Notes on Ordinary Differential Equations - VII

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1 Orbits of Flows ans Asymptotic Behaviour - I

After having set the theoretical basis of flows, we move to a more detailed study of their orbits. We recall that, given $x \in \Omega$, the orbit

$$O(x) = \{\Psi(t, x) : t \in I_x\}$$

is the union of all solutions equivalent to

$$u_x(t) = \Psi(t, x)$$

under time translations.The phase space Ω is the disjoint union of its orbits.

We give a first characterization of orbits.

Proposition 1 *Let $x \in \Omega$. The orbit $O(x)$ satisfies exactly one of the following:*

i) $\Psi(t, x)$ is injective;

ii) $\Psi(t, x)$ is constant (an equilibrium);

iii) There exists a positive T such that $\Psi(t, x) = \Psi(s, x)$ if and only if $s = t + mT$ for some $m \in \mathbb{Z}$ (a periodic orbit).

Proof. Suppose that $\Psi(t, x)$ is not injective; then there exists $t_1 < t_2$ such that $\Psi(t_1, x) = \Psi(t_2, x)$. Let $c = t_2 - t_1$; we have then

$$\Psi(s + c, x) = \Psi(s - t_1, \Psi(t_2, x)) = \Psi(s - t_1, \Psi(t_1, x)) = \Psi(s, x),$$

ie, the orbit $\Psi(t, x)$ admits period c. Notice that the first expression is defined if and only if $s \in I_x - c$, while the last is defined if and only if $s \in I_x$, confirming that we must have $I_x = \mathbb{R}$: this was already clear from the fact that the orbit is equal to $\{\Psi(t, x) : 0 \leq t \leq c\}$, a compact set.
We show now that one of conditions ii) and iii) occurs, analysing the set of periods for an orbit

$$C = \{c \in \mathbb{R} : \Psi(t + c, x) = \Psi(t, x) \ \forall t \in \mathbb{R}\};$$

Exercise 2 *Verify that C is (if not empty) a closed subgroup of $\mathbb{R}$, ie*

a) If $c, d \in C$ then $-c \in C$ and $c + d \in C$;

b) If $c_n \in C$ and $\lim_n c_n = c$ then $c \in C$.

The proof is completed by means of the following:

Claim 3 *A closed subgroup C of $\mathbb{R}$ is either*

1) trivial: $C = \{0\}$,

2) $C = \mathbb{R}$, or

3) infinite discrete: $C = \{mT : m \in \mathbb{Z}\}$ for some $T > 0$.

Before we prove this claim, we notice that, when C is the set of periods of the orbit $O(x)$, cases 1), 2) and 3) correspond respectively to the cases i), ii) and iii) from the Proposition.

The claim follows from the following reasoning: suppose a subgroup C of $\mathbb{R}$ is not trivial; then we have two possibilities: either C has a least positive element T, or it doesn't.

If it does, let $c \in C$ be any element; there exists an integer m such that

$$mT \le c < (m + 1)T;$$

if $mT < c$ then we would have

$$0 < c - mT < T,$$

a contradiction; so, $c = mT$ and we have case 3).

If it doesn't, then for every $\delta > 0$ there exists some $c \in C$ such that $0 < c < \delta$; but then, for every $z \in \mathbb{R}$,

$$mc \le z < (m + 1)c$$

for some integer m. We thus have that, for every $z \in \mathbb{R}$, and for every $\delta > 0$, there exists an element $d \in C$ such that $|z - d| < \delta$, ie, C is dense.

But if C is closed and dense it must coincide with $\mathbb{R}$ and we have case 2). ■

In order to study the asymptotic behaviour of orbits, we introduce the following definition. The positive (resp. negative) semi-orbit of x is $O^+(x) = \{\Psi(t, x) : t \ge 0\}$ (resp. $O^-(x) = \{\Psi(t, x) : t \le 0\}$.

Definition 4 *Let $x \in \Omega$ be such that $\Psi(t, x)$ is defined for all $t \ge 0$. The omega-limit of $x \in \Omega$ is the set of acumulation points of the positive semi-orbit of x, ie,*

$$\omega(x) = \{y \in \Omega : \exists\, t_n \to +\infty \text{ such that } \lim_{n\to+\infty} \Psi(t_n, x) = y\}.$$

Similarly, if $\Psi(t, x)$ is defined for all $t \le 0$, the alpha-limit of x is

$$\alpha(x) = \{y \in \Omega : \exists\, t_n \to -\infty \text{ such that } \lim_{n\to+\infty} \Psi(t_n, x) = y\}.$$

Notice that the α-limit set of x with respect to the flow of $\dot{x} = F(x)$ coincides with the ω-limit of x for $\dot{x} = -F(x)$; so, for many general statements, it is enough to consider ω-limit sets.

Example 5 *Some basic examples are:*

1 - *If $\dot{x} = Ax$ where all eigenvalues have negative real part, then $\omega(x) = \{0\}$, for every $x \in \mathbb{R}^n$.*

2 - *If the orbit of x is periodic then its ω-limit (and α-limit) coincides with the orbit itself.*

3 - *For the flow associated to the vector field given in polar coordinates as $G(r, \theta) = (r(1-r), 1)$, the ω-limit of an orbit with initial condition (r_0, θ_0), where $0 < r_0 < 1$, is the unit circle $r = 1$, while the α-limit is the origin.*

4 - *for a different kind of example, consider the linear vector field presented in 1.4.2 of Notes V*

$$\begin{cases} \dot{x}_1 = \beta_1 y_1 \\ \dot{y}_1 = -\beta_1 x_1 \\ \dot{x}_2 = \beta_2 y_2 \\ \dot{y}_2 = -\beta_2 x_2 \end{cases}$$

with β_2/β_1 irrational. The discussion therein shows that the ω-limit set of a non-zero point is a two dimensional torus.

Proposition 6 *Let $x \in \Omega$ be such that $\Psi(t, x)$ is defined for all $t \geq 0$.*

i) *$\omega(x) = \omega(\Psi(t, x))$, for all $t \in I_x$;*

ii) *$\omega(x)$ is closed: if $y_n \in \omega(x)$ and $y_n \to y$ then $y \in \omega(x)$;*

iii) *$\omega(x)$ is invariant under the flow: if $y \in \omega(x)$ then $\Psi(t, y) \in \omega(x)$ for all $t \in I_y$.*

If $O^+(x)$ is contained in a compact set $C \subset \Omega$, then $\omega(x)$ is not empty, and

iv) *$\omega(x)$ is compact;*

v) *for all $y \in \omega(x)$, $\Psi(t, y)$ is defined for all $t \in \mathbb{R}$;*

vi) *$\omega(x)$ is connected.*

Exercise 7 *Prove items i) through v) from the proposition.*

Proof. We prove the last item. Suppose that $\omega(x)$ is not connected; then it must be the union of two disjoint closed sets A and B, both contained in C. Let d be the distance between these two sets, ie,

$$d = \inf\{\|x - y\| : x \in A,\ y \in B\};$$

as A and B are compact, because C is, there are even points $a \in A$ and $b \in B$ such that $\|a-b\| = d$ (why?). We may choose an unbounded increasing sequence t_n of time values such that

$$\|\Psi(t_{2k}, x) - a\| < d/2 \ \text{ and } \|\Psi(t_{2k+1}, x) - b\| < d/2,\ \forall k.$$

But then, by continuity, there exists a sequence s_k such that

$$t_{2k} < s_k < t_{2k+1} \ \text{ and } \|\Psi(s_k, x) - a\| = \|\Psi(s_k, x) - b\| = d/2;$$

because we are in a compact set C, we may extract a subsequence $\Psi(s_{k_j}, x)$ converging to some c. This c is obviously in $\omega(x)$ but not in $A \cup B$, a contradiction. ■

1.1 The Poincaré-Bendixon Theorem

Appart from the general characterization given in the last proposition, ω-limit sets may be extremely complicated sets. In dimension 2, however, the characterization is far simpler.

Theorem 8 (Poncaré-Bendixon) *Let $F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field with finitely many equilibrium points. For any $x \in \Omega$, $\omega(x)$ is one of the following*

a) the empty set;

b) a equilibrium;

c) a periodic orbit;

d) a union of equilibrium points $p_1, \cdots, p_N$ with orbits γ_m such that $\omega(\gamma_m) = p_i$ and $\alpha(\gamma_m) = p_j$, for some $1 \leq i, j \leq N$.

The proof is derived from a sequence of lemmas, all based on the use of a transversal section (see the statement and proof of the Tubular Flow Theorem in Notes VI). The statements and proofs will be better understood with the help of a schematic drawing.

Lemma 9 *Let S be a transversal section. The intersections of an orbit $O(p)$ with S form a monotonous sequence of points in S, ie, there is a continuous bijection g from S to an open real interval such that, if those intersections happen for times*

$$t_1 < t_2 < \cdots$$

then

$$g(\Psi(t_i, p)) < g(\Psi(t_{i+1}, p))$$

for all i.

Proof. It is enough to discuss the situation for three consecutive intersections p_1, p_2, p_3 of $O(p)$ with S. Suppose that, with respect to a fixed orientation on S, p_2 is to the right of p_1; the region bounded by the segment of orbit joining p_1 and p_2 and the segment of S between these two points is positively invariant; so the next intersection p_3 must be to the right of p_2. ■

Lemma 10 *Suppose that $q \in \omega(x)$ and S is a transversal section containing q. Then there exists an unbounded increasing sequence t_n such that $\Psi(t_n, x) \in S$ and $\lim_{n\to+\infty} \Psi(t_n, x) = q$.*

Proof. Let V be an open neighborhood of q as in the statement of the Tubular Flow Theorem: There exists then an open neighborhood V of x_0, a $\varepsilon > 0$, an open ball $B \subset U$ centered at 0, and a diffeomorphism

$$h : V \to]-\varepsilon, \varepsilon[\times B$$

such that $h(S) = \{0\} \times B$ and $h(\Psi(t, x)) = h(x) + (t, 0)$; in particular, each connected component of $O(x) \cap V$ is mapped bijectively by h to a straight "horizontal" segment.
Notice that this implies that there exists a function $\tau : V \to]-\varepsilon, \varepsilon[$ such that $\Psi(\tau(x), x) \in S$.
We may assume we have an unbounded increasing sequence s_n, such that $x_n = \Psi(s_n, x) \in V$. But then $\Psi(\tau(x_n), x_n) \in S$, ie, the sequence $t_n = s_n + \tau(x_n)$ satisfies the condition stated. ■

Lemma 11 *Given a transversal section S , $\omega(x)$ intersects S in at most one point.*

Proof. Suppose $\omega(x) \cap S$ contains two points p and q and that, with respect to a fixed orientation in S, q is to the left of p. By Lemma 10, the orbit of x intersects S in a sequence of points converging to p and also in a sequence of points converging to q; but, by Lemma 9, the intersections of the orbit of x with S constitute a monotonous sequence (from left to right, say). This leads to a contradiction: if $\Psi(s, x) \in S$ is between p and q, all further intersections of the orbit with S would be to the right of this one and so there can not be a sequence $\Psi(t_n, x) \cap S$ converging to p. ■

Lemma 12 *Suppose* $p \in \omega(x)$*. If either* $\omega(p)$ *or* $\alpha(p)$ *contains a regular point then the orbit* $O(p)$ *is periodic and* $\omega(x) = O(p)$*.*

Proof. We do the proof for the case when $\omega(p)$ contains a regular point, as the other case is identic. Let $q \in \omega(p)$ be a regular point, and S a transversal section containing q. By Lemma 10, the orbit of p intersects S in a sequence of points $\Psi(t_n, p)$ converging to q. But all these are also in $\omega(x)$, as ω-limit sets are invariant; by Lemma 11, there may be only one point in $\omega(x) \cap S$; so, all $\Psi(t_n, p)$ must coincide with q and $O(p)$ is thus periodic.
We now show that $\omega(x) = O(p)$. For each $q \in O(p)$ there is a transversal section S containing q and a neighborhood V_q as in the proof of Lemma 10; for any point $z \in V_q \cap \omega(x)$ there is a t such that $\Psi(t, z) \in S$; but $\Psi(t, z) \in \omega(x)$ also, so, by Lemma 11, we must have $\Psi(t, z) = q$, ie, $z \in O(p)$.
We conclude that the intersection of the open set $\cup_{q \in O(p)} V_q$ with $\omega(x)$ is $O(p)$ itself. As $\omega(x)$ is connected, it must coincide with $O(p)$. ■

Proof. (of the Poincaré-Bendixon Theorem) Suppose that $\omega(x)$ is not empty. If it contains no regular points, as F has finitely many equilibrium points and $\omega(x)$ is a connected set, it must consist of a unique equilibrium (case b) of the theorem).
If $\omega(x)$ does not contain any singular points, then by Lemma 12 $\omega(x)$ is a periodic orbit (case c) of the theorem).
Finally, suppose $\omega(x)$ contains both singular and regular points. If p is a regular point in $\omega(x)$, then, by Lemma 12 again, its α and ω-limit sets must consist entirely of singular points (otherwise $\omega(x)$ would consist of a periodic orbit) and so must be each one a unique singular point. We are thus in case d) of the theorem. ■

Poincaré-Bendixon Theorem has some usefull corollaries. The first one is an immediate consequence:

Corollary 13 *If* $C \subset \Omega$ *is compact, positively invariant, and contains no equilibrium points, then it contains at least one periodic orbit.*

Example 14 *Consider*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + y(1 - x^2 - 2y^2) \end{cases}$$

It is easy to see that $(0, 0)$ *is the only equilibrium. If* $f(x, y) = \frac{x^2+y^2}{2}$*, a computation shows that*

$$\partial_t f(x, y) = y^2(1 - x^2 - 2y^2);$$

if C_1 is a circle $f(x,y) = c_1$ contained in the interior of

$$E = \{(y) : x^2 + 2y^2 \leq 1\}$$

(take any $c_1 < 1/4$), and C_2 is a circle $f(x,y) = c_2$ contained in the interior of $\mathbb{R}^2 \setminus E$ (take $c_2 > 1/2$), we have that the region

$$C = \{(x,y) : c_1 \leq f(x,y) \leq c_2\}$$

is clearly compact; let's confirm that it is positively invariant: if not there exists a point $(x,y) \in C$ and a $t > 0$ such that

1. *either $\Psi(s,(x,y))$ belongs to the interior of E for all $0 \leq s \leq t$ and $f(\Psi(t,(x,y)) < c_1$,*
2. *or $\Psi(s,(x,y))$ belongs to the interior of $\mathbb{R}^2 \setminus E$ for all $0 \leq s \leq t$ and $f(\Psi(t,(x,y)) > c_2$;*

consider, for instance the first case: the function

$$g : [0,t] \to \mathbb{R},\ g(s) = f(\Psi(s,(x,y)))$$

satisfies

$$0 > g(t) - g(0) = tg'(s^*)$$

by Lagrange's Theorem, but by the above computation $g'(s) \geq 0$ for all $s \in [0,t]$. The other case is similar.
We conclude that the system has a periodic orbit in C.

The following exercise uses the same idea in a more elaborate way:

Exercise 15 *Consider the equation*

$$\begin{cases} \dot{x} = x - y - x(x^2 + \frac{3}{2}y^2) \\ \dot{y} = x + y - y((x^2 + \frac{1}{2}y^2) \end{cases}$$

Show that the flow has at least one periodic orbit.

Corollary 16 *Suppose that the periodic orbit O is the ω-limit of some point $x \notin O$. Then there is an open neighborhood U of x such that $\omega(y) = O$ for every $y \in U$ and the set*

$$\{x \in \Omega : \omega(x) = O\}$$

contains all points sufficiently close to O that are in the same side of O as x.

Proof. We'll assume, with no loss of generality, and just for the sake of clarity, that x is on the "outside" of O. Suppose that S is a transversal section crossing O at a point p; if $\Psi(t_1, x)$ and $\Psi(t_2, x)$ are two consecutive intersections of the orbit of x with S, the region W bounded by the segment of that orbit

$$\{\Psi(t, x) : t_1 \leq t \leq t_2\},$$

the segment of S between those two points, and O, is positively invariant and contains all points sufficiently close to O that are in the same side of O as x.
We may suppose that this region is contained in the open neighborhood of O given by the union $\cup_{q \in O(p)} V_q$, where each V_q is an open set as in the Tubular Flow Theorem. This implies that the region does not contain any singular points; if it contained a periodic orbit different from O, it would have to contain O in its interior; but this is impossible, because $\omega(x) = O$.

Exercise 17 *Justify the statement in the last paragraph.*

So, by the Poincaré-Bendixon theorem, any $w \in W$ must have $\omega(w) = O$, which proves the last part of the corollary. To prove the first part, take for instance a neighborhood U' of $\Psi(t_2 + 1, x)$ contained in W; by continuity, there is a neighborhood U of x such that, for all $y \in U$, $\Psi(t_2 + 1, y) \in U'$, and so $\omega(y) = \omega(\Psi(t_2 + 1, y)) = O$. ■

A periodic orbit that satisfies the condition stated in the last corollary is called a **limit cycle**.

Corollary 18 *If the flow has a periodic orbit such that the region bounded by it is contained in Ω then that region contains an equilibrium.*

Proof. The proof is by contradiction, so we suppose that O is a periodic orbit such that the region $int(O)$ bounded by it is contained in Ω and doesn't contains an equilibrium.
We first notice that a periodic orbit O can not be at the same time the α-limit and the ω-limit of a point x: this is a consequence of Lemma 9.

Consider the (possibly uncountable) set of periodic orbits contained in $\overline{int(O)}$; the areas of the regions bounded by each of these orbits has an infimum a; we may take a sequence of orbits O_i bounding regions $int(O_i)$ such that $area(int(O_i))$ is a decreasing sequence converging to a.
If $x_i \in C_i$, there exists a convergent subsequence with limit, say, y. Then the orbit of y must also be periodic, for otherwise its ω-limit would be a periodic orbit and, by the last corollary, all points sufficiently close to y should have this same periodic orbit as ω-limit; in particular, all x_i, for sufficiently large i, would have the same ω-limit, a contradiction. Also, the region bounded by the orbit of y must have area a.
Now let z be in the region bounded by the orbit of y; because we assume there

are no equilibrium points, both the α and ω-limit of z must be periodic orbits, and one of them will be strictly contained in the region bounded by the orbit of y and bound a region with area smaller than a, which is impossible.

■

We give also another similar proof, which is maybe more elegant but makes use of some deep concepts.
We consider again the set Γ of all periodic orbits contained in $\overline{int(O)}$. This set may be partially ordered by the order relation

$$O_1 \prec O_2 \Leftrightarrow O_2 \subset int(O_1).$$

We show that if T is a chain, ie, a totally ordered subset of $Gamma$, then there exists an upper bound of T: let $C = \cap \overline{int(O_s)}$ where the intersection is taken over all $O_s \in T$; C is a compact set and, in particular nonempty.
Suppose $q \in C$; from the assumption that there are no equilibrium points in $int(O)$, $\omega(q)$ is a periodic orbit O_l which must be contained in $int(O_s)$ for all $O_s \in T$; so, O_l is an upper bound for T.
An ordered set where any chain has an upper bound is called **inductive** and a fundamental result, Zorn's Lemma, states that any nonempty inductive set has a maximal element. In our case, a maximal element μ of Γ is a periodic orbit contained in $int(O)$ such that $int(\mu)$ does not contain a periodic orbit. But, again under the assumption that there are no equilibrium points in $int(O)$, and so also not in $int(\mu)$, this contradicts the Poincaré-Bendixon Theorem.

Exercise 19 *Prove that the system*

$$\begin{cases} \dot{x} = 2x - x^5 - xy^4 \\ \dot{y} = y - y^3 - yx^2 \end{cases}$$

has no periodic orbits.
Hint: show that all equilibrium points are contained in the coordinate axes and study the restriction of the flow to these axes.