

HEARING THE WEIGHTS OF WEIGHTED PROJECTIVE PLANES

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ABSTRACT. Which properties of an orbifold can we “hear,” i.e., which topological and geometric properties of an orbifold are determined by its Laplace spectrum? We consider this question for a class of four-dimensional Kähler orbifolds: weighted projective planes $M := \mathbb{C}P^2(N_1, N_2, N_3)$ with three isolated singularities. We show that the spectra of the Laplacian acting on 0- and 1-forms on M determine the weights N_1 , N_2 , and N_3 . The proof involves analysis of the heat invariants using several techniques, including localization in equivariant cohomology. We show that we can replace knowledge of the spectrum on 1-forms by knowledge of the Euler characteristic and obtain the same result. Finally, after determining the values of N_1 , N_2 , and N_3 , we can hear whether M is endowed with an extremal Kähler metric.

1. INTRODUCTION

An orbifold consists of a Hausdorff topological space together with an atlas of coordinate charts satisfying certain equivariance conditions (cf. §2.1). We will be interested in compact Kähler orbifolds; in particular, we focus on weighted projective planes, asking what the Laplace spectrum of such a space can tell us about its properties.

For manifolds, the asymptotic expansion of the heat kernel can be used to connect the geometry of the manifold to its spectrum. The so-called *heat invariants* appearing in the asymptotic expansion tell us the dimension, volume, and various quantities involving the curvature of the manifold. We are interested in the Kähler setting; using the heat invariants, results have been obtained on spectral determination of complex projective spaces [5], Einstein manifolds [24, 30], and general Kähler manifolds [29, 30]. These results are all for smooth manifolds, while we will be interested in the effects of the presence of singularities.

Orbifolds began appearing sporadically in the spectral theory literature in the early 1990’s, and have been receiving more dedicated attention in the last five years. It is known that the volume and dimension of an orbifold are spectrally determined [15]. The nature of the singularities allowed to appear in an isospectral set has been studied; in general, there can be at most finitely many isotropy types (up to isomorphism) in a set of isospectral Riemannian orbifolds that share a uniform lower bound on Ricci curvature [28]. On the other hand, N. Shams, E. Stanhope and D. Webb [27] constructed arbitrarily large (but always finite) isospectral sets

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which satisfy this curvature condition, where each element in a given set has points of distinct isotropy. One interesting question is whether the spectrum determines the orders of the singular points for large classes of orbifolds; we show that this is the case for weighted projective planes. In particular, we prove (see §6)

Theorem 1. *Let $M := \mathbb{C}P^2(N_1, N_2, N_3)$ be a four-dimensional weighted projective space with isolated singularities, equipped with a Kähler metric. Then the spectra of the Laplacian acting on 0- and 1-forms on M determine the weights N_1, N_2, N_3 .*

Note that we need to consider the spectrum of the Laplacian acting on both 0- and 1-forms. We conjecture that the spectrum of the Laplacian acting on 0-forms determines the weights, and have verified this with extensive computer testing. However, we cannot prove Theorem 1 using only this information at this point (cf. Section 6.2)

The proof of Theorem 1 involves introducing new techniques into the analysis of the heat invariants. In particular, we use localization in equivariant cohomology to obtain expressions for topological invariants of $\mathbb{C}P^2(N_1, N_2, N_3)$ in terms of N_1, N_2 , and N_3 . We also use a decomposition of the Riemannian curvature tensor inside the space of Kählerian curvature tensors to exploit the complex structure on our weighted projective spaces. We show that we can replace knowledge of the spectrum on 1-forms by knowledge of the Euler characteristic and obtain the same result. Finally, after determining the values of N_1, N_2 , and N_3 , we can hear whether the metric on $\mathbb{C}P^2(N_1, N_2, N_3)$ is an extremal Kähler metric in the sense of Calabi [10].

The same methods can be used with higher-dimensional weighted projective spaces to obtain conditions on the weights (cf. Remark 2 in §6). However, these conditions do not uniquely determine the weights. To obtain additional information we would need to impose restrictions on the metric and to use higher-order terms of the asymptotic expansion of the heat trace.

The paper is organized as follows. We begin with the necessary background on orbifolds and orbi-bundles, then introduce the localization formula for orbifolds and give the setup necessary to apply it in our case. The decomposition of the Riemannian curvature in terms of Kählerian curvature tensors is presented. Localization is then used to compute various integrals whose values we will need in the proof of Theorem 1. In §4, we use a certain polytope associated to a given symplectic toric orbifold to give a description of extremal metrics on weighted projective planes, and to calculate the integral of the square of the scalar curvature on a weighted projective plane. In §5, we recall the asymptotic expansion of the heat trace of an orbifold as given in [14], and we calculate the first few heat invariants for $\mathbb{C}P^2(N_1, N_2, N_3)$. Finally, we bring all of these tools together to prove Theorem 1 and related results in §6.

2. PRELIMINARIES

2.1. Orbifolds and Orbi-bundles. An *orbifold* M is a singular manifold whose singularities are locally isomorphic to quotient singularities of the form \mathbb{R}^n/Γ , where Γ is a finite subgroup of $GL(n, \mathbb{R})$. An *orbifold chart* on M is a triple (U, Γ, V) consisting of an open subset U of M , a finite group $\Gamma \subset GL(n, \mathbb{R})$, an open subset V of \mathbb{R}^n and a homeomorphism $U = V/\Gamma$. An orbifold structure on the paracompact Hausdorff space $|M|$ is then a collection of orbifold charts $\{(U_i, \Gamma_i, V_i)\}$ covering

$|M|$, subject to appropriate compatibility conditions. In particular, these conditions ensure that on each connected component of M , the generic stabilizers of the Γ_i -actions are isomorphic. For each singularity $p \in M$, there is a finite subgroup Γ_p of $GL(n, \mathbb{R})$, unique up to conjugation, such that open neighborhoods of p in M are homeomorphic to neighborhoods of the origin in \mathbb{R}^n/Γ_p . We call Γ_p the *orbifold structure group* of p .

Example 1. Let $\mathbf{N} = (N_1, \dots, N_{m+1})$ be a vector of positive integers which are pairwise relatively prime. The weighted projective space

$$\mathbb{C}P^m(\mathbf{N}) := \mathbb{C}P^m(N_1, \dots, N_{m+1}) := (\mathbb{C}^{m+1})^*/\sim,$$

where

$$((z_1, \dots, z_{m+1}) \sim (\lambda^{N_1} z_1, \dots, \lambda^{N_{m+1}} z_{m+1}), \lambda \in \mathbb{C}^*),$$

is a compact orbifold. It has $m + 1$ isolated singularities at the points $[1 : 0 : \dots : 0], \dots, [0 : \dots : 0 : 1]$, with orbifold structure groups $\mathbb{Z}_{N_1}, \dots, \mathbb{Z}_{N_{m+1}}$. Note that $\mathbb{C}P^m(\mathbf{1})$ is the usual smooth projective space $\mathbb{C}P^m$.

The definitions of vector fields, differential forms, metrics, group actions, etc. naturally extend to orbifolds. For instance, a *symplectic orbifold* is an orbifold equipped with a closed non-degenerate 2-form ω .

Given any orbifold M , *fiber orbi-bundles* (or *orbifold bundles*) $\pi : E \rightarrow M$ are defined by Γ -equivariant fiber bundles $Z \rightarrow E_V \rightarrow V$ in orbifold charts (U, Γ, V) , together with suitable compatibility conditions. Each fiber $\pi^{-1}(p)$ is not, in general, diffeomorphic to Z , but to some quotient of Z by the action of the orbifold structure group Γ_p .

Line orbi-bundles $\pi : L \rightarrow \Sigma$ over a 2-dimensional orbifold Σ (also called an *orbi-surface*) of genus g with k cone singularities with orders N_1, \dots, N_k , correspond to Seifert fibrations (3-dimensional manifolds together with an S^1 action with finite stabilizers [26, 25, 16]) when we take the corresponding circle bundles. Hence, to each line orbi-bundle we can associate a collection of integers (b, m_1, \dots, m_k) , called the *Seifert invariant* of L over Σ , as well as its *Chern number* or *degree* defined by the formula $\deg(L) := b + \sum_{i=1}^k \frac{m_i}{N_i}$. Let us briefly recall how to obtain this invariant (see for example [22] and [16] for a detailed construction). If x_1, \dots, x_k are the orbifold singularities of Σ and N_1, \dots, N_k are the orders of the corresponding orbifold structure groups, then a neighborhood of x_i in L is of the form $(D^2 \times \mathbb{C})/\mathbb{Z}_{N_i}$, where D^2 is a 2-disk and where \mathbb{Z}_{N_i} acts on $D^2 \times \mathbb{C}$ by

$$\xi_{N_i} \cdot (z, w) = (\xi_{N_i} z, \xi_{N_i}^{m_i} w),$$

for some integer $0 \leq m_i < N_i$, and ξ_{N_i} a primitive N_i -th root of unity. Given Σ and its orbifold singularities x_1, \dots, x_k , we define special line orbi-bundles H_{x_i} as follows. Let Σ_{x_i} be the orbi-surface obtained from Σ by deleting a small open neighborhood U_i around x_i , where U_i is isomorphic to D^2/\mathbb{Z}_{N_i} . Take the trivial bundle over Σ_{x_i} , and over U_i take the line orbi-bundle $(D^2 \times \mathbb{C})/\mathbb{Z}_{N_i}$, for a \mathbb{Z}_{N_i} -action given by

$$\xi_{N_i} \cdot (z, w) = (\xi_{N_i} z, \xi_{N_i} w)$$

with gluing map $\alpha : \partial\Sigma_{x_i} \times \mathbb{C} \rightarrow (\partial D^2 \times \mathbb{C})/\mathbb{Z}_{N_i}$,

$$\alpha(e^{i\theta}, w) = (e^{-i\theta}, e^{-i\theta} w).$$

That is,

$$H_{x_i} := (\Sigma_{x_i}^+ \times \mathbb{C}) \bigcup_{\alpha} (D^{2^-} \times \mathbb{C}) / \mathbb{Z}_{N_i},$$

where $\Sigma_{x_i}^+$ is positively oriented and D^{2^-} is negatively oriented. The bundle $L \otimes H_{x_1}^{-m_1} \otimes \dots \otimes H_{x_k}^{-m_k}$, called the *de-singularization* of L , is a trivial line orbi-bundle over each neighborhood of the x_i 's and is naturally isomorphic to a smooth line bundle $|L|$ over the (smooth) surface $|\Sigma|$. Taking the first Chern number b of $|L|$, the collection of integers (b, m_1, \dots, m_k) is then the *Seifert invariant* of L over Σ .

This invariant of an orbifold line bundle classifies it. Indeed, denoting by $\text{Pic}^t(\Sigma)$ the *Picard group* of topological isomorphism classes of line orbi-bundles over Σ (with group operation the tensor product), the map

$$\begin{aligned} \text{Pic}^t(\Sigma) &\longrightarrow \mathbb{Q} \oplus \bigoplus_{i=1}^k \mathbb{Z}_{N_i} \\ L &\mapsto (\deg(L), m_1, \dots, m_k), \end{aligned}$$

is an injection with image the set of tuples (c, m_1, \dots, m_k) with $c = \sum_{i=1}^k m_i / N_i \pmod{\mathbb{Z}}$ (cf. [16]). In particular, if the N_i 's are pairwise relatively prime, then $\text{Pic}^t(\Sigma) \cong \mathbb{Z}$ is generated by a line orbi-bundle L_0 with $\deg(L_0) = \frac{1}{N_1 \dots N_k}$.

An important class of examples of line orbi-bundles are those over weighted projective spaces $\Sigma = \mathbb{C}P^1(p, q)$. Just as any line bundle over a sphere is isomorphic to some line bundle $\mathcal{O}(r)$ with first Chern number $r \in \mathbb{Z}$, defined by $(S^3 \times \mathbb{C}) / \sim$, where $(z_1, z_2, w) \sim (\lambda z_1, \lambda z_2, \lambda^r w)$ for $\lambda \in S^1$, any line orbi-bundle with isolated singularities over an orbifold ‘‘sphere’’ $\mathbb{C}P^1(p, q)$ with p and q relatively prime is isomorphic to some $\mathcal{O}_{p,q}(r) = L_0^r$, defined by $(S^3 \times \mathbb{C}) / \sim$, where now $(z_1, z_2, w) \sim (\lambda^p z_1, \lambda^q z_2, \lambda^r w)$ for $\lambda \in S^1$. Note that $\deg(\mathcal{O}_{p,q}(r)) = \frac{r}{pq}$ and that, if $r > 0$, $\mathcal{O}_{p,q}(r)$ is isomorphic to the normal orbi-bundle of $\mathbb{C}P^1(p, q)$ inside the weighted projective space $\mathbb{C}P^2(p, q, r)$.

2.2. Group actions and equivariant cohomology. Let (M, ω) be a symplectic orbifold, and G a compact, connected Lie group acting smoothly on M . Since G is connected, the components of the fixed point set M^G are suborbifolds of M . Moreover, if G is abelian, the normal orbi-bundle ν_F of each fixed point component F is even dimensional, and admits an invariant Hermitian structure. If M is oriented, any choice of such a Hermitian structure defines an orientation in F . The definition of *equivariant differential forms*, $\mathcal{A}_G(M)$ (that is, polynomial G -equivariant mappings α from the Lie algebra \mathfrak{g} of G to the space of differential forms in M) extends naturally to orbifolds equipped with a group action, as does the equivariant differential $d_G : \mathcal{A}_G(M) \rightarrow \mathcal{A}_G(M)$ defined as

$$d_G(\alpha)(\xi) = d\alpha(\xi) - 2\pi i \iota(\xi_M)\alpha(\xi),$$

where ξ_M is the fundamental vector field corresponding to ξ .

Let us assume now that $G = T$ is abelian, that M is compact, connected and oriented and consider the integration mapping $\int : \mathcal{A}_T(M) \rightarrow \mathcal{A}_T(M)$ and the embeddings $\iota_F : F \rightarrow M$ of the connected components of the fixed point set. The Atiyah-Bott and Berline-Vergne localization formula ([4, 6]), generalized to orbifolds by Meinrenken [21], states that

Proposition 2.1. (*Localization formula for orbifolds*) Suppose $G = T$ is abelian, and let $\alpha \in \mathcal{A}_T(M)$ be d_T -closed. Then

$$\frac{1}{d_M} \int_M \alpha = \sum_F \frac{1}{d_F} \int_F \frac{\iota_F^* \alpha}{e(\nu_F)},$$

where the sum is over the connected components of the fixed point set, where, for a connected orbifold X , d_X is the order of the orbifold structure group of a generic point of X , and where $e(\nu_F)$ is the equivariant Euler class of ν_F , the normal orbibundle to F .

The right hand side of this equation is very simple if M admits an invariant almost complex structure. In this case, the computation of the equivariant Chern classes of ν_F is given by the corresponding *equivariant Chern series* $c^T(\nu_F) := \sum_i t^i c_i^T(\nu_F)$. Moreover (using the splitting principle if necessary) we can assume, without loss of generality, that ν_F splits into a direct sum of invariant line orbibundles L_i with first Chern classes $c_1(L_i)$ where T acts with rational *orbi-weights*¹ λ_i , and we have:

$$(2.1) \quad c_t^T(\nu_F) = \prod_i (1 + t(c_1(L_i) + \lambda_i)).$$

As an example, the *equivariant Euler class* $e(\nu_F)$ is

$$(2.2) \quad e(\nu_F) = \prod_i (c_1(L_i) + \lambda_i)$$

and the *first equivariant Chern class* $c_1^T(\nu_F)$ is

$$(2.3) \quad c_1^T(\nu_F) = \sum_i (c_1(L_i) + \lambda_i).$$

2.3. Circle actions. Let M be a 4-dimensional symplectic orbifold with isolated cone singularities equipped with a Hamiltonian S^1 -action, and let F be a fixed point. A neighborhood of F is modeled by some quotient $\mathbb{C}^2/\mathbb{Z}_N$ for a \mathbb{Z}_N action given by

$$\xi_N \cdot (z, w) = (\xi_N z, \xi_N^m w),$$

with $1 \leq m < N$ and $(m, N) = 1$ (we are assuming that the orbifold singularities are isolated). The circle action on this neighborhood will be given by

$$e^{ix} \cdot (z, w) = (e^{i\frac{k_1}{N}x} z, e^{i\frac{k_2}{N}x} w),$$

for some integers k_1 and k_2 (see [19] for details). If F is an isolated fixed point then the greatest common divisor (k_1, k_2) is equal to 1 or N (we are assuming the action to be effective), and $k_2 = mk_1 \pmod{N}$ (in order to have a well-defined action). The numbers $\frac{k_1}{N}x$ and $\frac{k_2}{N}x$ are the orbi-weights of the action at F and, in these coordinates, the moment map is given by $\phi(z, w) = \phi(F) + \frac{1}{2}(\frac{k_1}{N}|z|^2 + \frac{k_2}{N}|w|^2)$. If F is not an isolated fixed point, then the orbi-weight tangent to the fixed surface containing F is equal to zero while the normal one is equal to $\pm x$ (again for the action to be effective).

¹Given an orbifold chart (U, Γ, V) around a point in F it is not always true that the G -action on U lifts to V but some finite covering $\hat{G} \rightarrow G$ does. The weights for this action of \hat{G} on ν_F are called the *orbi-weights* of G .

Example 2. Consider Example 1 with $n = 2$, i.e., let M be the weighted projective space $\mathbb{C}P^2(N_1, N_2, N_3)$ where the positive integers N_i are pairwise relatively prime, now equipped with the S^1 -action given by

$$e^{ix} \cdot [z_0 : z_1 : z_2] = [z_0 : z_1 : e^{ix} z_2].$$

This action is Hamiltonian with respect to the standard symplectic form ω . Moreover, it fixes the point $F_3 := [0 : 0 : 1]$ as well as the orbi-surface $\Sigma := \mathbb{C}P^1(N_1, N_2)$. A coordinate system centered at F_3 is given by

$$(z, w) \in \mathbb{C}^2 \mapsto [z : w : 1] \in M,$$

with $(z, w) \sim (\xi_{N_3}^{N_1} z, \xi_{N_3}^{N_2} w)$, where ξ_{N_3} is a primitive N_3 -th root of unity. The action of S^1 on this coordinate system is given by

$$e^{ix} \cdot [z : w : 1] = [z : w : e^{ix}] = [e^{-i\frac{N_1}{N_3}x} z : e^{-i\frac{N_2}{N_3}x} w : 1],$$

implying that the weights of the isotropy representation of S^1 on $T_{F_3}M$ are $-\frac{N_1}{N_3}x$ and $-\frac{N_2}{N_3}x$. On the other hand, the orbi-weights at every point in Σ are $(x, 0)$. Note that the degree of the normal orbi-bundle of Σ inside M is $b_\Sigma = \frac{N_3}{N_1 N_2}$.

2.4. Riemann curvature tensor. We now review the well-known decomposition of the Riemann curvature tensor of a Riemannian metric. A detailed exposition for the manifold case can be found for instance in [8, 17, 2]. Since we are interested in weighted projective spaces, which admit a Kähler metric, our goal is to connect this standard decomposition of the Riemannian curvature to its Kählerian decomposition.

In what follows, we extract the necessary background from [2]. The curvature R on a Riemannian orbifold (M, g) is defined, as usual, by

$$R_{X,Y}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

for all vector fields X, Y, Z , where ∇ is the Levi-Civita connection. It is a 2-form with values in the adjoint orbi-bundle AM (the bundle of skew-symmetric endomorphisms of the tangent orbi-bundle) and satisfies the Bianchi identity². Using the metric g we can identify AM with Ω^2M and so R can be viewed as a section of $\Omega^2M \otimes \Omega^2M$. Moreover, it follows from the Bianchi identity that R belongs to the symmetric part $S^2\Omega^2M$ of $\Omega^2M \otimes \Omega^2M$ as well as to the kernel of the map

$$\beta : S^2\Omega^2M \longrightarrow \Omega^4M$$

determined by the wedge product. The kernel of β , $\mathcal{R}M$, is called the orbi-bundle of abstract curvature tensors.

In addition, we have the *Ricci contraction*, a linear map

$$c : \mathcal{R}M \longrightarrow SM,$$

where SM denotes the orbi-bundle of symmetric bilinear forms on M , which sends R to the Ricci form *Ric*.³ Hence, we obtain an orthogonal decomposition

$$\mathcal{R}M = c^*(SM) \oplus WM,$$

where WM , called the orbi-bundle of *abstract Weyl tensors*, is the kernel of c in $\mathcal{R}M$. Consequently, R can be written as $R = c^*(h) + W$, where W is called the

² $R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y = 0$.

³ $Ric_{X,Y} = \text{tr}(Z \mapsto R_{X,Z}Y)$.

Weyl tensor and h is such that $c \circ c^*(h) = Ric$. Since, for $n \geq 3$ the map c is surjective, c^* is injective and h is given by

$$h = \frac{\tau}{2n(n-1)}g + \frac{Ric_0}{n-2},$$

where τ is the scalar curvature (the trace of Ric with respect to g), and where Ric_0 denotes the traceless part of Ric (i.e. $Ric = \frac{\tau}{n}g + Ric_0$). Hence, the curvature R , viewed as a symmetric endomorphism of $\Omega^2 M$ using g , decomposes as

$$(2.4) \quad R = \frac{\tau}{n(n-1)}Id_{|\Omega^2 M} + \frac{1}{n-2}\{Ric_0, \cdot\} + W,$$

where $\{Ric_0, \cdot\}$ acts on $\alpha \in \Omega^2 M$ as the anti-commutator, $\{Ric_0, \alpha\} := Ric_0 \circ \alpha + \alpha \circ Ric_0$. We will use the standard notation

$$U := \frac{\tau}{n(n-1)}Id_{|\Omega^2 M}$$

$$Z := \frac{1}{n-2}\{Ric_0, \cdot\}$$

for the first two terms of this orthogonal decomposition (note that in this notation $|R|^2 = |U|^2 + |Z|^2 + |W|^2$).

Since we are interested in Kähler orbifolds, we can consider an orthogonal complex structure J parallel with respect to the Levi-Civita connection and obtain a Kähler form ω as $\omega(X, Y) = g(JX, Y)$. We can also define the *Ricci form* and its primitive part ρ_0 (that is, such that $(\rho_0, \omega) = 0$) in the same way⁴. Note that this isomorphism $S \mapsto S(J\cdot, \cdot) =: S \circ J$ from the J -invariant part of $S^2 \Omega^2 M$ to $\Omega^{1,1} M$ is not an isometry. In fact, $|S|^2 = 2|S \circ J|^2$ (for example $|g|^2 = n = 2m$ while $|\omega|^2 = m$). Since the Riemannian curvature R has values in $\Omega^{1,1} M$, the J -invariant part of $\Omega^2 M$, it can be viewed as a section of the suborbi-bundle of *abstract Kählerian curvature tensors*, \mathcal{KM} , defined as the intersection of \mathcal{RM} with $\Omega^{1,1} M \otimes \Omega^{1,1} M$. However, in general, none of the components of the decomposition (2.4) is in \mathcal{KM} , and we have a new decomposition of R inside \mathcal{KM}

$$R = \frac{\tau}{2m(m+1)}(Id_{|\Omega^{J,+}} + \omega \otimes \omega) + \frac{1}{m+2}(\{Ric_0, \cdot\}_{|\Omega^{J,+}} + \rho_0 \otimes \omega + \omega \otimes \rho_0) + W^{\mathcal{K}},$$

where $|\Omega^{J,+}$ is the orthogonal projection of $\Omega^2 M$ onto its J -invariant part and $W^{\mathcal{K}}$ is the so-called Bochner tensor.

Alternatively, we have (cf. [8], p. 77)

$$(2.5) \quad R = \frac{\tau}{2m^2}\omega \otimes \omega + \frac{1}{m}\rho_0 \otimes \omega + \frac{1}{m}\omega \otimes \rho_0 + B,$$

where B decomposes as

$$B = \frac{\tau}{2m(m+1)}Id_{|\Omega^{1,1} M} + B_0.$$

Both decompositions (2.4) and (2.5) are related by

$$|B_0|^2 = -\frac{3(m-1)}{m+1}|U|^2 - \frac{m-2}{m}|Z|^2 + |W|^2$$

$$|\rho_0|^2 = (m-1)|Z|^2$$

$$\tau^2 = 4m(2m-1)|U|^2.$$

⁴ $\rho(X, Y) = Ric(JX, Y)$ and $\rho_0(X, Y) = Ric_0(JX, Y)$

In the particular case where $m = 2$, (2.5) becomes

$$(2.6) \quad R = \frac{\tau}{8}\omega \otimes \omega + \frac{1}{2}\rho_0 \otimes \omega + \frac{1}{2}\omega \otimes \rho_0 + B,$$

with

$$(2.7) \quad |B_0|^2 = |W|^2 - |U|^2$$

$$(2.8) \quad |\rho_0|^2 = |Z|^2$$

$$(2.9) \quad \tau^2 = 24|U|^2.$$

3. TOPOLOGICAL INTEGRALS ON WEIGHTED PROJECTIVE PLANES

In this section we use localization in equivariant cohomology to obtain expressions for several integrals, topological invariants of the weighted projective space $\mathbb{C}P^2(N_1, N_2, N_3)$, in terms of the N_i 's. These will be used in the proof of Theorem 1 in §6.

Let us consider $M := \mathbb{C}P^2(N_1, N_2, N_3)$ and take the Hamiltonian S^1 -action from Example 2. Then Proposition 2.1 becomes

$$(3.1) \quad \int_M \alpha = \frac{N_3 \iota_{F_3}^* \alpha}{N_1 N_2 x^2} + \int_\Sigma \frac{\iota_\Sigma^* \alpha}{b_\Sigma u + x},$$

where: $F_3 = [0 : 0 : 1]$ is the isolated fixed point (at which the S^1 -moment map ϕ takes its maximum value); $\Sigma = \mathbb{C}P^1(N_1, N_2)$ is the orbi-surface on which ϕ takes its minimum value; and b_Σ is the degree of ν_Σ , the normal orbi-bundle of Σ , i.e., $b_\Sigma u = c_1(\nu_\Sigma)$ for a generator $u \in H^2(\Sigma)$ (cf. §2). Here we used the fact that the orbi-weight over a fiber of ν_Σ is equal to x , implying that $c_1^{S^1}(\nu_\Sigma) = b_\Sigma u + x$.

Applying (3.1) to $\alpha = 1$ yields

$$0 = \frac{N_3}{N_1 N_2 x^2} + \int_\Sigma \frac{1}{b_\Sigma u + x} = \frac{N_3}{N_1 N_2 x^2} + \frac{1}{x} \int_\Sigma \sum_{j=0}^{\infty} (-1)^j \left(\frac{b_\Sigma u}{x} \right)^j,$$

and we conclude, as expected, that $b_\Sigma = \frac{N_3}{N_1 N_2}$.

Using (3.1) with the equivariant symplectic form $\omega^\sharp := \omega - \phi x$ (cf. [4]) gives

$$0 = \int_M \omega^\sharp = \frac{N_3 \iota_{F_3}^* \omega^\sharp}{N_1 N_2 x^2} + \int_\Sigma \frac{\iota_\Sigma^* \omega^\sharp}{b_\Sigma u + x} = -\frac{N_3}{N_1 N_2 x} y_{\max} + \int_\Sigma \frac{\omega - y_{\min} x}{b_\Sigma u + x}$$

where y_{\max} and y_{\min} denote the values $\phi(F_3)$ and $\phi(\Sigma)$, i.e., the maximum and minimum values of ϕ . Hence,

$$0 = -\frac{N_3}{N_1 N_2 x} y_{\max} + \int_\Sigma \left(\frac{\omega}{x} - y_{\min} \right) \sum_{j=0}^{\infty} (-1)^j \left(\frac{b_\Sigma u}{x} \right)^j$$

and we have

$$(3.2) \quad \text{area}(\Sigma) = b_\Sigma (y_{\max} - y_{\min}) = \frac{N_3}{N_1 N_2} (y_{\max} - y_{\min}).$$

Moreover, using (3.1) with $\alpha = (\omega^\sharp)^2$ yields

$$\begin{aligned}
\int_M (\omega^\#)^2 &= \frac{N_3 (\iota_F^* \omega^\#)^2}{N_1 N_2 x^2} + \int_\Sigma \frac{(\iota_\Sigma^* \omega^\#)^2}{b_\Sigma u + x} = \frac{N_3}{N_1 N_2} y_{\max}^2 + \int_\Sigma \frac{(\omega - y_{\min} x)^2}{b_\Sigma u + x} \\
&= \frac{N_3}{N_1 N_2} y_{\max}^2 + \frac{1}{x} \int_\Sigma (-2y_{\min} \omega x + y_{\min}^2 x^2) \sum_{j=0}^{\infty} (-1)^j \left(\frac{b_\Sigma u}{x} \right)^j \\
&= \frac{N_3}{N_1 N_2} y_{\max}^2 - 2y_{\min} \text{area}(\Sigma) - y_{\min}^2 b_\Sigma \\
&= \frac{N_3}{N_1 N_2} (y_{\max}^2 - 2y_{\min} (y_{\max} - y_{\min}) - y_{\min}^2) \\
&= \frac{N_3}{N_1 N_2} (y_{\max} - y_{\min})^2.
\end{aligned}$$

On the other hand, since

$$\int_M (\omega^\#)^2 = \int_M (\omega - \phi x)^2 = \int_M \omega^2 = 2 \text{vol}(M),$$

we conclude that

$$(3.3) \quad 2 \text{vol}(M) = \frac{N_3}{N_1 N_2} (y_{\max} - y_{\min})^2.$$

If instead we use (3.1) with $\alpha = c_1^{S^1}(TM) \wedge \omega^\#$, we obtain (cf. §2.3)

$$\begin{aligned}
\int_M c_1^{S^1} \wedge \omega^\# &= \frac{N_1 + N_2}{N_1 N_2} y_{\max} + \int_\Sigma \frac{(c_1(T\Sigma) + b_\Sigma u + x)(\omega - \phi x)}{b_\Sigma u + x} \\
&= \frac{N_1 + N_2}{N_1 N_2} y_{\max} + \int_\Sigma \frac{((\frac{1}{N_1} + \frac{1}{N_2})u + b_\Sigma u + x)(\omega - y_{\min} x)}{b_\Sigma u + x}.
\end{aligned}$$

Here we used the fact that

$$\iota_\Sigma^* c_1^{S^1}(TM) = c_1^{S^1}(T\Sigma) + b_\Sigma u + x$$

and that, for a general orbi-surface Σ with cone singularities,

$$c_1^{S^1}(T\Sigma) = c_1(T\Sigma) = \chi(\Sigma) u = (2 - 2g + \sum_{i=1}^k (\frac{1}{\alpha_i} - 1)) u,$$

where the α_i 's are the orders of orbifold structure groups of the singularities of Σ , g is the genus of the underlying topological surface $|\Sigma|$ and $\chi(\Sigma)$ is the orbifold Euler characteristic of Σ (cf. [16]). Then we have

$$\begin{aligned}
\int_M c_1^{S^1} \wedge \omega^\# &= \frac{N_1 + N_2}{N_1 N_2} y_{\max} + \int_\Sigma \omega - \frac{N_1 + N_2}{N_1 N_2} \int_\Sigma \frac{y_{\min} x u}{b_\Sigma u + x} = \\
&= \frac{N_1 + N_2}{N_1 N_2} y_{\max} + \text{area}(\Sigma) - y_{\min} \frac{N_1 + N_2}{N_1 N_2} \int_\Sigma \sum_{j=0}^{\infty} (-1)^j \left(\frac{b_\Sigma}{x} \right)^j u^{j+1} \\
&= \frac{N_1 + N_2}{N_1 N_2} (y_{\max} - y_{\min}) + \text{area}(\Sigma) = \frac{N_1 + N_2 + N_3}{N_1 N_2} (y_{\max} - y_{\min}).
\end{aligned}$$

On the other hand, since $c_1^{S^1}(TM) = c_1(TM) + fx$ where c_1 is the ordinary first Chern class and f is a global function on M (cf. [4]), we have, for dimensional

reasons,

$$\int_M c_1^{S^1} \wedge \omega^\sharp = \int_M c_1 \wedge \omega,$$

and, using (3.3), we obtain

$$(3.4) \quad \int_M c_1 \wedge \omega = \frac{N_1 + N_2 + N_3}{N_1 N_2} (y_{\max} - y_{\min}) = \frac{N_1 + N_2 + N_3}{\sqrt{N_1 N_2 N_3}} \sqrt{2 \operatorname{vol}(M)}.$$

Applying again (3.1) now to $\alpha = (c_1^{S^1})^2(TM)$ yields

$$\begin{aligned} \int_M (c_1^{S^1})^2 &= \frac{(N_1 + N_2)^2}{N_1 N_2 N_3} + \int_\Sigma \frac{(c_1(T\Sigma) + b_\Sigma u + x)^2}{b_\Sigma u + x} \\ &= \frac{(N_1 + N_2)^2}{N_1 N_2 N_3} + \int_\Sigma \frac{((\frac{1}{N_1} + \frac{1}{N_2})u + b_\Sigma u + x)^2}{b_\Sigma u + x} \\ &= \frac{(N_1 + N_2)^2}{N_1 N_2 N_3} + 2 \left(\frac{1}{N_1} + \frac{1}{N_2} \right) + \int_\Sigma b_\Sigma u + x \\ &= \frac{(N_1 + N_2 + N_3)^2}{N_1 N_2 N_3}. \end{aligned}$$

Moreover, since we also have $\int_M (c_1^{S^1})^2 = \int_M c_1^2$, we conclude that

$$(3.5) \quad \int_M c_1^2 = \frac{(N_1 + N_2 + N_3)^2}{N_1 N_2 N_3}.$$

Finally, using (3.1) with $\alpha = c_2^{S^1}(TM)$, we get

$$\begin{aligned} \int_M c_2^{S^1} &= \frac{1}{N_3} + \int_\Sigma \frac{c_1(T\Sigma) \cdot (b_\Sigma u + x)}{b_\Sigma u + x} \\ &= \frac{1}{N_3} + \int_\Sigma \frac{(\frac{1}{N_1} + \frac{1}{N_2})u \cdot (b_\Sigma u + x)}{b_\Sigma u + x} \\ &= \frac{1}{N_3} + \frac{1}{N_1} + \frac{1}{N_2}. \end{aligned}$$

Here we used the fact that

$$\iota_\Sigma^* c_2^{S^1}(TM) = \iota_\Sigma^* e(TM) = e(T\Sigma) \cdot e(\nu_\Sigma) = c_1^{S^1}(T\Sigma) \cdot (b_\Sigma u + x).$$

Since, on the other hand, $\int_M c_2^{S^1} = \int_M c_2$, we obtain

$$(3.6) \quad \int_M c_2 = \frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3}.$$

4. SCALAR CURVATURE OF EXTREMAL METRICS ON WEIGHTED PROJECTIVE PLANES

In this section we follow [1] to give a convenient description of the scalar curvature of extremal Kähler metrics on weighted projective planes, using toric geometry. We compute in particular the square of its L^2 -norm on $\mathbb{C}P^2(N_1, N_2, N_3)$, which is determined by the weights N_1, N_2, N_3 .

4.1. Weighted and labeled projective spaces. The weighted projective space $\mathbb{C}P^m(\mathbf{N})$ described in Example 1 is in fact a compact *complex* orbifold. Consider the finite group $\Gamma_{\mathbf{N}}$ defined by

$$\Gamma_{\mathbf{N}} = \left(\mathbb{Z}_{\hat{N}_1} \times \cdots \times \mathbb{Z}_{\hat{N}_{m+1}} \right) / \mathbb{Z}_{\hat{N}},$$

where

$$\hat{N}_r = \prod_{k=1, k \neq r}^{m+1} N_k, \quad \hat{N} = \prod_{k=1}^{m+1} N_k$$

and

$$\begin{aligned} \mathbb{Z}_{\hat{N}} &\hookrightarrow \mathbb{Z}_{\hat{N}_1} \times \cdots \times \mathbb{Z}_{\hat{N}_{m+1}} \\ \zeta &\mapsto (\zeta^{N_1}, \dots, \zeta^{N_{m+1}}) \end{aligned}$$

($\mathbb{Z}_q \equiv \mathbb{Z}/q\mathbb{Z}$ is identified with the group of q -th roots of unity in \mathbb{C}). $\Gamma_{\mathbf{N}}$ acts on $\mathbb{C}P^m(\mathbf{N})$ via

$$[\eta] \cdot [z] = [\eta_1 z_1 : \cdots : \eta_{m+1} z_{m+1}], \text{ for all } [\eta] \in \Gamma_{\mathbf{N}}, [z] \in \mathbb{C}P^m(\mathbf{N}),$$

and we define the labeled projective space $\mathbb{C}P^m[\mathbf{N}]$ as the quotient

$$\mathbb{C}P^m[\mathbf{N}] := \mathbb{C}P^m(\mathbf{N}) / \Gamma_{\mathbf{N}},$$

and denote its points by $[[z_1 : \cdots : z_{m+1}]] \in \mathbb{C}P^m[\mathbf{N}]$.

By definition, the induced orbifold structure on $\mathbb{C}P^m[\mathbf{N}]$ is such that the quotient map

$$\pi_{\mathbf{N}} : \mathbb{C}P^m(\mathbf{N}) \longrightarrow \mathbb{C}P^m[\mathbf{N}]$$

is an orbifold covering map. This means that any orbifold geometric structure on $\mathbb{C}P^m[\mathbf{N}]$ (e.g., symplectic, complex, Kähler, etc) lifts through $\pi_{\mathbf{N}}$ to a $\Gamma_{\mathbf{N}}$ -invariant orbifold geometric structure on $\mathbb{C}P^m(\mathbf{N})$.

The action of $\Gamma_{\mathbf{N}}$ is free on

$$\check{\mathbb{C}}P^m(\mathbf{N}) := \{[z_1 : \cdots : z_{m+1}] \in \mathbb{C}P^m(\mathbf{N}) : z_k \neq 0 \text{ for all } k\}.$$

In particular, $\pi_{\mathbf{N}}$ has degree $|\Gamma_{\mathbf{N}}| = (\hat{N})^{m-1}$ and

$$\check{\mathbb{C}}P^m[\mathbf{N}] := \{[[z_1 : \cdots : z_{m+1}]] \in \mathbb{C}P^m[\mathbf{N}] : z_k \neq 0 \text{ for all } k\}$$

is an open dense smooth subset of $\mathbb{C}P^m[\mathbf{N}]$. On the other hand, one can check (see [1]) that the orbifold structure group of any point in

$$\mathbb{C}P^m[\mathbf{N}]_r := \{[[z_1 : \cdots : z_{m+1}]] \in \mathbb{C}P^m[\mathbf{N}] : z_r = 0 \text{ and } z_k \neq 0 \text{ for all } k \neq r\}$$

is isomorphic to $\mathbb{Z}_{\hat{N}_r}$.

Both $\mathbb{C}P^m(\mathbf{N})$ and $\mathbb{C}P^m[\mathbf{N}]$ are examples of compact Kähler toric orbifolds, i.e., Kähler orbifolds of real dimension $n = 2m$ equipped with an effective holomorphic and Hamiltonian action of the standard real m -torus $\mathbb{T}^m = \mathbb{R}^m / 2\pi\mathbb{Z}^m$. In fact, the standard Kähler structure and \mathbb{T}^{m+1} -action on \mathbb{C}^{m+1} induce a suitable Kähler structure and \mathbb{T}^m -action on these quotients (see [1]). For the labeled projective space $\mathbb{C}P^m[\mathbf{N}]$ the moment map of the Hamiltonian \mathbb{T}^m -action, $\phi : \mathbb{C}P^m[\mathbf{N}] \longrightarrow (\mathbb{R}^m)^*$, has image the simplex $P_{\lambda}^m \subset (\mathbb{R}^m)^*$ defined by

$$P_{\lambda}^m = \bigcap_{r=1}^{m+1} \{x \in (\mathbb{R}^m)^* : \ell_r(x) := \langle x, \mu_r \rangle - \lambda \geq 0\},$$

where: $\lambda \in \mathbb{R}^+$, $\mu_r = e_r$, $r = 1, \dots, m$, $\mu_{m+1} = -\sum_{j=1}^m e_j$ and (e_1, \dots, e_m) denotes the standard basis of \mathbb{R}^m (see Figure 1 for the case $m = 2$). The positive real number $\lambda \in \mathbb{R}^+$ parametrizes the cohomology class of the Kähler form on $\mathbb{C}P^m[\mathbf{N}]$.

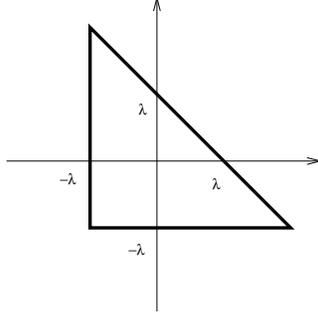


FIGURE 1. The simplex $P_\lambda^2 = \phi(\mathbb{C}P^2[\mathbf{N}])$.

4.2. Scalar curvature of extremal metrics. As shown by Bryant in [9], weighted projective spaces have Bochner-Kähler metrics, i.e., Kähler metrics with vanishing Bochner curvature, which are unique in each Kähler class. These metrics are in particular extremal in the sense of Calabi [10], i.e. minimize the L^2 -norm of the scalar curvature within Kähler metrics in a fixed Kähler class (see also [20]). Moreover, they are always toric and turn out to have a very simple description in terms of explicit data on the simplex P_λ^m (see [1]).

In particular, the scalar curvature τ_λ of these extremal Bochner-Kähler metrics ω_λ , being a \mathbb{T}^m -invariant function on $\mathbb{C}P^m(\mathbf{N})$ and $\mathbb{C}P^m[\mathbf{N}]$, descends to a function on the simplex P_λ^m which is given by

$$(4.1) \quad \tau_\lambda(x) = \frac{1}{\lambda} \left(\frac{n}{m+1} \left(\sum_{r=1}^{m+1} \frac{1}{\hat{N}_r} \right) + \frac{2(m+2)}{m+1} \sum_{j=1}^m \left(\frac{1}{\hat{N}_{m+1}} - \frac{1}{\hat{N}_j} \right) \frac{x_j}{\lambda} \right).$$

Since the orbifold covering map $\pi_{\mathbf{N}} : \mathbb{C}P^m(\mathbf{N}) \rightarrow \mathbb{C}P^m[\mathbf{N}]$ has degree $(\hat{N})^{m-1}$ and the push-forward by the moment map ϕ of the volume form $\mu_\lambda := \omega_\lambda^m / m!$ on $\mathbb{C}P^m[\mathbf{N}]$ is $(2\pi)^m dx$ on P_λ^m , we have that

$$\begin{aligned} \frac{1}{4\pi} \int_{\mathbb{C}P^m(\mathbf{N})} \tau_\lambda \mu_\lambda &= \frac{1}{4\pi} (\hat{N})^{m-1} \int_{\mathbb{C}P^m[\mathbf{N}]} \tau_\lambda \mu_\lambda \\ &= \frac{1}{4\pi} (\hat{N})^{m-1} (2\pi)^m \int_{P_\lambda^m} \tau_\lambda(x) dx. \end{aligned}$$

Using the fact that

$$\int_{P_\lambda^m} x_j dx = 0, \quad \forall \lambda \in \mathbb{R}^+, j = 1, \dots, m,$$

we then have

$$\begin{aligned}
\frac{1}{4\pi} \int_{\mathbb{C}P^m(\mathbf{N})} \tau_\lambda \mu_\lambda &= (2\pi)^{m-1} (\hat{N})^{m-1} \frac{m}{\lambda(m+1)} \left(\sum_{r=1}^{m+1} \frac{1}{\hat{N}_r} \right) \text{vol}(P_\lambda^m) \\
&= (2\pi)^{m-1} (\hat{N})^{m-2} \frac{m}{m+1} \left(\sum_{r=1}^{m+1} N_r \right) \frac{\text{vol}(P_\lambda^m)}{\lambda} \\
&= (2\pi)^{m-1} (\hat{N})^{m-2} \frac{m}{m+1} \left(\sum_{r=1}^{m+1} N_r \right) \lambda^{m-1} \text{vol}(P_1^m) \\
&= \frac{\text{vol}(P_1^m)^{\frac{1}{m}}}{\hat{N}^{\frac{1}{m}}} \frac{m}{m+1} \left(\sum_{r=1}^{m+1} N_r \right) \left((2\pi\lambda)^m \hat{N}^{m-1} \text{vol}(P_1^m) \right)^{\frac{m-1}{m}} \\
&= \frac{\text{vol}(P_1^m)^{\frac{1}{m}}}{\hat{N}^{\frac{1}{m}}} \frac{m}{m+1} \left(\sum_{r=1}^{m+1} N_r \right) (\text{vol}_\lambda(\mathbb{C}P^m(\mathbf{N})))^{\frac{m-1}{m}}.
\end{aligned}$$

When $m = 2$ we have that $\text{vol}(P_1^2) = 9/2$ and so

$$\int_{\mathbb{C}P^2(\mathbf{N})} c_1 \wedge \omega_\lambda = \frac{1}{4\pi} \int_{\mathbb{C}P^2(\mathbf{N})} \tau_\lambda \mu_\lambda = \frac{N_1 + N_2 + N_3}{\sqrt{N_1 N_2 N_3}} \sqrt{2 \text{vol}_\lambda(\mathbb{C}P^2(\mathbf{N}))},$$

which agrees with (3.4) as expected. Note that our normalization of the scalar curvature is the usual one in Riemannian geometry, which is twice the one often used in Kähler geometry.

When $m = 2$ the integral of the square of the scalar curvature is scale invariant, and so

$$\int_{\mathbb{C}P^2(\mathbf{N})} (\tau_\lambda)^2 \mu_\lambda = \int_{\mathbb{C}P^2(\mathbf{N})} (\tau_1)^2 \mu_1.$$

The scalar curvature can be written in this case as

$$\tau_1(x) = \frac{4}{3N_1 N_2 N_3} ((N_1 + N_2 + N_3) + 2N_3(x_1 + x_2) - 2N_1 x_1 - 2N_2 x_2).$$

Using the fact that

$$\int_{P_1^2} (x_i)^2 dx = \frac{9}{4} \quad \text{and} \quad \int_{P_1^2} x_i x_j dx = -\frac{9}{8}, \quad i \neq j,$$

one gets

$$\begin{aligned}
\int_{\mathbb{C}P^2(\mathbf{N})} (\tau_\lambda)^2 \mu_\lambda &= (N_1 N_2 N_3) (2\pi)^2 \int_{P_1^2} (\tau_1(x))^2 dx \\
&= 96\pi^2 \frac{N_1^2 + N_2^2 + N_3^2}{N_1 N_2 N_3}.
\end{aligned}$$

5. HEAT INVARIANTS

To study the relationship between the geometry and the Laplace spectrum of orbifolds with isolated singularities, we will consider the asymptotic expansion of the heat trace $K(x, x, t)$ as $t \rightarrow 0^+$. The so-called *heat invariants* appearing in this expansion involve geometric quantities such as the dimension, the volume, and the curvature. For good orbifolds (i.e., those arising as global quotients of manifolds), Donnelly [11] proved the existence of the heat kernel and constructed the asymptotic expansion for the heat trace. This work was extended to general orbifolds in [14],

where the expressions obtained also clarify the contributions of the various pieces of the singular set. While [11] and [14] treat the heat trace asymptotics for functions, we will also require the asymptotics for 1-forms. We begin with the function case, extracting the necessary background from [14, §4] and simplifying the expressions when possible to reflect the fact that we are only interested in isolated singularities.

Definition. [11] Let h be an isometry of a Riemannian manifold X and let $\Omega(h)$ denote the set of (isolated) fixed points of h . For $x \in \Omega(h)$, define $A_h(x) := h_* : T_x(X) \rightarrow T_x(X)$; note that $A_h(x)$ is non-singular. Set

$$B_h(x) = (I - A_h(x))^{-1}.$$

Proposition 5.1. [11] *Let (X, g) be a closed Riemannian manifold, let $K(t, x, y)$ be the heat kernel of X , and let h be a nontrivial isometry of X with isolated fixed points. Then $\int_X K(t, x, h(x)) \mu_g(x)$ has an asymptotic expansion as $t \rightarrow 0^+$ of the form*

$$\sum_{x \in \Omega(h)} \sum_{k=0}^{\infty} t^k \tilde{b}_k(h).$$

For $x \in \Omega(h)$, $\tilde{b}_k(h)$ depends only on the germs of h and of the Riemannian metric of X at x .

Donnelly gives explicit formulae for \tilde{b}_0 and \tilde{b}_1 [11, Thm. 5.1] as follows. All indices run from 1 to n , where n is the dimension of the Riemannian manifold X . At each point $x \in \Omega(h)$, choose an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x(X)$. The sign convention on the curvature tensor R of X is chosen so that R_{abab} is the sectional curvature of the 2-plane spanned by e_a and e_b . Set

$$\tau = \sum_{a,b=1}^n R_{abab}$$

and

$$Ric_{ab} = \sum_{c=1}^n R_{acbc}.$$

Thus τ is the scalar curvature and Ric the Ricci tensor of X . Then

$$(5.1) \quad \tilde{b}_0(h) = |\det(B_h(x))|$$

and, summing over repeated indices,

$$(5.2) \quad \tilde{b}_1(h) = |\det(B_h(x))| \left(\frac{1}{6} \tau + \frac{1}{6} Ric_{kk} + \frac{1}{3} R_{iksj} B_{ki} B_{js} + \frac{1}{3} R_{iktj} B_{kt} B_{ji} - R_{kaja} B_{ks} B_{js} \right).$$

We want to see what the analogous expressions are for orbifolds which are not necessarily global quotients. We again restrict to the case of orbifolds which have only isolated singularities and present the appropriately simplified expressions. Let M be such an orbifold, and choose a singularity $x \in M$. Let (U, V, Γ) be an orbifold chart for a neighborhood of x , and let $\gamma \in \Gamma$. Define

$$b_k(\gamma) = \tilde{b}_k(\sigma(\gamma)),$$

where σ is an isomorphism from Γ to the isotropy group of a point in the preimage of x under the homeomorphism given by the orbifold chart. That is, we use the charts to calculate the value of b_k locally. Set

$$I_\gamma := \sum_{k=0}^{\infty} t^k b_k(\gamma),$$

and

$$I_x := \sum_{\gamma \in \Gamma} I_\gamma.$$

Also set

$$I_0 := (4\pi t)^{-\dim(M)/2} \sum_{k=0}^{\infty} a_k(M) t^k$$

where the $a_k(M)$ (which we will usually write simply as a_k) are the familiar heat invariants. In particular, $a_0 = \text{vol}(M)$, $a_1 = \frac{1}{6} \int_M \tau(x) \mu_g$, etc. Note that if M is finitely covered by a Riemannian manifold X , say $M = G \backslash X$, then $a_k(M) = \frac{1}{|G|} a_k(X)$.

We can now give an expression for the asymptotic expansion of the heat trace of an orbifold M with isolated singularities (see [14] for more details).

Theorem 5.1. [14] *Let M be a Riemannian orbifold and let $\lambda_1 \leq \lambda_2 \leq \dots$ be the spectrum of the associated Laplacian acting on smooth functions on M . The heat trace $\sum_{j=1}^{\infty} e^{-\lambda_j t}$ of M is asymptotic as $t \rightarrow 0^+$ to*

$$(5.3) \quad I_0 + \sum_{x \in \Omega} \frac{I_x}{|\Gamma|}$$

where Ω is the set of isolated singularities of M . This asymptotic expansion is of the form

$$(5.4) \quad (4\pi t)^{-\dim(M)/2} \sum_{j=0}^{\infty} c_j t^{\frac{j}{2}}.$$

Using this expression, we calculate the first few terms in the asymptotic expansion of the heat trace of a 4-dimensional Riemannian orbifold M with isolated singularities. Let $x \in M$ be a cone point of order N . For any chart (U, V, Γ) about x , let γ generate Γ ; for $j = 1, \dots, N-1$ we have (cf. §2.3)

$$A_{\gamma^j} = \gamma_*^j = \begin{bmatrix} \cos(\frac{2j\pi}{N}) & -\sin(\frac{2j\pi}{N}) & 0 & 0 \\ \sin(\frac{2j\pi}{N}) & \cos(\frac{2j\pi}{N}) & 0 & 0 \\ 0 & 0 & \cos(\frac{2mj\pi}{N}) & -\sin(\frac{2mj\pi}{N}) \\ 0 & 0 & \sin(\frac{2mj\pi}{N}) & \cos(\frac{2mj\pi}{N}) \end{bmatrix},$$

for some integer $1 \leq m < N$ with $(m, N) = 1$. Thus

$$b_0(\gamma^j) = |\det((I - A_{\gamma^j})^{-1})| = \frac{1}{16 \sin^2(\frac{j\pi}{N}) \sin^2(\frac{mj\pi}{N})}.$$

Hence

$$I_x = \sum_{j=1}^{N-1} \frac{1}{16 \sin^2(\frac{j\pi}{N}) \sin^2(\frac{mj\pi}{N})} + O(t).$$

Now consider our weighted projective space $M := \mathbb{C}P^2(N_1, N_2, N_3)$. Note that

$$\begin{aligned} I_0 &= (4\pi t)^{-2} \sum_{k=0}^{\infty} a_k(M) t^k \\ &= \frac{1}{16\pi^2} (a_0 t^{-2} + a_1 t^{-1} + a_2 + a_3 t + \dots). \end{aligned}$$

By Theorem 5.1, we see that the heat trace of M is asymptotic as $t \rightarrow 0^+$ to

$$\frac{1}{16\pi^2} (a_0 t^{-2} + a_1 t^{-1} + a_2) + T + O(t),$$

where

$$\begin{aligned} T &= \frac{1}{N_1} \sum_{j=1}^{N_1-1} \frac{1}{16 \sin^2(\frac{N_2 j \pi}{N_1}) \sin^2(\frac{N_3 j \pi}{N_1})} + \frac{1}{N_2} \sum_{j=1}^{N_2-1} \frac{1}{16 \sin^2(\frac{N_1 j \pi}{N_2}) \sin^2(\frac{N_3 j \pi}{N_2})} \\ &\quad + \frac{1}{N_3} \sum_{j=1}^{N_3-1} \frac{1}{16 \sin^2(\frac{N_1 j \pi}{N_3}) \sin^2(\frac{N_2 j \pi}{N_3})} \end{aligned}$$

and we have replaced the factor m by the appropriate ratios of the N_i 's (see Example 2). Hence the coefficient of the term of degree -2 is $\frac{a_0}{16\pi^2}$, the coefficient of the term of degree -1 is $\frac{a_1}{16\pi^2}$, and that of the term of degree 0 is $\frac{a_2}{16\pi^2} + T$. This means that the spectrum determines

$$\begin{aligned} a_0 &= \text{vol}(M) \\ a_1 &= \frac{1}{6} \int_M \tau(x) \mu_g \end{aligned}$$

and

$$\frac{1}{360 * 16\pi^2} \int_M (2\|R\|^2 - 2|Ric|^2 + 5\tau^2) \mu_g + T.$$

Remark 1. Here, the norm $\|\cdot\|$ is obtained by contracting the tensor with itself, while the norm $|\cdot|$ used in [8] and in §2.4 is the usual norm for antisymmetric products of symmetric tensors. Therefore, $\|T \wedge S\| = 2|T \wedge S|$ for any pair of rank-2 symmetric tensors and $\|T\| = |T|$. In particular, $\|R\| = 2|R|$, while $\|Ric\| = |Ric|$.

To complete our study of the heat trace asymptotics, we briefly discuss the case of forms. As Donnelly notes in [12], the results therein “generalize easily to the Laplacian with coefficients in a bundle.” Indeed, by examining [11, 13], we see that the appropriate coefficient $k(p)$ on the singular part of the heat expansion for forms is $\binom{n}{p}$, where n is the dimension of the orbifold M and p is the level of form. On the smooth part of the expansion, Theorem 3.7.1 of [18] gives a coefficient of $k(p) = \binom{n}{p}$ on a_0 , a coefficient of $k_0(p) = \binom{n}{p} - 6\binom{n-2}{p-1}$ on a_1 , and coefficients on the curvature terms appearing in a_2 of $k_1(p) = 2\binom{n}{p} - 30\binom{n-2}{p-1} + 180\binom{n-4}{p-2}$, $k_2(p) = -2\binom{n}{p} + 180\binom{n-2}{p-1} - 720\binom{n-4}{p-2}$, $k_3(p) = 5\binom{n}{p} - 60\binom{n-2}{p-1} + 180\binom{n-4}{p-2}$ on $\|R\|^2$, $|Ric|^2$, and τ^2 , respectively (i.e., $a_2 = \frac{1}{360}(4\pi)^{-n/2} \int_M \{k_1(p)\|R\|^2 + k_2(p)|Ric|^2 + k_3(p)\tau^2\} \mu_g$). For $n = 4$, the following table gives the values for these coefficients for $p = 0$ and $p = 1$; we will use these values in computations in §6.

	$k(p)$	$k_0(p)$	$k_1(p)$	$k_2(p)$	$k_3(p)$
$p = 0$	1	1	2	-2	5
$p = 1$	4	-2	-22	172	-40

6. HEARING THE WEIGHTS AND EXTREMAL METRICS

Using the background and tools developed in the preceding sections, we now present and prove several results related to spectral determination of the weights of four-dimensional weighted projective spaces.

6.1. Proof of Theorem 1. Let R be the full curvature of M , let Ric be the full Ricci curvature, and let τ be the scalar curvature. We know from §5 that the spectrum of the form-valued Laplacian determines

$$(6.1) \quad \alpha_0(\Delta_p) = \frac{1}{16\pi^2} k(p) \text{Vol}(M)$$

$$(6.2) \quad \alpha_1(\Delta_p) = \frac{1}{16\pi^2} \frac{1}{6} k_0(p) \int_M \tau \mu_g$$

$$(6.3) \quad \alpha_2(\Delta_p) = \frac{1}{16\pi^2} \frac{1}{360} \int_M (k_1(p)|R|^2 + k_2(p)|Ric|^2 + k_3(p)\tau^2) \mu_g + k(p)T,$$

where T is the trigonometric sum

$$T := \frac{1}{16} \left(\frac{1}{N_1} \sum_{j=1}^{N_1-1} \frac{1}{\sin^2(\frac{N_2 j \pi}{N_1}) \sin^2(\frac{N_3 j \pi}{N_1})} + \frac{1}{N_2} \sum_{j=1}^{N_2-1} \frac{1}{\sin^2(\frac{N_1 j \pi}{N_2}) \sin^2(\frac{N_3 j \pi}{N_2})} + \frac{1}{N_3} \sum_{j=1}^{N_3-1} \frac{1}{\sin^2(\frac{N_1 j \pi}{N_3}) \sin^2(\frac{N_2 j \pi}{N_3})} \right).$$

From decompositions (2.4) and (2.6) and relations (2.7) through (2.9) we have (see also Remark 1)

$$\begin{aligned} k_1(p)|R|^2 + k_2(p)|Ric|^2 + k_3(p)\tau^2 &= \\ &= 4k_1(p)(|U|^2 + |Z|^2 + |W|^2) + k_2(p)|Ric|^2 + k_3(p)\tau^2 \\ &= 4k_1(p)(|U|^2 + |\rho_0|^2 + |W|^2) + k_2(p)(|Ric_0|^2 + \frac{\tau^2}{4}) + k_3(p)\tau^2 \\ &= \left(\frac{k_1(p)}{6} + \frac{k_2(p)}{4} + k_3(p) \right) \tau^2 + 2(2k_1(p) + k_2(p))|\rho_0|^2 + 4k_1(p)(|B_0|^2 + |U|^2) \\ &= \left(\frac{k_1(p)}{3} + \frac{k_2(p)}{4} + k_3(p) \right) \tau^2 + 2(2k_1(p) + k_2(p))|\rho_0|^2 + 4k_1(p)|B_0|^2, \end{aligned}$$

where we used that $Ric = \frac{\text{tr} Ric}{2m} g + Ric_0 = \frac{\tau}{4} g + Ric_0$ (implying that $|Ric|^2 = \frac{\tau^2}{4} + |Ric_0|^2$), and that $2|\rho_0|^2 = |Ric_0|^2$. Moreover, we know that on a complex orbifold of (real) dimension 4,

$$(6.4) \quad 4\pi^2 \int_M c_1^2 = \int_M \left(\frac{\tau^2}{8} - |\rho_0|^2 \right) \mu_g$$

and

$$(6.5) \quad 8\pi^2 \int_M c_2 = \int_M \left(\frac{\tau^2}{12} - |\rho_0|^2 + |B_0|^2 \right) \mu_g$$

(cf. [8, p. 80] for the proof in the manifold case), and so (6.3) becomes

$$\begin{aligned} \mathfrak{a}_2(\Delta_p) &= \frac{1}{360 * 16\pi^2} \left(k_1(p) + \frac{1}{2} k_2(p) + k_3(p) \right) \int_M \tau^2 + \\ &+ \frac{1}{180} k_1(p) \int_M c_2 - \frac{1}{720} (4k_1(p) + k_2(p)) \int_M c_1^2 + k(p) T. \end{aligned}$$

Using the values of $k_i(p)$ for $p = 0, 1$ (cf. Table 5), we obtain

$$(6.6) \quad \begin{bmatrix} \frac{1}{960\pi^2} & \frac{1}{90} & 1 \\ \frac{1}{240\pi^2} & -\frac{11}{90} & 4 \end{bmatrix} \begin{bmatrix} \int_M \tau^2 \\ \int_M c_2 \\ T \end{bmatrix} = \begin{bmatrix} \mathfrak{a}_2(\Delta_0) + \frac{1}{120} \int_M c_1^2 \\ \mathfrak{a}_2(\Delta_1) + \frac{7}{60} \int_M c_1^2 \end{bmatrix},$$

implying that

$$(6.7) \quad \int_M c_2 = -6 \left(\mathfrak{a}_2(\Delta_1) - 4\mathfrak{a}_2(\Delta_0) + \frac{1}{12} \int_M c_1^2 \right).$$

On the other hand, the integral of the scalar curvature is a topological invariant, depending only on the Kähler class represented by ω and on the first Chern class in the following way,

$$\int_M \tau \mu_g = 2\pi \int_M c_1 \wedge \omega.$$

Hence, using for instance $\mathfrak{a}_0(\Delta_0)$ and $\mathfrak{a}_1(\Delta_0)$, we have by (3.4)

$$(6.8) \quad \mathfrak{a}_1(\Delta_0) = \frac{1}{12} \sqrt{2\mathfrak{a}_0(\Delta_0) b},$$

where

$$(6.9) \quad b := \frac{(N_1 + N_2 + N_3)^2}{N_1 N_2 N_3} = \int_M c_1^2$$

(cf. (3.5)). Therefore, we conclude that we can hear b from the heat invariants and then, using (6.7), we can also hear

$$(6.10) \quad c := \int_M c_2 = \frac{N_1 N_2 + N_1 N_3 + N_2 N_3}{N_1 N_2 N_3}$$

(see (3.6)), as well as

$$(6.11) \quad d := \frac{N_1^2 + N_2^2 + N_3^2}{N_1 N_2 N_3} = b - 2c.$$

We will now see that these three numbers b , c , d determine N_1 , N_2 and N_3 . First, we note that in c , the numerator $N_1 N_2 + N_1 N_3 + N_2 N_3$ is relatively prime with the denominator $N_1 N_2 N_3$. Indeed, if these two integers had a common divisor l , this would have to divide one of the N_i 's; let us assume without loss of generality that l divided N_1 ; then l would necessarily divide $N_2 N_3$ (since l would also be a divisor of $N_1 N_2 + N_1 N_3$) which is impossible since the N_i 's are pairwise relatively prime. Knowing this, we conclude that we can also hear $s := N_1 N_2 N_3$, the smallest

integer that multiplied by c produces an integer, and then, multiplying b , c and d by this number, we hear the integers:

$$\begin{aligned} p &= N_1 + N_2 + N_3 \\ q &= N_1^2 + N_2^2 + N_3^2 \\ r &= N_1N_2 + N_1N_3 + N_2N_3. \end{aligned}$$

Consider the monic polynomial with roots N_1, N_2, N_3 :

$$(6.12) \quad (y - N_1)(y - N_2)(y - N_3) = y^3 - py^2 + ry - s.$$

Since we can hear all the coefficients of this polynomial, we conclude that we can hear the roots, namely N_1, N_2, N_3 , up to permutation. Note that (6.12) has a unique triple root if and only if M is a smooth manifold, thus finishing the proof of Theorem 1.

Remark 2. Using the methods of §3 with higher-dimensional weighted projective spaces $M = \mathbb{C}P^m(\mathbf{N})$ we can obtain the values of the topological integrals

$$\begin{aligned} \int_M c_2 \wedge \omega^{m-2} &= (m! \text{Vol}(M))^{\frac{m-2}{m}} \frac{\sum_{i,j=1}^{m+1} N_i N_j}{(N_1 \cdots N_{m+1})^{\frac{2}{m}}}, \\ \int_M c_1^2 \wedge \omega^{m-2} &= (m! \text{Vol}(M))^{\frac{m-2}{m}} \frac{(N_1 + \cdots + N_{m+1})^2}{(N_1 \cdots N_{m+1})^{\frac{2}{m}}}. \end{aligned}$$

Moreover, using $\mathbf{a}_i(\Delta_p)$ for $i, p = 0, 1, 2$, the curvature decompositions of §2.4 and the expressions

$$\begin{aligned} \frac{4\pi^2}{(m-2)!} \int_M c_1^2 \wedge \omega^{m-2} &= \int_M \left(\frac{m-1}{4m} \tau^2 - |\rho_0|^2 \right) \mu_g \\ \frac{8\pi^2}{(m-2)!} \int_M c_2 \wedge \omega^{m-2} &= \int_M \left(\frac{m-1}{4(m+1)} \tau^2 - \frac{2(m-1)}{m} |\rho_0|^2 + |B_0|^2 \right) \mu_g \end{aligned}$$

found for instance in [8], we conclude that we can hear

$$\int_M \tau^2, \quad \int_M c_2 \wedge \omega^{m-2} \quad \text{and} \quad \int_M c_1^2 \wedge \omega^{m-2},$$

implying that we can hear

$$\frac{\sum_{i,j=1}^{m+1} N_i N_j}{(N_1 \cdots N_{m+1})^{\frac{2}{m}}} \quad \text{and} \quad \frac{(N_1 + \cdots + N_{m+1})^m}{N_1 \cdots N_{m+1}}.$$

To obtain more information on the weights from the Laplace spectra, one could impose restrictions on the metric and use higher-order terms of the asymptotic expansion of the heat trace.

6.2. Hearing weights from the function spectrum. As mentioned in §1, we are unable to extract the weights using only the information provided by the first few heat invariants for the 0-form spectrum. The recourse to 1-forms was necessitated by the lack of a closed formula for the value of the trigonometric sum T in terms of the N_i 's. Note that, since

$$\frac{1}{\sin^2 \alpha \sin^2 \beta} = (1 + \cot^2 \alpha)(1 + \cot^2 \beta) = 1 + \cot^2 \alpha + \cot^2 \beta + \cot^2 \alpha \cot^2 \beta$$

and

$$\sum_{j=1}^{N-1} \cot^2 \left(\frac{j\pi}{N} \right) = \frac{1}{3}(N-1)(N-3)$$

(cf. [7]), we have

$$\begin{aligned} (6.13) \quad T &= \frac{1}{16} \left(\frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} + \frac{(N_1-1)(N_1-2)}{3N_1} + \frac{(N_2-1)(N_2-2)}{3N_2} + \frac{(N_3-1)(N_3-2)}{3N_3} \right) \\ &+ \frac{1}{N_1} d(N_1; N_2, N_2, N_3, N_3) + \frac{1}{N_2} d(N_2; N_1, N_1, N_3, N_3) + \frac{1}{N_3} d(N_3; N_1, N_1, N_2, N_2) \\ &= \frac{1}{16} \left(-3 + 3 \left(\frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} \right) + \frac{1}{3}(N_1 + N_2 + N_3) + \frac{1}{N_1} d(N_1; N_2, N_2, N_3, N_3) + \right. \\ &\left. + \frac{1}{N_2} d(N_2; N_1, N_1, N_3, N_3) + \frac{1}{N_3} d(N_3; N_1, N_1, N_2, N_2) \right), \end{aligned}$$

where, for positive integers p_i , $d(p_0; p_1, p_2, p_3, p_4)$ is the higher-dimensional Dedekind sum

$$(6.14) \quad d(p_0; p_1, p_2, p_3, p_4) := \sum_{j=1}^{p_0-1} \left(\prod_{i=1}^4 \cot \left(\frac{j\pi p_i}{p_0} \right) \right)$$

(see for example [31]). Nevertheless, the denominators of each Dedekind sum in (6.13) are known (see [31]) and allow us, in most cases, to determine the product $N_1 N_2 N_3$. Then, using the expression for b given by (6.9), we are able to determine the sum $N_1 + N_2 + N_3$. However, these two values alone are not enough to determine the N_i 's uniquely.

If we assume the weights N_1 , N_2 and N_3 to be prime then we can determine the orders of the singularities using only \mathbf{a}_0 and \mathbf{a}_1 for functions.

Theorem 6.1. *Let $M := \mathbb{C}P^2(N_1, N_2, N_3)$ be a four-dimensional weighted projective space with isolated singularities. Assume the weights N_1 , N_2 and N_3 are all prime. Then the spectrum of the Laplacian acting on functions on M determines the weights.*

Proof. From the values of \mathbf{a}_0 and \mathbf{a}_1 for functions we can determine the value of $b = \frac{(N_1 + N_2 + N_3)^2}{N_1 N_2 N_3}$ in (6.9) using equation (6.8). Note that we can hear the smallest integer which when multiplied by b produces an integer. Call this smallest integer D and note that it is a product of at most three prime numbers.

If D is a product of three primes then they are necessarily equal to N_1 , N_2 and N_3 .

If D is a product of two primes, we immediately know the values of the orders of two of the singularities, say N_1 and N_2 . To obtain the value of the third we first multiply b by the product $N_1 N_2$ to determine the value of

$$l := \frac{(N_1 + N_2 + N_3)^2}{N_3} \in \mathbb{Z};$$

then N_3 necessarily divides $(N_1 + N_2 + N_3)^2$ and consequently divides $N_1 + N_2 + N_3$. Thus

$$l = N_3 p_1^{2m_1} \cdots p_k^{2m_k}$$

for some prime numbers p_1, \dots, p_k and some positive integers m_1, \dots, m_k ; we conclude that we can hear $(p_1^{m_1} \cdots p_k^{m_k})^2$ (the greatest perfect square that divides l) and hence N_3 .

Similarly, if D is a product of just one prime, say N_1 , then we know the value of the order of one singularity. To retrieve the values of the other two we multiply b by N_1 to obtain

$$l := \frac{(N_1 + N_2 + N_3)^2}{N_2 N_3} \in \mathbb{Z}.$$

Since $(N_2, N_3) = 1$ and $l = N_2 N_3 (p_1^{m_1} \cdots p_k^{m_k})^2$ for some prime numbers p_1, \dots, p_k and some positive integers m_1, \dots, m_k , we can hear the value of the product $N_2 N_3$ and consequently of N_2 and N_3 .

Finally, if $D = 1$ (i.e., if b is an integer) then $(N_1 + N_2 + N_3)^2$ is the product of $N_1 N_2 N_3$ by a perfect square and we can again determine $N_1 N_2 N_3$ and from that N_1, N_2 and N_3 . \square

If we remove the restriction that N_1, N_2 and N_3 be prime, we can still determine the orders of the singularities from \mathfrak{a}_0 and \mathfrak{a}_1 for functions if we fix the Euler characteristic of the orbifold. Indeed, if we are given $\chi(M) := \frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3}$, we know the value of c in (6.10) and so, following the argument in the proof of Theorem 1, we are able to determine $s = N_1 N_2 N_3$. Assuming we know \mathfrak{a}_0 and \mathfrak{a}_1 we can again determine the value of b in (6.9), using equation (6.8). Then, knowing b and c allows us to determine $d := \frac{N_1^2 + N_2^2 + N_3^2}{N_1 N_2 N_3} = b - 2c$ and so we again have the integers p, q, r and s from the proof of Theorem 1 which allow us to determine N_1, N_2 and N_3 . In summary, we have the following theorem.

Theorem 6.2. *Let $M := \mathbb{C}P^2(N_1, N_2, N_3)$ be a four-dimensional weighted projective space with isolated singularities. If we fix the Euler characteristic of M (i.e., fix the topological class), then the spectrum of the Laplacian acting on functions on M determines the weights N_1, N_2, N_3 .*

6.3. Hearing extremal metrics. After determining the values of N_1, N_2 and N_3 we can also hear if a certain metric is extremal (cf. §4.2) from the value of \mathfrak{a}_2 .

Theorem 6.3. *Let $M := \mathbb{C}P^2(N_1, N_2, N_3)$ be a four-dimensional weighted projective space with isolated singularities. Suppose that the values of N_1, N_2 , and N_3 are known. Then the value of \mathfrak{a}_2 for the Laplacian acting on functions determines whether M is endowed with the extremal metric.*

Proof. As shown in the proof of Theorem 1 (cf. (6.6)),

$$(6.15) \quad \mathfrak{a}_2(\Delta_0) = \frac{1}{960\pi^2} \int_M \tau^2 + \frac{1}{90} \int_M c_2 - \frac{1}{120} \int_M c_1^2 + T.$$

We have explicit expressions for $\int_M c_2$, $\int_M c_1^2$, and T in terms of N_1, N_2 , and N_3 (see (6.9), (6.10)), so using (6.15) we see that we can hear $\int_M \tau^2$ from the value of \mathfrak{a}_2 . If the metric is extremal, we have seen in §4.2 that

$$(6.16) \quad \int_M \tau^2 = 96\pi^2 \frac{N_1^2 + N_2^2 + N_3^2}{N_1 N_2 N_3}.$$

Hence, by checking the value we hear for $\int_M \tau^2$ against that given by (6.16), we can see whether M is endowed with the extremal metric. Note that we are using the fact that the extremal metric is an absolute minimum for the L^2 -norm of the scalar curvature among Kähler metrics in a fixed Kähler class. \square

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