Exercises

September 6, 2019

1. Show that if, in a network with source a and sink z, vertices with different voltages are glued together, then the effective resistance from a to z will strictly decrease.

2. Suppose that Z is a set of states in a Markov chain and that x_0 is a state not in Z. Assume that when the Markov chain is started in x_0, then it visits Z with probability 1. Define the random path $Y_0, Y_1, \dots$ by $Y_0 := x_0$ and then recursively by letting Y_{n+1} have the distribution of one step of the Markov chain starting from Y_n given that the chain will visit Z before visiting any of $Y_0, Y_1, \dots, Y_n$ again. However, if $Y_n \in Z$, then the path is stopped and Y_{n+1} is not defined. Show that (Y_n) has the same distribution as loop-erasing a sample of the Markov chain started from x_0 and stopped when it reaches Z. In the case of a random walk, the conditioned path (Y_n) is called the Laplacian random walk from x_0 to Z.

3. Suppose that the graph G has a Hamiltonian path, i.e. there exists a path $(x_k : 1 \le k \le n)$ that is a spanning tree. Let X be a simple random walk on G and let $T(A) = \min\{t \ge 0 : X_t \in A\}$ and $T^+(A) = \min\{t \ge 1 : X_t \in A\}$ be the first hitting time and the first return time respectively to the set A. Define

$$q_k = \mathbb{P}_{x_k}\big(T^+(\{x_k\}) > T(\{x_{k+1}, \dots, x_n\})\big)$$

and show that the number of spanning trees of G equals $\prod_{k<n} q_k \deg(x_k)$.

4. How efficient is Wilson's method? What takes time is to generate a random successor state of a given state. Call this a step of the algorithm. Show that the expected number of steps to generate a random spanning tree rooted at r is

$$\sum_x \frac{\deg(x)}{2|E|}(\mathbb{E}_x[\tau_r] + \mathbb{E}_r[\tau_x]),$$

where $|E|$ is the set of edges and $\deg(x)$ is the degree of the vertex x.