EQUIVARIANT COHOMOLOGY OF S^1-ACTIONS ON 4-MANIFOLDS

LEONOR GODINHO

ABSTRACT. Let M be a symplectic 4-dimensional manifold equipped with a Hamiltonian circle action with isolated fixed points. We describe a method for computing its integral equivariant cohomology in terms of fixed point data. We give some examples of these computations.

1. INTRODUCTION

It is known that the equivariant cohomology of a symplectic manifold equipped with a Hamiltonian torus action is determined by circle actions on certain submanifolds. Indeed, Goresky, Kottwitz and MacPherson [GKM], using the work of Chang and Skjelbred [CS], showed that the computation of the cohomology rings $H_T^*(M)$ for a torus action of at least dimension two reduces to computation of $H^*_{T/H}(M^H)$ for all codimension-1 subtori H, where M^H denotes the fixed point set of H.

However, not much is known on how to compute the equivariant cohomology ring in the case of a circle action. In this paper, we study the special case of integral equivariant cohomology of a Hamiltonian circle action with isolated fixed points on a compact 4-dimensional manifold M. It is known that, in this case, the inclusion map $i : M^{S^1} \longrightarrow M$ of the fixed point set into M induces an injection $i^* : H^*_{S^1}(M,\mathbb{Z}) \longrightarrow H^*_{S^1}(M^{S^1},\mathbb{Z})$ (see [TW3] and [Ki] for details). We give a very simple combinatorial method of describing the image of i^*.

In the case of a torus action with at least dimension two, Goresky, Kottwitz and MacPherson in [GKM], and Tolman and Weitsman in [TW2], use the one-skeleton, that is, the subspace consisting on the closure of all points whose orbit under the action is one-dimensional, to describe this image. Here, with an S^1-action, we will use what we call the *isotropy skeleton* of the S^1-space, also formed by the fixed points and a set of connecting spheres. These spheres will now be the connected components of the fixed point set of subgroups of the circle different from $\{1\}$ (*isotropy $\mathbb{Z}_k$-spheres*), or *free gradient spheres*, obtained by choosing an invariant almost complex structure and an invariant Riemannian metric and taking the flow of the Hamiltonian function (see Section 3 for a detailed definition). This isotropy skeleton was already used in [H] in a more general context. Indeed, they compute its equivariant cohomology and use it to determine the signature of the manifold. However, they do not relate the equivariant cohomology of the skeleton with that of the manifold. Here we do determine $H^*_{S^1}(M,\mathbb{Z})$ (based only on information contained on the isotropy skeleton).

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Note that, just as in the GKM case [GZ], we can also obtain a graph from this skeleton with vertices the fixed points, pairwise connected by labeled edges whenever they are both in one of the above spheres (the corresponding labels are the orders of the stabilizers of these spheres). These graphs have a very simple shape: there is a unique top vertex and a unique bottom vertex and the edges occur in two branches which connect both extremal vertices (cf. Figure 1).

The equivariant cohomology $H^*_{S^1}(M,\mathbb{Z})$ is an algebra: it is a ring under the cup product and it is a module over the symmetric algebra $\mathbb{S} = S(\mathfrak{s}^*)$ of polynomial functions on the Lie algebra $\mathfrak{s}$ of S^1 ($\mathbb{S} \cong \mathbb{Z}[u]$ is the polynomial ring in a single generator $u \in H^2_{S^1}(pt,\mathbb{Z})$). Using the isotropy skeleton of M, we show that $H^*_{S^1}(M,\mathbb{Z})$ is isomorphic to the following subalgebra of the direct sum of N copies of $\mathbb{S}$, where N is the number of fixed points:

Theorem 1.1. *Let M be a 4-dimensional compact symplectic manifold equipped with a Hamiltonian circle action with finitely many fixed points and let (J,g) be a compatible*[1] *pair of an almost complex structure and an invariant Riemannian metric. Then, the restriction map*

$$H^*_{S^1}(M,\mathbb{Z}) \longrightarrow H^*_{S^1}(M^{S^1},\mathbb{Z}) \cong \bigoplus_{F_i \in M^{S^1}} \mathbb{S}$$

is injective, and its image is the subalgebra

$$\begin{aligned}
\mathcal{A} = \{f \;&=\; (f_0,\, f_1,\dots,f_l,\, f_{l+1},\dots,f_{l+m},\, f_{N-1}) \in \bigoplus_{k=1}^{N} \mathbb{S} :\\
f_1 \;&=\; f_0 - a_1\, p_1\, u; \qquad f_{l+1} = f_0 - b_1\, q_1\, u;\\
f_i \;&=\; f_0 + (\, a_{i+1}\, p_{i-1} - a_i\, p_i\,)\, u,\;\; i = 2,\dots,l;\\
f_{l+j} \;&=\; f_0 + (\, b_{j+1}\, q_{j-1} - b_j\, q_j\,)\, u,\;\; j = 2,\dots m;\\
f_{N-1} \;&=\; f_0 - (b_{m+1}\, p_l + a_{l+1}\, q_m)\, u + a_{l+1}\, b_{m+1}\, p_{N-1}\, u^2,\;\; \textit{for}\\
&\quad \textit{arbitrary polynomials}\;\; f_0,\, p_1 \dots, p_l,\, q_1, \dots q_m, p_{N-1} \in \mathbb{S}\},
\end{aligned}$$

where

(i) *l, m are integers with $l + m + 2 = N$, the number of fixed points;*
(ii) *$(a_{i+1}, -a_i)$, $i = 2,\dots l$, are the isotropy weights of the action on the normal bundles of the index-2 fixed points F_i on one branch of the isotropy skeleton associated to (J,g);*
(iii) *$(b_{j+1}, -b_j)$, $j = 2,\dots m$, are the isotropy weights of the action on the normal bundles of the index-2 fixed points F'_j on the other branch of the isotropy skeleton associated to (J,g);*
(iv) *(a_1, b_1) are the isotropy weights of the action on the normal bundle of the index-0 fixed point;*
(v) *$(-a_{l+1}, -b_{m+1})$ are the isotropy weights of the action on the normal bundle of the index-4 fixed point (cf. Figure 1).*

[1]A pair (J,g) of an S^1-invariant Riemannian metric g and an almost complex structure J is said to be *compatible* if $g(u,v) = \omega(u, Jv)$.

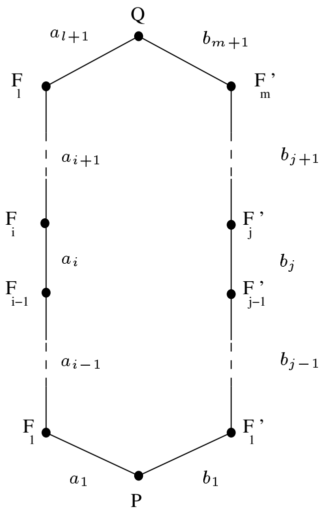

FIGURE 1. Isotropy skeleton of M

Remark. The image of $i^* : H^*_{S^1}(M, \mathbb{C}) \longrightarrow H^*_{s^1}(M^{S^1}, \mathbb{C})$ was already described by Goldin and Holm in [GH] as the set

$$(1.1) \qquad \mathcal{A} = \{(f_1, \ldots, f_N) \in \bigoplus_{i=1}^{N} \mathbb{S} : \; f_i - f_j \in u \cdot \mathbb{C}[u], \; \sum_{i=1}^{N} \frac{f_i}{a_1^i \, a_2^i u^2} \in \mathbb{S}\},$$

where a_1^i, a_2^i are the isotropy weights of the action of S^1 on the normal bundle of the fixed point F_i. An easy example where integral coefficients give more information than complex ones is that of S^1 acting on $S^2 \times S^2$ with speed one on one sphere, and speed $m > 1$ on the other. Theorem 1.1, implies that a class vanishing on the fixed point P of index 0, and on one of the index-2 fixed points, F_2, must be a multiple of mu when restricted to the fixed point of index 4, Q (cf. Figure 5 and Table 1). In contrast, it is easy to check from (1.1), that there is a class on $i^*(H^*_{S^1}(M, \mathbb{C}))$ which is equal to u when restricted to Q while vanishing on P and F_2, and so equivariant complex cohomology does not distinguish between these spaces.

We prove Theorem 1.1 in Section 4 and, on Section 5, we give some examples of the computations allowed by this result.

2. Preliminaries

2.1. **Equivariant Cohomology.** The *equivariant cohomology* of an S^1-manifold is defined as the ordinary cohomology of the space $M_{S^1} = M \times_{S^1} ES^1$, where ES^1 is a contractible space on which S^1 acts freely. The projection $M \times_{S^1} ES^1 \longrightarrow BS^1$, where the *classifying space* BS^1 is equal to ES^1/S^1, induces a pullback map $H^*(BS^1, \mathbb{Z}) \longrightarrow H^*_{S^1}(M, \mathbb{Z})$, making $H^*_{S^1}(M, \mathbb{Z})$ into a module over

$$\mathbb{S} = H^*_{S^1}(pt, \mathbb{Z}) \cong H^*(BS^1, \mathbb{Z}),$$

the polynomial ring in a single generator $u \in H^2(BS^1, \mathbb{Z})$.

Let us now consider an S^1-equivariant line bundle L over a fixed point F. The S^1-action on L is conjugate to some circle action on $\mathbb{C}$, $z \mapsto e^{2\pi i a t} z$, and the first equivariant Chern class of L is given by $c_1(L) = au$. Any S^1-equivariant vector bundle E over F decomposes equivariantly into complex line bundles ($E = L_1 \oplus \cdots \oplus L_m$) on which the circle acts with isotropy weights $a_1, \ldots, a_m$. The *equivariant Euler class* of E is then given by $e(E) = (\prod_{i=1}^m a_i)\, u^m$.

2.2. **Morse Theory.** Kirwan [Ki] proved that, if M is a symplectic manifold equipped with a Hamiltonian circle action, the Hamiltonian function $H : M \longrightarrow \mathbb{R}$ is a perfect Morse function on M. The critical points of this function are the fixed points of the action and the *index* of a critical point F is equal to twice the number of negative isotropy weights of the circle action on the normal bundle of F.

Taking a compatible pair (J, g) of an invariant almost complex structure and an invariant Riemannian metric, we have $\nabla H = -J\xi_M$ where ξ_M is the vector field that generates the S^1 action. Moreover, for a critical point F of H, the *stable manifold*, $W^S(F)$ and the *unstable manifold*, $W^U(F)$ of F are given by

$$\begin{aligned} W^S(F) =& \{p \in M : \lim_{t \to -\infty} \gamma(t) = F\} \\ W^U(F) =& \{p \in M : \lim_{t \to +\infty} \gamma(t) = F\}, \end{aligned}$$

where $\gamma(t)$ is a solution of the gradient flow equation $\gamma'(t) = \nabla H(\gamma(t))$.

3. Isotropy Skeleton

In this section, we consider some facts about gradient spheres and define the isotropy skeleton of a manifold. Let M be a compact symplectic 4-dimensional manifold equipped with an effective circle action with isolated fixed points. As in [K], a *$\mathbb{Z}_k$-sphere*, ($k \geq 2$), is defined as the connected component of the set of points which are fixed by the finite subgroup $\mathbb{Z}_k \subset S^1$. Each of these components is a closed symplectic 2-sphere on which $S^1/\mathbb{Z}_k$ acts effectively (cf. [K] for details).

Let us now choose a compatible pair (J, g) of an almost complex structure and an invariant Riemannian metric. The S^1-action on the underlying almost complex manifold extends to an action of $\mathbb{C}^*$ in the following way ([K], [AH]): identifying $\mathbb{C}^*$ to $S^1 \times \mathbb{R}^+$, and denoting by ξ_M the vector field on M which generates the circle action, we define the action of $\mathbb{R}^+$ to be the flow of the vector field $\eta_M := J\xi_M$ and so, since ξ_M and η_M commute, we have an action of $\mathbb{C}^*$. We will call a *gradient sphere* in M, the closure of a non trivial $\mathbb{C}^*$-orbit. Its poles are the limits at times ∞ and $-\infty$ of the gradient flow (that is, the flow generated by the vector field $-J\xi_M$ which coincides with the gradient flow of the Hamiltonian function with respect to the metric) inside this orbit, which, in turn, are fixed points of the circle action. A *free* gradient sphere is a gradient sphere whose stabilizer is trivial. Every $\mathbb{Z}_k$-sphere is a gradient sphere and every non-free gradient sphere is a $\mathbb{Z}_k$-sphere. Note that different compatible metrics may lead to different arrangements of free gradient spheres. However, non-free gradient spheres ($\mathbb{Z}_k$-spheres) are independent of this choice since they are defined only in terms of the S^1-action.

The (J, g)-*isotropy skeleton* of M is then defined in the following way:

Definition. For (J, g) as above, the (J, g)-isotropy skeleton of M is the set X formed by fixed points and the following set of connecting spheres:

1. if two fixed points are connected by a $\mathbb{Z}_k$-sphere then we include this sphere;
2. if two fixed points are the north and south poles of a unique free gradient sphere then we include this sphere;

(3) if the fixed point of index 0 has $a = 1$ as an isotropy weight and is not the south pole of a $\mathbb{Z}_k$-sphere having the fixed point of index 4 as north pole, nor the south pole of two gradient spheres having index-2 fixed points as north poles, then we take one of the infinite free gradient spheres which connect it to the index-4 fixed point.

Remarks. For each fixed point of index 2, the closure $\overline{W}^S(F)$ of its stable (unstable) manifold is a gradient sphere and any free gradient sphere having as pole the fixed point of index 0 or the fixed point of index 4 is smooth at this point ([AH]-Lemma 4.9). Consequently, every sphere in the isotropy skeleton is smooth at its poles.

The (J, g)-isotropy skeleton X can be represented by a graph Γ_X with edges representing gradient spheres (labeled by the orders of their stabilizers), and vertices representing the fixed points. These graphs have a simple shape: there is a unique top vertex and a unique bottom vertex and the edges occur in two branches which connect both extremal vertices. Indeed, they differ from the *extended graph* defined in [K] only in the fact that they may include an edge labeled 1 connecting the bottom and the top vertices.

Again, the (J, g)-isotropy skeleton of M and the associated graph Γ_X may depend on the choice of (J, g) in the arrangement of free gradient spheres (cf. Example 1. in Section 5). Indeed, it is shown in [K] that if, for some metric, there is a free gradient sphere having two index-2 fixed points as poles, then there is a small perturbation of the (invariant and compatible) metric for which the gradient sphere ascending from the south pole becomes disjoint from the gradient sphere descending from the north pole. These two new gradient spheres connect the index-2 fixed points directly to the fixed points of index 4 and 0 respectively. Since Γ_X only has two branches, this implies that there can be at most one free gradient sphere between index-2 fixed points, in which case Γ_X also has one edge labeled 1 connecting the bottom vertex directly to the top vertex. Moreover, each (J, g)-isotropy skeleton can have at most two additional free gradient spheres per branch, connecting the fixed points of index 0 and 4 to index-2 fixed points. The arrangement of these spheres may depend on the choice of metric only if the smallest value of the Hamiltonian function on the interior vertices (i.e. that correspond to index-2 fixed points) of one branch is greater than the highest value of the Hamiltonian function on the interior vertices of the other branch.

Remarks. We know that any compact 4-dimensional manifold equipped with a Hamiltonian circle action with isolated fixed points can be obtained from either a $\mathbb{C}P^2$ or a Hirzebruch surface by a sequence of symplectic blow-ups at fixed points of index 0 or 2 ([K]). Consequently, any graph Γ_X can be obtained from the graphs in Figure 2 by a sequence of blow-ups where, at each step, the blown-up vertex gives place to two vertices and an edge as depicted in Figure 3.

4. Proof of Theorem 1.1

To prove Theorem 1.1 we need a version of Kirwan's injectivity Theorem [Ki] relating the equivariant integral cohomology of the manifold M with the equivariant integral cohomology of its fixed point set. Note that this result still holds for circle actions with isolated fixed points since the cohomology of the fixed point set has no torsion (see [TW3] for details):

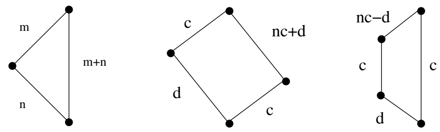

FIGURE 2. Minimal graphs for an isotropy skeleton

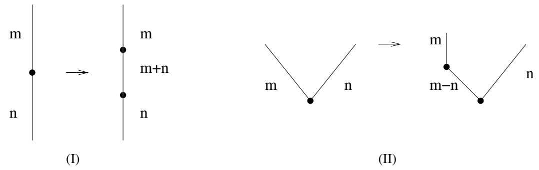

FIGURE 3. Blow up at (I) an index-2 fixed point, (II) a minimum.

Theorem 4.1. *Let (M,ω) be a compact symplectic manifold equipped with a Hamiltonian action of a circle S^1 with isolated fixed points and let M^{S^1} be its fixed point set. Then, the inclusion map $i : M^{S^1} \longrightarrow M$ induces an injection*

$$i^* : H^*_{S^1}(M,\mathbb{Z}) \longrightarrow H^*_{S^1}(M^{S^1},\mathbb{Z}).$$

From the proof of this theorem in [Ki] we can take the following result whose detailed proof can be found in [G]:

Proposition 4.1. *Let M be a symplectic manifold with a Hamiltonian action with isolated fixed points. Let (J,g) be a compatible pair of an invariant almost complex structure and an invariant Riemannian metric. For any $F \in M^{S^1}$ of index $2d$ there exists a class $\alpha_F \in H^{2d}_{S^1}(M,\mathbb{Z})$ that,*

- *(i) restricted to F, is equal to the equivariant Euler class of the negative normal bundle of F, that is, $\alpha_F|_F = \left(\prod_{i=1}^d a_i\right) u^d$, where $a_1,\ldots,a_d$ are the negative isotropy weights of the circle action on the normal bundle of F;*
- *(ii) vanishes when restricted to any other fixed point F' which cannot be joined to F along a sequence of integral lines of the negative gradient field $-\nabla H$ with respect to the metric g (that is, by a sequence of downward spheres in X).*

Moreover, taken together over all fixed points, these classes form a basis for the cohomology $H^*_{S^1}(M,\mathbb{Z})$ as a $H^*(BS^1,\mathbb{Z})$-module. We will call them generating classes.

Finally, we can prove Theorem 1.1:

Proof. For each fixed point F of index 2 we consider the closure $Z_F := \overline{W}^S(F)$ of its stable manifold which we know to be a gradient sphere of the isotropy skeleton. The circle acts on its normal bundle ν_{Z_F} keeping Z_F invariant. Therefore, we can define the equivariant Euler class of ν_{Z_F}, as well as the equivariant Thom class of Z_F, $\alpha_F := \tau_{S^1}(Z_F) \in H^2_{S^1}(M,\mathbb{Z})$. The support of α_F can be shrunk into a neighborhood of Z_F, implying that its restriction to Z_F agrees with $e_1(\nu_{Z_F})$, while its restriction to any other fixed point of M^{S^1} outside Z_F vanishes. Consequently,

$\tau_{S^1}(Z_F)$ is supported on Z_F and its restriction to each fixed point $\tilde{F}$ of Z_F is equal to $\tau_{S^1}(Z_F)|_{\tilde{F}} = au$, where a is the isotropy weight of the circle action on the fiber of the normal bundle of $\tilde{F}$ which is not tangent to Z_F (and u is the generator of $H^*_{S^1}(\tilde{F},\mathbb{Z})$). We conclude that these classes satisfy all the conditions of Proposition 4.1. In addition to the classes α_F, we also consider the class $\alpha_P := 1$ and the class α_Q given by Proposition 4.1 which, restricted to the fixed point of index 4, is equal to the equivariant Euler class of its negative normal bundle, and vanishes on any other fixed point. Taking the classes α_P, α_F (for every index-2 fixed point F) and α_Q, we obtain a basis of $H^*_{S^1}(M,\mathbb{Z})$. Hence, if we have the isotropy skeleton depicted in Figure 1, the image of the restriction map $H^*_{S^1}(M,\mathbb{Z}) \longrightarrow H^*_{S^1}(M^{S^1},\mathbb{Z})$ is the subalgebra $\mathcal{A}$ formed by classes $f = (f_0, f_1, \ldots, f_l, f_{l+1}, \ldots, f_{l+m}, f_{N-1}) \in \bigoplus_{j=1}^N \mathbb{S}$ where

$$\begin{aligned}
f_0 &= p_0\,\alpha_P\,|_P = p_0;\\
f_1 &= p_0\,\alpha_P\,|_{F_1} + p_1\,\alpha_{F_1}\,|_{F_1} = f_0 - a_1\,p_1\,u;\\
f_i &= p_0\,\alpha_P\,|_{F_i} + p_{i-1}\,\alpha_{F_{i-1}}\,|_{F_i} + p_i\,\alpha_{F_i}\,|_{F_i} = f_0 + (a_{i+1}\,p_{i-1} - a_i\,p_i)\,u,\\
f_{l+1} &= p_0\,\alpha_P\,|_{F'_1} + q_1\,\alpha_{F'_1}\,|_{F'_1} = f_0 - b_1\,q_1\,u;\\
f_{l+j} &= p_0\,\alpha_P\,|_{F'_j} + q_{j-1}\,\alpha_{F'_{j-1}}\,|_{F'_j} + q_j\,\alpha_{F'_j}\,|_{F'_j} = f_0 + (b_{j+1}\,q_{j-1} - b_j\,q_j)\,u;\\
f_{N-1} &= p_0\,\alpha_P\,|_Q + p_l\,\alpha_{F_l}\,|_Q + q_m\,\alpha_{F'_m}\,|_Q + p_{N-1}\,\alpha_Q\,|_Q =\\
&= f_0 - (b_{m+1}\,p_l + a_{l+1}\,q_m)\,u + a_{l+1}\,b_{m+1}\,p_{N-1}\,u^2, \text{ for arbitrary}\\
&\quad \text{polynomials } p_0,\, p_1, \ldots, p_l,\, q_1, \ldots, q_m,\, p_{N-1} \in \mathbb{S};\ \ i = 2, \ldots, l;\ \ j = 2, \ldots m,
\end{aligned}$$

where we denote by $F_1, \ldots, F_l$ the index-2 fixed points on one branch of the isotropy skeleton and by $F'_1, \ldots, F'_m$ the index-2 fixed points on the other branch.

Indeed, for f_0, we see that the only generating class that does not vanish when restricted to P is α_P ($\alpha_P\,|_F = 1$ for every $F \in M^{S^1}$) and so $f_0 = p_0\,\alpha_P\,|_P = p_0$. For f_1, only α_P and $\alpha_{F_1} := \tau_{S^1}(Z_{F_1})$ do not vanish when restricted to F_1 ($\alpha_{F_1}\,|_{F1} = -a_1\,u$), and so $f_1 = p_0\,\alpha_P\,|_{F_1} + p_1\,\alpha_{F_1}\,|_{F_1} = p_0 - a_1\,p_1\,u$. Similarly, for f_2, only α_P, α_{F_1} and $\alpha_{F_2} := \tau_{S^1}(Z_{F_2})$ do not vanish when restricted to F_2 ($\alpha_{F_1}\,|_{F_2} = a_3 u$ and $\alpha_{F_2}\,|_{F_2} = -a_2 u$) and so $f_2 = p_0\,\alpha_P\,|_{F_2} + p_1\,\alpha_{F_1}\,|_{F_2} + p_2\,\alpha_{F_2}\,|_{F_2} = p_0 + (a_3\,p_1 - a_2\,p_2)\,u$, and so on. Finally, the only generating classes that do not vanish on the fixed point of index 4, Q, are α_{F_l}, $\alpha_{F'_m}$ and α_Q (cf. Figure 1) with $\alpha_{F_l}\,|_Q = -b_{m+1} u$, $\alpha_{F'_l}\,|_Q = -a_{l+1}\,u$ and $\alpha_Q\,|_Q = a_{l+1}\,b_{m+1}\,u^2$, and so $f_{N-1} = p_0\,\alpha_P\,|_Q + p_l\,\alpha_{F_l}\,|_Q + q_m\,\alpha_{F'_m}\,|_Q + p_{N-1}\,\alpha_Q\,|_Q = p_0 - (b_{m+1}\,p_l + a_{l+1}\,q_m)\,u + a_{l+1}\,b_{m+1}\,p_{N-1}\,u^2$. $\square$

Remark. Note that, even though the arrangement of free gradient spheres in the (J,g)-isotropy skeleton may depend on the choice of (J,g), this does not affect the $\mathcal{A}$, since the equivariant cohomology of M does not depend on (J,g).

5. Examples

1. We first show an example of how $H^*_{S^1}(X,\mathbb{Z})$ is independent of the choice of (J,g). In a Kähler toric variety, the preimage of an edge of the moment map polytope under the moment map for the torus action is a 2-sphere which is complex and invariant. Therefore, when we consider the space as an S^1-space, this 2-sphere is either fixed by the action or is a gradient sphere for the Kähler metric. Hence, the arrangement of the gradient spheres with respect to the the Käler metric is given exactly by the arrangement of the non-horizontal edges of the moment map polytope.

Let us consider for example the Hirzebruch surface

$$W_{2m} = \{([a:b],[x:y:z]) \in \mathbb{C}P^1 \times \mathbb{C}P^2 : \ a^{2m}y - b^{2m}x = 0\}$$

(which we know to be diffeomorphic to $S^2 \times S^2$), equipped with the symplectic form induced by multiples of the Fubini-Study forms on $\mathbb{C}P^1$ and on $\mathbb{C}P^2$, $C_1\omega_1 \oplus C_2\omega_2$, and the Hamiltonian $\mathbb{T}^2$-action induced by the action

$$(\lambda_1,\lambda_2)\cdot([a:b],[x:y:z]) = ([\lambda_2 a : b],[\lambda_2^m\, x : \lambda_2^{-m}\, y : \lambda_1\, z]), \quad (\lambda_1,\lambda_2) \in \mathbb{T}^2$$

on $\mathbb{C}P^1 \times \mathbb{C}P^2$. The corresponding moment map is given by

$$\phi(([a:b],[x:y:z])) = (C_2\frac{|z|^2}{|x|^2+|y|^2+|z|^2}, C_1\frac{|a|^2}{|a|^2+|b|^2} + C_2\, m\frac{|x|^2-|y|^2}{|x|^2+|y|^2+|z|^2}).$$

The four fixed points of this action are $P = ([0:1],[0:1:0])$, $F_1 = ([0:1],[0:0:1])$, $F_2 = ([1:0],[0:0:1])$ and $Q = ([1:0],[1:0:0])$, and the moment map image (the convex-hull of the images of the fixed points) is represented in Figure 4.

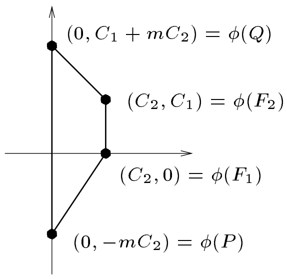

FIGURE 4. Moment map polytope for the $\mathbb{T}^2$-action on W_{2m}

From this polytope we can obtain the isotropy skeleton corresponding to the action of the second circle,

$$\lambda_2 \cdot ([a:b],[x:y:z]) = ([\lambda_2 a : b],[\lambda_2^m\, x : \lambda_2^{-m}\, y : z]),$$

and the Kähler metric, as can be seen in Figure 5 (a). Note that there are two $\mathbb{Z}_m$-spheres connecting F_2 to Q and F_1 to P respectively.

We know from [K] that, when the isotropy skeleton has a free gradient sphere whose north and south poles, F_1 and F_2, are both interior fixed points (i.e. which are neither a maximum nor a minimum), there exists a small perturbation of the metric within the space of compatible metrics, for which there exists one free gradient sphere whose north and south poles are F_2 and the minimum for the Hamiltonian function, and another free gradient sphere whose south pole is F_1 and whose north pole is the maximum of this function. All other gradient spheres remain unchanged. For this perturbed metric, we obtain the isotropy skeleton in Figure 5 (b). Hence, we have here an example where the same Hamiltonian S^1-space admits two different compatible metrics for which the gradient spheres are arranged differently.

Let us see how to compute the classes α_{F_1} and α_{F_2} for both isotropy skeletons. In the first case, $\overline{W}^S(F_1)$ is a sphere connecting F_1 to F_2. Hence, $\alpha_{1,P} := \alpha_{F_1}|_P = 0$, $\alpha_{1,1} := \alpha_{F_1}|_{F_1} = -mu$, $\alpha_{1,2} := \alpha_{F_1}|_{F_2} = mu$ and $\alpha_{1,Q} := \alpha_{F_1}|_Q = 0$. The closure

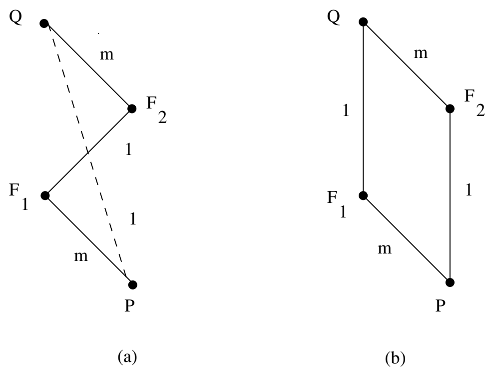

FIGURE 5. Isotropy skeletons for W_{2m}

$\overline{W}^S(F_2)$ of the stable manifold of F_2 is the $\mathbb{Z}_m$-sphere connecting F_2 to Q and so $\alpha_{2,P} = \alpha_{2,1} = 0$, while $\alpha_{2,2} = \alpha_{2,Q} = -u$. In the case of the perturbed metric, $\overline{W}^S(F_1)$ is a sphere connecting F_1 to Q and so $\alpha_{1,P} = \alpha_{1,2} = 0$ and $\alpha_{1,1} = \alpha_{1,Q} = -mu$. Similarly, we can conclude that $\alpha_{2,P} = \alpha_{2,1} = 0$ and $\alpha_{2,2} = \alpha_{2,Q} = -u$. These results are listed in Table 1 (a) and (b).

F_j	α_P	α_{F_1}	α_{F_2}	α_Q
P	1	0	0	0
F_1	1	$-mu$	0	0
F_2	1	mu	$-u$	0
Q	1	0	$-u$	mu^2

(a)

F_j	α_P	α_{F_1}	α_{F_2}	α_Q
P	1	0	0	0
F_1	1	$-mu$	0	0
F_2	1	0	$-u$	0
Q	1	$-mu$	$-u$	mu^2

(b)

TABLE 1. Generating classes for the isotropy skeletons of W_{2k}

We can easily check that these two sets of generating functions generate the same subalgebra of $\bigoplus_{k=1}^4 \mathbb{S}$. In Figure 6, we show the general form of an equivariant class in $H^2_{S^1}(W_{2k}, \mathbb{Z})$, shown as an element of the equivariant cohomology of the fixed points.

2. Let M be $\mathbb{C}P^2$ equipped with the circle action

$$\lambda \cdot [z_1 : z_2 : z_3] = [\lambda^p z_1 : \lambda^q z_2 : z_3],$$

with integers $p > q > 0$. If we blow up the interior fixed point $[0 : 1 : 0]$ we obtain a new circle action on the resulting manifold. If we repeat this three more times, always blowing up the interior fixed point with the lowest value of the Hamiltonian function, we get $\mathbb{C}P^2$ blown up four times equipped with the circle action with the isotropy skeleton shown in Figure 7.

The closures $\overline{W}^S(F_i)$ of the stable manifolds for $i = 1, \ldots, 5$ are the isotropy spheres connecting F_i to F_{i+1} and so we obtain the values $\alpha_{i,j} := \alpha_{F_i}|_{F_j}$ listed in the following table:

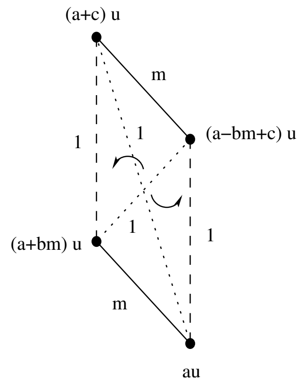

FIGURE 6. General form of a class in $H^2_{S^1}(W_{2k}, \mathbb{Z})$ $(a, b, c \in \mathbb{Z})$

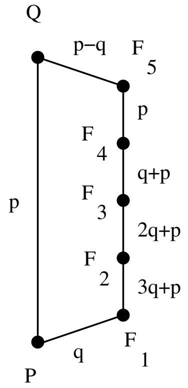

FIGURE 7. Isotropy skeleton of $\mathbb{C}P^2$ blown up four times

F_j	α_P	α_{F_1}	α_{F_2}	α_{F_3}	α_{F_4}	α_{F_5}	α_Q
P	1	0	0	0	0	0	0
F_1	1	$-qu$	0	0	0	0	0
F_2	1	$(2q+p)u$	$-(3q+p)u$	0	0	0	0
F_3	1	0	$(q+p)u$	$-(2q+p)u$	0	0	0
F_4	1	0	0	pu	$-(q+p)u$	0	0
F_5	1	0	0	0	$(p-q)u$	$-pu$	0
Q	1	0	0	0	0	$-pu$	$p(p-q)u^2$

TABLE 2

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Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisbon, Portugal, Fax: (351) 218417598

E-mail address: lgodin@math.ist.utl.pt