Numerical scheme for quadratic hamiltonian MFG

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Introduction

- Uniform subdivision $(t_0, \dots, t_I)$ of $(0, T)$ where $t_i = i\Delta t$
- Uniform subdivision $(x_0, \dots, x_J)$ of $(0, 1)$ where $x_j = j\Delta x$
- Finite difference scheme: $\hat{\psi}^n_{i,j}$ and $\hat{\phi}^{n+\frac{1}{2}}_{i,j}$
- Neumann conditions: $\hat{\psi}^n_{i,-1} = \hat{\psi}^n_{i,1}$ and $\hat{\psi}^n_{i,J+1} = \hat{\psi}^n_{i,J-1}$
- Neumann conditions: $\hat{\phi}^{n+\frac{1}{2}}_{i,-1} = \hat{\phi}^{n+\frac{1}{2}}_{i,1}$ and $\hat{\phi}^{n+\frac{1}{2}}_{i,J+1} = \hat{\phi}^{n+\frac{1}{2}}_{i,J-1}$

$$\mathcal{M} = M_{I+1,J+1}(\mathbb{R})$$

$$\mathcal{M}_\epsilon = \{(m_{i,j})_{i,j} \in \mathcal{M}, \quad \forall i, j, m_{i,j} \geq \epsilon\}$$

Numerical scheme

$$\hat{\psi}^0_{i,j} = 0$$

Completely implicit scheme for $\hat{\phi}^{n+\frac{1}{2}}$:

$$\frac{\hat{\phi}^{n+\frac{1}{2}}_{i+1,j} - \hat{\phi}^{n+\frac{1}{2}}_{i,j}}{\Delta t} + \frac{\sigma^2}{2}\frac{\hat{\phi}^{n+\frac{1}{2}}_{i,j+1} - 2\hat{\phi}^{n+\frac{1}{2}}_{i,j} + \hat{\phi}^{n+\frac{1}{2}}_{i,j-1}}{(\Delta x)^2} = -\frac{1}{\sigma^2} f(x_j, \hat{\phi}^{n+\frac{1}{2}}_{i,j}\hat{\psi}^n_{i,j})\hat{\phi}^{n+\frac{1}{2}}_{i,j}$$

$$\hat{\phi}^{n+\frac{1}{2}}_{I,j} = \exp\left(\frac{u_T(x_j)}{\sigma^2}\right)$$

Completely implicit scheme for $\hat{\psi}^{n+1}$:

$$\frac{\hat{\psi}^{n+1}_{i+1,j} - \hat{\psi}^{n+1}_{i,j}}{\Delta t} - \frac{\sigma^2}{2}\frac{\hat{\psi}^{n+1}_{i+1,j+1} - 2\hat{\psi}^{n+1}_{i+1,j} + \hat{\psi}^{n+1}_{i+1,j-1}}{(\Delta x)^2} = \frac{1}{\sigma^2} f(x_j, \hat{\phi}^{n+\frac{1}{2}}_{i+1,j}\hat{\psi}^{n+1}_{i+1,j})\hat{\psi}^{n+1}_{i+1,j}$$

$$\hat{\psi}^{n+1}_{0,j} = \frac{m_0(x_j)}{\hat{\phi}^{n+\frac{1}{2}}_{0,j}}$$

Monotonicity of the scheme and limit behavior

Theorem

Assume that m_0 is bounded. The numerical scheme verifies the following properties:

- $(\hat{\phi}^{n+\frac{1}{2}})_n$ *is a decreasing sequence of* $\mathcal{M}_\epsilon$.
- $(\hat{\psi}^n)_n$ *is an increasing sequence of* $\mathcal{M}_0$, *bounded from above, independently of the subdivision.*
- $(\hat{\phi}^{n+\frac{1}{2}}, \hat{\psi}^n)_n$ *converges towards a couple* $(\hat{\phi}, \hat{\psi}) \in \mathcal{M}_\epsilon \times \mathcal{M}_0$.

Convergence of the scheme

Definition of the norm $\|\cdot\|$

$$\forall m = (m_{i,j})_{i,j} \in \mathcal{M}, \|m\|^2 = \sup_{0 \le i \le I} \frac{1}{J+1} \sum_{j=0}^{J} m_{i,j}^2$$

Hypothesis

- We suppose that f, u_T and m_0 are bounded.

Convergence of the scheme

$$\tilde{\phi}_{i,j}^{n+\frac{1}{2}} = \phi^{n+\frac{1}{2}}(t_i, x_j)$$

$$\tilde{\psi}_{i,j}^{n+1} = \psi^{n+1}(t_i, x_j)$$

Consistency errors

$$\tilde{\eta}_{i,j}^{n+\frac{1}{2}} = \frac{\tilde{\phi}_{i+1,j}^{n+\frac{1}{2}} - \tilde{\phi}_{i,j}^{n+\frac{1}{2}}}{\Delta t} + \frac{\sigma^2}{2}\frac{\tilde{\phi}_{i,j+1}^{n+\frac{1}{2}} - 2\tilde{\phi}_{i,j}^{n+\frac{1}{2}} + \tilde{\phi}_{i,j-1}^{n+\frac{1}{2}}}{(\Delta x)^2} + \frac{1}{\sigma^2} f(x_j, \tilde{\phi}_{i,j}^{n+\frac{1}{2}} \tilde{\psi}_{i,j}^{n})\tilde{\phi}_{i,j}^{n+\frac{1}{2}}$$

$$\tilde{\eta}_{i,j}^{n+1} = \frac{\tilde{\psi}_{i+1,j}^{n+1} - \tilde{\psi}_{i,j}^{n+1}}{\Delta t} - \frac{\sigma^2}{2}\frac{\tilde{\psi}_{i+1,j+1}^{n+1} - 2\tilde{\psi}_{i+1,j}^{n+1} + \tilde{\psi}_{i+1,j-1}^{n+1}}{(\Delta x)^2} - \frac{1}{\sigma^2} f(x_j, \tilde{\phi}_{i+1,j}^{n+\frac{1}{2}} \tilde{\psi}_{i+1,j}^{n+1})\tilde{\psi}_{i+1,j}^{n+1}$$

Theorem (Stability bounds)

Let us suppose, in addition to the hypotheses made above, that f is uniformly Lipschitz with respect to its second variable, i.e.

$$\exists K, \forall x \in (0,1), \forall \xi_1, \xi_2 \in \mathbb{R}_+, |f(x,\xi_2) - f(x,\xi_1)| \leq K|\xi_2 - \xi_1|$$

Then, $\forall \nu > 0$, $\exists C > 0$, $\forall I, J \in \mathbb{N}$ *such that* $\frac{1}{\Delta t} > 1 + \frac{K}{\sigma^2} \max\left(\left\|e^{\frac{u_T}{\sigma^2}}\right\|_\infty^2, \frac{\|m_0\|_\infty^2}{\epsilon^2}\right) + \nu$, *we have* $\forall n \in \mathbb{N}$:

$$\|\widehat{\phi}^{n+\frac{1}{2}} - \tilde{\phi}^{n+\frac{1}{2}}\|^2 \leq C\|\widehat{\psi}^n - \tilde{\psi}^n\|^2 + C\|\tilde{\eta}^{n+\frac{1}{2}}\|^2$$

$$\|\widehat{\psi}^{n+1} - \tilde{\psi}^{n+1}\|^2 \leq C\|\widehat{\phi}^{n+\frac{1}{2}} - \tilde{\phi}^{n+\frac{1}{2}}\|^2 + C\|\tilde{\eta}^{n+1}\|^2$$

Convergence of the scheme

Theorem (Convergence)

Let us suppose, in addition to the hypotheses made above that u_T, m_0 and f are so that $\forall n \in \mathbb{N}, \phi^{n+\frac{1}{2}}, \psi^n \in C^{1,2}([0,T] \times [0,1])$. Then $\forall n \in \mathbb{N}$:

$$\lim_{\Delta t, \Delta x \to 0} \|\widehat{\phi}^{n+\frac{1}{2}} - \tilde{\phi}^{n+\frac{1}{2}}\| = 0$$

$$\lim_{\Delta t, \Delta x \to 0} \|\widehat{\psi}^{n+1} - \tilde{\psi}^{n+1}\| = 0$$

Convergence of the scheme

Theorem

Let us suppose, in addition to the hypotheses made above that u_T, m_0 and f are so that $\forall n \in \mathbb{N}, \phi^{n+\frac{1}{2}}, \psi^n \in C^{1,2}([0,T] \times [0,1])$ and $\phi, \psi \in C^{1,2}([0,T] \times [0,1])$
Then:

$$\lim_{n\to\infty} \limsup_{\Delta t, \Delta x \to 0} \|\widehat{\phi}^{n+\frac{1}{2}} - \tilde{\phi}\| = 0$$

$$\lim_{n\to\infty} \limsup_{\Delta t, \Delta x \to 0} \|\widehat{\psi}^{n+1} - \tilde{\psi}\| = 0$$

Example

People willing to live at the center, but not together.

$$\Omega = (0,1), \quad T = 0.5, \quad \sigma = 1 \quad u_T = 0$$

$$f(x,\xi) = -16(x-1/2)^2 - 0.1\min(5,\max(0,\xi))$$

$$m_0(x) = \frac{\mu(x)}{\int_0^1 \mu(x')dx'}$$

$$\text{where} \quad \mu(x) = 1 + 0.2\cos\left(\pi\left(2x - \frac{3}{2}\right)\right)^2$$

51 points in time and 51 points in space.

Convergence after 5 iterations for n (threshold on m: 10^{-7}).

Solution ϕ

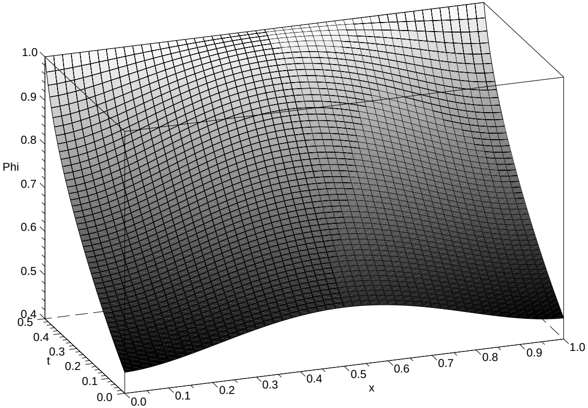

Figure: Solution for ϕ. Convergence after 5 iterations for n

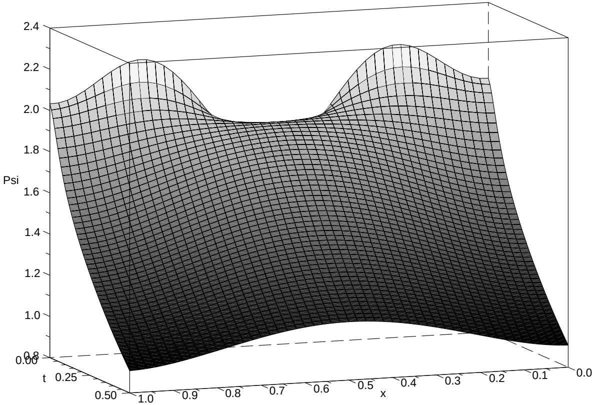

Figure: Solution for ψ. Convergence after 5 iterations for n

Solution u

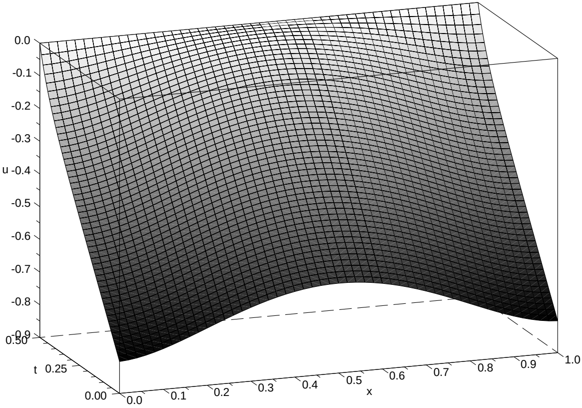

Figure: Solution for u. Convergence after 5 iterations for n

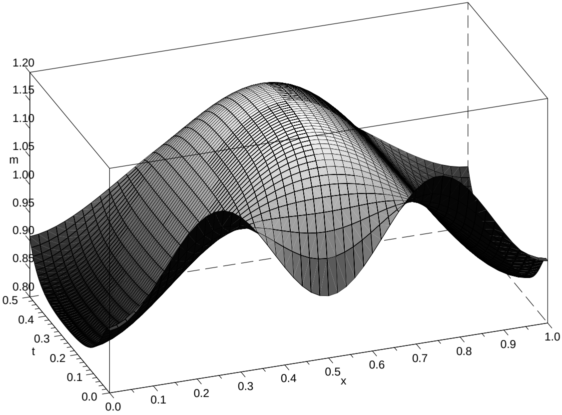

Figure: Solution for m. Convergence after 5 iterations for n

Solution optimal control $\alpha = \nabla u$

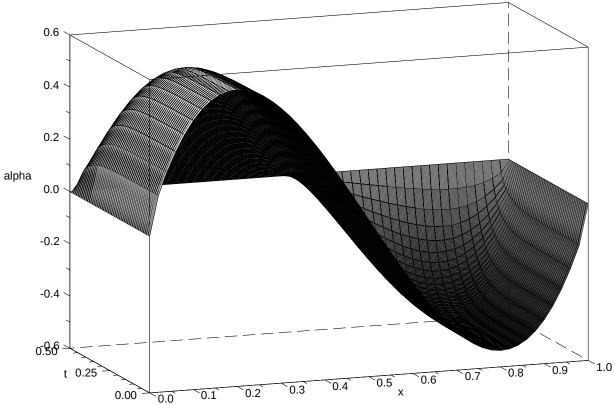

Figure: Solution for α. Convergence after 5 iterations for n

Computation time

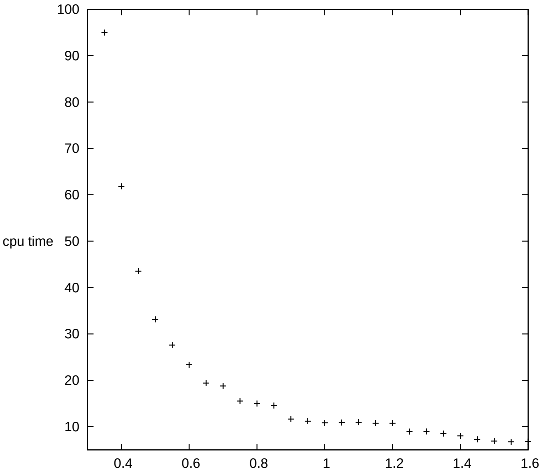

Figure: Computation time as a function of σ

Conclusion

THANK YOU.