrm{Var}_{P^t}(f_2) = P_0^t(f_2^2) - (\underset{\substack{\uparrow \\ \text{mean } = n(1-\frac{1}{n})^t}}{P_0^t(f_2)})^2 = P_0^t(n f_1 + 2 f_3) - n^2(1-\frac{1}{n})^{2t} = n + n(n-1)(1-\frac{2}{n})^t - n^2(1-\frac{2}{n}+\frac{1}{n^2})^t$ $= 0$

$\simeq n$ if $t = \frac{1}{2} n\log n - cn$



for appropriate those intervals are disjoint $\Rightarrow$ the measures are distinct.

Chebyshov

Chebyshev $A = \{x : |f_2(x)| \leq b\}$

$U(A) \geq 1 - \frac{1}{b^2}$

$U(A) - P_0^t(A) \geq 1 - \frac{1}{b^2} - \frac{1}{e^{2c}}$

$P_0^t(A) \leq \frac{1}{b^2} + \frac{1}{e^{2c}}$

(This is in Chapter 3 of Persi's book)

$\|P_0^t - U\|_{TV} = \max_{B\subseteq X} \{|P_0^t(B) - U(B)|\} \geq U(A) - P^t(A) \approx 1$ if $t = \frac{1}{2}n\log n - cn$

This method is ~~[illegible]~~ due to Diaconis + S and it is called the <u>2nd moment method</u> (with the first moment we could get a lower bound of n)

<u>Wilson's lemma</u>: P transition matrix of a Markov chain on X. $f: X \to \mathbb{R}$, $Pf = \lambda f$ and

$\mathbb{E}((f(X_1) - f(X_0))^2 \mid X_0) \leq R$ then if $t = \frac{1}{2\log(1/\lambda)}\left[\log \frac{(1-\lambda)(f(x_0))^2}{2R} + \log \frac{\varepsilon}{1-\varepsilon}\right]$ then

$\|P_x^t - M\|_{TV} \geq 1-\varepsilon$ [no need for reversibility, only need an eigenvalue and an eigenfunction]

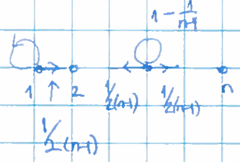
TÉCNICO LISBOA

Consider random adjacent transpositions:

$$S = \{ (a\,a{+}1),\ a = 1,2,\dots,n-1 \} \cup \{ id \}$$

$$P(x,y) = \begin{cases} \frac{1}{2} & y = x \\ \frac{1}{2(n-1)}, & y = x\,(a, a{+}1) \\ 0 & \text{otherwise} \end{cases}$$

Can project this to a much smaller Markov chain by looking at what happens to the first card. This has only n states rather than $n!$



The eigenvectors and eigenvalues are similar to the example about the circle:

$$\varphi(k) = \cos \frac{\pi(2k-1)}{2n} \quad \text{w/ eigenvalue} \quad \frac{n-2}{n-1} + \frac{1}{n}\cos\left(\frac{\pi}{n}\right)$$

$$\varphi_k(\sigma) = \cos \frac{\pi(2\sigma(k)-1)}{n} \qquad \sigma(k) = \text{card at position } k.$$

We could have considered any other card than the first. Let

$$\Phi(\sigma) = \sum_{k=1}^{n/2} \varphi(k)\, \varphi_k(\sigma)$$

$$\Phi(id) = \sum_{k=1}^{n/2} (\varphi(k))^2 = \sum_{k=1}^{n/2} \cos^2 \frac{\pi(2k-1)}{2n} \geq \frac{n}{4}$$

If we perform $(a, a{+}1)$ ~~$(\sigma(a), \sigma(a{+}1))$~~

(all other terms cancel)

$$\left| \Phi(\sigma \circ (a, a{+}1)) - \Phi(\sigma) \right| = \varphi(a)\varphi(\sigma(a{+}1)) + \varphi(a{+}1)\varphi(\sigma(a)) - \varphi(a)\varphi(\sigma(a)) - \varphi(a{+}1)\varphi(\sigma(a{+}1))$$

$$= |\varphi(a) - \varphi(a{+}1)| \underbrace{|\varphi(\sigma(a)) - \varphi(\sigma(a{+}1))|}_{\leq 2 \text{ (difference of cosines)}}$$

$$|\varphi(a) - \varphi(a{+}1)| = |\varphi'(\xi)| \leq \frac{\pi}{n}$$

Hence $\mathbb{E}\left((\Phi(\sigma_1) - \Phi(\sigma_0))^2 / \sigma_0 \right) \leq \frac{1}{2(n-1)} \frac{2\pi}{n} \sim \frac{1}{n^2} = R$ in Wilson's theorem.

Using next term in Taylor series of cosine we see that $\lambda \approx 1 - \frac{d}{n^3}$ for $\frac{1}{n}\cos\frac{\pi}{n} = 1 - \frac{\pi}{n^2}$

$$t \geq -\frac{1}{2\log\left(1 - \frac{d}{n^3}\right)} \left[\log \frac{\frac{1}{n^3}\, n^2}{n^2} + \log \varepsilon \right] \approx \frac{n^3}{2}\log n + n^3 \log \frac{\varepsilon}{\pi}$$