INSTITUTO SUPERIOR TÉCNICO
LEFT – LEBL – LQ – LEAM – LEMAT
Ano Lectivo: 2006/2007

MATEMÁTICA COMPUTACIONAL

Resolução do Exame de 17 de Julho de 2007

[1][20]

$$z_1 = e^x, \qquad z_2 = e^{-x}, \qquad z_3 = z_1 - z_2, \qquad z = \sinh x = \frac{z_3}{2}$$

$$\delta_{\tilde{z}_1} = x\,\delta_{\tilde{x}} + \delta_1$$

$$\delta_{\tilde{z}_2} = -x\,\delta_{\tilde{x}} + \delta_2$$

$$\begin{aligned}
\delta_{\tilde{z}_3} &= \frac{z_1}{z_3}\,\delta_{\tilde{z}_1} - \frac{z_2}{z_3}\,\delta_{\tilde{z}_2} + \delta_3 \\
&= \frac{x(z_1+z_2)}{z_3}\,\delta_{\tilde{x}} + \frac{z_1}{z_3}\,\delta_1 - \frac{z_2}{z_3}\,\delta_2 + \delta_3 \\
\delta_{\tilde{z}} &= \delta_{\tilde{z}_3} + \delta_4 \\
&= \frac{x\cosh x}{\sinh x}\,\delta_{\tilde{x}} + \frac{e^x}{2\sinh x}\,\delta_1 - \frac{e^{-x}}{2\sinh x}\,\delta_2 + \delta_3 + \delta_4 \\
&=: p(x)\,\delta_{\tilde{x}} + q_1(x)\,\delta_1 + q_2(x)\,\delta_2 + \delta_3 + \delta_4
\end{aligned}$$

O problema é bem posto para qualquer $x \in \mathbb{R}$ pois, pondo $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$, obtém-se $\delta_{\tilde{z}} = p(x)\delta_{\tilde{x}}$, e p, com $p(0) = \lim_{x\to 0} p(x) = 1$, é limitada em qualquer vizinhança finita de x.

O algoritmo para o cálculo de $\sinh x$ é numericamente instável para $x \approx 0$ pois q_1 e q_2 são singulares para $x = 0$.

[2]

(a)[15]

$$f'(x) = 2x - \cos x, \qquad f''(x) = 2 + \sin x$$

$$f''(x) > 0, \qquad \forall x \in \mathbb{R}$$

$\Rightarrow$ f' tem quanto muito um zero em $\mathbb{R}$

$\Rightarrow$ f tem quanto muito dois zeros em $\mathbb{R}$

$$\left.\begin{aligned} f(-0.8) &= +0.357356 \\ f(-0.6) &= -0.0753575 \end{aligned}\right\} \Rightarrow \; f \text{ tem pelo menos um zero em } [-0.8, -0.6]$$

$$\left.\begin{array}{l} f(1.3) = -0.273558 \\ f(1.5) = +0.252505 \end{array}\right\} \Rightarrow \ f \ \text{ tem pelo menos um zero em } [1.3, 1.5]$$

$\Rightarrow$ f tem exactamente dois zeros em $\mathbb{R}$, $w \in [-0.8, -0.6]$, $z \in [1.3, 1.5]$.

(b)[20]

$f(z) = 0 \quad \Leftrightarrow \quad z = g(z)$

$g(x) = x - 0.6f(x), \qquad I := [1.3, 1.5]$

$g'(x) = 1 - 0.6f'(x), \qquad g''(x) = -0.6f''(x)$

Condições suficientes de convergência do método do ponto fixo para z, $\forall x_0 \in I$:

(i) $g \in C^1(I)$

(ii) $\max_{x \in I} |g'(x)| = |g'(1.5)| = 0.757558 < 1$,

pois $g'(1.3) = -0.399501, \qquad g'(1.5) = -0.757558, \qquad g''(x) < 0, \ \forall x \in I$

(iii) $g(I) \subset I$,

pois $g(1.3) = 1.46413 \in I, \qquad g(1.5) = 1.34850 \in I, \qquad g'(x) < 0, \ \forall x \in I$

(c)[15]

$|z - x_m| \leq L^m |z - x_0| \leq \varepsilon$

$$m \log L \leq \log \frac{\varepsilon}{|z - x_0|}$$

$$m \geq \frac{\log \dfrac{\varepsilon}{|z - x_0|}}{\log L} \qquad (0 < L < 1 \ \Rightarrow \ \log L < 0)$$

$$x_0 = 1.4 \ \Rightarrow \ |z - x_0| \leq \frac{1.5 - 1.3}{2} = 0.1$$

$L = \max_{x \in I} |g'(x)| = |g'(1.5)| = 0.757558$

$\varepsilon = 10^{-6}$

$m \geq 41.4648$

$M = 42$

(d)[10]

O método do ponto fixo com função iteradora g não pode convergir para w pois $|g'(w)| > 1$. Com efeito:

$$g'(-0.8) = 2.37802, \qquad g'(-0.6) = 2.21520, \qquad g''(x) < 0, \;\; \forall x \in [-0.8, -0.6]$$

$$\Rightarrow \quad \min_{x \in [-0.8,-0.6]} |g'(x)| = |g'(-0.6)| = 2.21520 \quad \Rightarrow \quad |g'(w)| > 1.$$

[3]

(a)[15]

Fórmula de interpolação de Lagrange:

$$q_2(x) = \sum_{j=0}^{2} y(x_j)\, l_j(x), \qquad l_j(x) = \prod_{i=0, i\neq j}^{2} \frac{x - x_i}{x_j - x_i}$$

$$l_0(x) = \frac{(x-0.4)(x-0.6)}{(0.2-0.4)(0.2-0.6)} = \frac{1}{0.08}\,(x-0.4)(x-0.6)$$

$$l_1(x) = \frac{(x-0.2)(x-0.6)}{(0.4-0.2)(0.4-0.6)} = -\frac{1}{0.04}\,(x-0.2)(x-0.6)$$

$$l_2(x) = \frac{(x-0.2)(x-0.4)}{(0.6-0.2)(0.6-0.4)} = \frac{1}{0.08}\,(x-0.2)(x-0.4)$$

$$q_2(x) = \frac{17.5}{0.08}\,(x-0.4)(x-0.6) - \frac{15.0}{0.04}\,(x-0.2)(x-0.6) + \frac{13.125}{0.08}\,(x-0.2)(x-0.4)$$

(b)[15]

Fórmula de Newton às diferenças divididas:

i	x_i	$y[x_i]$	$y[\cdot,\cdot]$	$y[\cdot,\cdot,\cdot]$
0	0.0	21.0		
			-17.5	
1	0.2	17.5		+12.5
			-12.5	
2	0.4	15.0		

$$\begin{aligned} p_2(x) &= y[x_0] + y[x_0,x_1](x-x_0) + y[x_0,x_1,x_2](x-x_0)(x-x_1) \\ &= 21.0 - 17.5x + 12.5x(x-0.2) \end{aligned}$$

(c)[20]

$$e_2(x) = y(x) - p_2(x) = \frac{1}{6}\,y'''(\xi) W_3(x)$$

$$\xi \in]x_0; x_1; x_2; x[, \qquad W_3(x) = (x-x_0)(x-x_1)(x-x_2)$$

$$|e_2(x)| \le \frac{1}{6}\,\|y'''\|_\infty \|W_3\|_\infty, \qquad \forall x \in [0.0, 0.4]$$

$$\|y'''\|_\infty = \max_{x\in[0,0.4]} |y'''(x)|, \qquad \|W_3\|_\infty = \max_{x\in[0,0.4]} |W_3(x)|$$

$$y''(x) = \frac{2}{21^2}\,[y(x)]^3, \qquad y'''(x) = -\frac{6}{21^3}\,[y(x)]^4, \qquad y^{(iv)}(x) = \frac{24}{21^4}\,[y(x)]^5$$

y''' é negativa e crescente em [0.0, 0.4].

$$\|y'''\|_\infty = |y'''(0.0)| = 126.0$$

$$W_3(x) = x(x-0.2)(x-0.4)$$

$$W_3'(x) = 3x^2 - 1.2x + 0.08 = 3(x-d_0)(x-d_1)$$

$$d_0 = 0.2\left(1 - \frac{1}{\sqrt{3}}\right) = 0.0845299, \qquad d_1 = 0.2\left(1 + \frac{1}{\sqrt{3}}\right) = 0.315470$$

$$\|W_3\|_\infty = |W_3(d_0)| = |W_3(d_1)| = \frac{0.016}{3\sqrt{3}} = 0.00307920$$

$$|e_3(x)| \le \frac{0.112}{\sqrt{3}} = 0.0646632, \qquad \forall x \in [0, 0.4]$$

(d)[20]

Melhor aproximação mínimos quadrados:

$$\phi^*(x) = a^*\phi_0(x) + b^*\phi_1(x), \qquad \phi_0(x) = 1, \qquad \phi_1(x) = x$$

$$\begin{bmatrix} \langle\bar\phi_0,\bar\phi_0\rangle & \langle\bar\phi_0,\bar\phi_1\rangle \\ \langle\bar\phi_1,\bar\phi_0\rangle & \langle\bar\phi_1,\bar\phi_1\rangle \end{bmatrix} \begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} \langle\bar f,\bar\phi_0\rangle \\ \langle\bar f,\bar\phi_1\rangle \end{bmatrix}$$

$$\langle\bar\phi,\bar\psi\rangle = \sum_{i=0}^{3} \bar\phi_i\bar\psi_i, \qquad \forall \bar\phi,\bar\psi \in \mathbb{R}^4$$

$$\bar\phi = [\phi(x_0) \quad \phi(x_1) \quad \phi(x_2) \quad \phi(x_3)]^T, \qquad \phi \in C(\mathbb{R})$$

$$\bar\phi_0 = [1 \quad 1 \quad 1 \quad 1]^T$$

$$\bar\phi_1 = [0 \quad 0.2 \quad 0.4 \quad 0.6]^T$$

$$\bar f = [0.0476190 \quad 0.0571429 \quad 0.0666667 \quad 0.0761905]^T$$

$$\langle\bar\phi_0,\bar\phi_0\rangle = 4$$

$\langle\bar\phi_0,\bar\phi_1\rangle = \langle\bar\phi_1,\bar\phi_0\rangle = 1.2$

$\langle\bar\phi_1,\bar\phi_1\rangle = 0.56$

$\langle\bar f,\bar\phi_0\rangle = 0.247619$

$\langle\bar f,\bar\phi_1\rangle = 0.0838095$

$$\begin{bmatrix} 4 & 1.2 \\ 1.2 & 0.56 \end{bmatrix}\begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} 0.247619 \\ 0.0838095 \end{bmatrix} \quad\Rightarrow\quad \begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} 0.0476190 \\ 0.0476190 \end{bmatrix}$$

$\phi^*(x) = 0.0476190(1.0 + x)$

[4]

$$W'(t) = F(t, W(t)), \quad W = \begin{bmatrix} y \\ z \end{bmatrix}, \quad F(t,W) = \begin{bmatrix} 1+\cos(\pi t) + y(yz-2) \\ y(1-yz) \end{bmatrix}$$

$$W(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(a)[13]

Método de Euler (passo h; dois passos):

$W_1 = W_0 + hF(0, W_0)$

$$W_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + h\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+h \\ 1 \end{bmatrix}$$

$W_2 = W_1 + hF(h, W_1)$

$$\begin{aligned} W_2 &= \begin{bmatrix} 1+h \\ 1 \end{bmatrix} + h\begin{bmatrix} 1+\cos(\pi h) + (1+h)(h-1) \\ (1+h)(-h) \end{bmatrix} \\ &= \begin{bmatrix} 1+h(1+\cos(\pi h)+h^2) \\ 1-h^2(1+h) \end{bmatrix} \end{aligned}$$

(b)[12]

Método de Euler modificado (passo $2h$; um passo):

$\tilde W_1 = \tilde W_0 + 2hF(h, \tilde W_0^E)$

$$\tilde W_0^E = \tilde W_0 + hF(0, \tilde W_0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + h\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+h \\ 1 \end{bmatrix}$$

$$\tilde{W}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2h \begin{bmatrix} 1+\cos(\pi h) + (1+h)(h-1) \\ (1+h)(-h) \end{bmatrix}$$
$$= \begin{bmatrix} 1 + 2h(\cos(\pi h) + h^2) \\ 1 - 2h^2(1+h) \end{bmatrix}$$

[5][20]

Método de Adams-Bashforth de ordem 2:

$$y_{n+1} = y_n + \frac{h}{2}\,[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})]$$

Erro de discretização local:

$$\tau(x, Y(x); h) = \frac{1}{h}[Y(x+h) - Y(x)] - \frac{1}{2}\,[3f(x, Y(x)) - f(x-h, Y(x-h))]$$

$$Y'(x) = f(x, Y(x)), \qquad Y(x) = y$$

$$\tau(x, Y(x); h) = \frac{1}{h}[Y(x+h) - Y(x)] - \frac{3}{2}\,Y'(x) + \frac{1}{2}\,Y'(x-h)$$

$$Y(x+h) = Y(x) + hY'(x) + \frac{h^2}{2}\,Y''(x) + \frac{h^3}{6}\,Y'''(x) + \mathcal{O}(h^4)$$

$$Y'(x-h) = Y'(x) - hY''(x) + \frac{h^2}{2}\,Y'''(x) + \mathcal{O}(h^4)$$

$$\tau(x, Y(x); h) = \frac{5h^2}{12}\,Y'''(x) + \mathcal{O}(h^3)$$

$$Y''(x) = (df)(x, Y(x)), \qquad Y'''(x) = (d^2 f)(x, Y(x))$$

$$\tau(x, y; h) = \frac{5h^2}{12}\,(d^2 f)(x, y) + \mathcal{O}(h^3)$$