Lecture 4 suggested problems

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Exercise on Total Variation distance: Let μ and ν be two probability measures on a finite space X. Prove that

$$\|\mu - \nu\|_{\text{T.V.}} = \max_{\{A \subset X\}} |\mu(A) - \nu(A)|.$$

Exercise on C_n: Use Wilson's lemma, to prove that the lazy, simple random walk on C_n needs at least order n^2 steps to reach stationarity.

Random-to-top: Imitate the lower bound argument for the lazy random walk on the hypercube, to prove that for random to top card shuffle on S_n, we have that

$$s(n \log n - cn) \geq 1 - e^{-c},$$

where $s(t)$ is the separation distance at time t.

Eigenvalue/eigenvectors exercise: Let (P, π) be a Markov chain on a finite space X. Let $\lambda \neq 1$ be an eigenvalue for P. Prove that

$$|\lambda|^t \leq 2d(t),$$

where $d(t)$ is the total variation distance at time t.

Random to random: Consider a deck of n cards. The random to random card shuffle suggests to pick a card uniformly at random, remove it from the deck and insert it to a uniformly random position. Let N_t be the number of cards that were never removed during the first t shuffles.

(a) Prove that $\mathbb{E}(N_t) = n\left(1 - \frac{1}{n}\right)^t$ and

$$\text{Var}(N_t) = n(n-1)\left(1 - \frac{2}{n}\right)^t + n\left(1 - \frac{1}{n}\right)^t + n^2\left(1 - \frac{1}{n}\right)^{2t}.$$

(b) Let $L : S_n \to \mathbb{N}$, where $L(\sigma)$ is the length of the longest increasing subsequence of σ. Take as a given that there are constants c_0 and c_1 such that under the uniform measure

$$\mathbb{E}(L) = 2\sqrt{n} + c_1 n^{1/6} + o(n^{1/6}) \text{ and } \text{Var}(L) = c_0 n^{1/3} + o(n^{1/3}).$$

Consider the set $A = \{\sigma \in S_n : L(\sigma) \geq \mathbb{E}(L) + t\sqrt{\text{Var}(L)}\}$ to prove that if we take $\lim_{c\to\infty} \lim_{n\to\infty} d(\left(\frac{1}{2} - \varepsilon\right) n(\log n - c) = 1$.