Lecture 2 and 3 suggested problems

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Exercise on C_n: Let C_n be the cycle of length n. For the the lazy, simple random walk on C_n, confirm that the eigenvalues are $\frac{1}{2} + \frac{1}{2}\cos\frac{2\pi j}{n}$, for $j = 1, \dots, n$ and the corresponding eigenvectors are $f_j(x) = \cos\frac{2\pi j x}{n}$.

Random to random: Consider a deck of n cards. The random to random card shuffle suggests to pick a card uniformly at random, remove it from the deck and insert it to a uniformly random position. Prove that random to random is aperiodic, irreducible and reversible. Prove that there is a constant c, such that

$$0 \leq \lambda_2 \leq 1 - \frac{c}{n},$$

where λ_2 is the second largest eigenvalue of the transition matrix.

Adjacent transpositions: Consider a deck of n cards. Let $S = \{(i, i+1),\ i \in \{1, \dots, n-1\}\}$. The adjacent transpositions card shuffle suggests to pick $i \in \{1, \dots, n-1\}$ uniformly at random and flip a fair coin. If heads, we do nothing. If tails, we perform $(i, i+1)$ to the deck (so we transpose the card in position i with the card in position $i+1$). Prove that the adjacent transposition card shuffle is aperiodic, irreducible and reversible. Prove that there is a constant c, such that

$$0 \leq \lambda_2 \leq 1 - \frac{c}{n^3},$$

where λ_2 is the second largest eigenvalue of the transition matrix.

Dirichlet forms exercise: Let (P, π) be a reversible Markov chain on a finite space X. Prove that

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{x, y \in X} (f(x) - f(y))^2 \pi(x) P(x, y),$$

for every $f : X \to \mathbb{R}$.

Eigenvalue/eigenvectors exercise: Let (P, π) be a Markov chain on a finite space X. Let ϕ be an eigenfunction of P corresponding to an eigenvalue $\lambda \neq 1$. Prove that $\pi(\phi) := \mathbb{E}_\pi(\phi) = 0$.