INSTITUTO SUPERIOR TÉCNICO
LEFT – LEBL – LQ – LEAM – LEMAT
Ano Lectivo: 2006/2007

MATEMÁTICA COMPUTACIONAL

Resolução do Exame de 3.JUL.2007

[1]

(a)[20]

$f(x) = x - 0.075 \sin x - 1.68$

$f'(x) = 1 - 0.075 \cos x, \qquad f''(x) = 0.075 \sin x$

Condições suficientes de convergência do método de Newton com $x_0 \in I = [1.7, 1.8] = [a, b]$ para $z \in I$:

$f \in C^2(I)$

(i) $f(a) = -0.0544, \quad f(b) = 0.0470 \quad \Rightarrow \quad f(a)f(b) < 0$

(ii) $f'(x) > 0, \ \forall x \in I$

(iii) $f''(x) > 0, \ \forall x \in I$

(iv) $\left| \dfrac{f(a)}{f'(a)} \right| = 0.0539 < b - a = 0.1$

$\left| \dfrac{f(b)}{f'(b)} \right| = 0.0462 < b - a = 0.1$

(b)[20]

Método de Newton:

$$x_{m+1} = x_m - \frac{f(x_m)}{f'(x_m)}, \qquad m \geq 0$$

$x_0 = 1.8$

$x_1 = 1.75383$

$x_2 = 1.75375$

Estimativa de erro:

$$|z - x_2| \leq \frac{1}{K} \left(K|z - x_0|\right)^4 = K^3 |z - x_0|^4$$

$$K = \frac{\max_{x \in I} |f''(x)|}{2 \min_{x \in I} |f'(x)|} = \frac{|f''(a)|}{2|f'(a)|} \overset{\approx}{<} 0.0369$$

$$|z - x_0| \leq b - a = 0.1$$

$$|z - x_2| < 5.02 \times 10^{-9}$$

[2][30]

Método de Gauss-Seidel:

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} = L + D + U = M_{GS} + N_{GS}$$

$$M_{GS} = L + D \qquad N_{GS} = U$$

$$x^{(m+1)} = D^{-1}\left(-Lx^{(m+1)} - Ux^{(m)} + b\right), \qquad m \geq 0$$

$$\begin{bmatrix} x_1^{(m+1)} \\ x_2^{(m+1)} \\ x_3^{(m+1)} \end{bmatrix} = \frac{1}{4}\left(- \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(m+1)} \\ x_2^{(m+1)} \\ x_3^{(m+1)} \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(m)} \\ x_2^{(m)} \\ x_3^{(m)} \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix} \right)$$

$$x^{(0)} = 0$$

$$\begin{cases} x_1^{(1)} = \frac{1}{4}(-0 - 0 + 5) = \frac{5}{4} = 1.25 \\ x_2^{(1)} = \frac{1}{4}\left(-\frac{5}{4} - 0 + 6\right) = \frac{19}{16} = 1.1875 \\ x_3^{(1)} = \frac{1}{4}\left(-\frac{19}{16} - 0 + 5\right) = \frac{61}{64} = 0.953125 \end{cases}$$

$$\begin{cases} x_1^{(2)} = \frac{1}{4}\left(-0 - \frac{19}{16} + 5\right) = \frac{61}{64} = 0.953125 \\ x_2^{(2)} = \frac{1}{4}\left(-\frac{61}{64} - \frac{61}{64} + 6\right) = \frac{131}{128} = 1.0234375 \\ x_3^{(2)} = \frac{1}{4}\left(-\frac{131}{128} - 0 + 5\right) = \frac{509}{512} = 0.994140625 \end{cases}$$

Estimativa de erro:

$$\|x - x^{(2)}\|_\infty \leq \frac{c}{1 - c}\|x^{(2)} - x^{(1)}\|_\infty, \qquad \|C_{GS}\|_\infty \leq \frac{1}{3} = c$$

$$x^{(2)} - x^{(1)} = \begin{bmatrix} -\frac{19}{64} \\ -\frac{21}{128} \\ +\frac{21}{512} \end{bmatrix} \quad \Rightarrow \quad \|x^{(2)} - x^{(1)}\|_\infty = \frac{19}{64} = 0.296875$$

$$\|x - x^{(2)}\|_\infty \leq \frac{19}{128} = 0.1484375$$

[3]

(a)[20]

Melhor aproximação mínimos quadrados:

$$\phi^*(x) = a^*\phi_0(x) + b^*\phi_1(x), \qquad \phi_0(x) = 1, \qquad \phi_1(x) = x^3$$

$$\begin{bmatrix} \langle \phi_0, \phi_0 \rangle & \langle \phi_0, \phi_1 \rangle \\ \langle \phi_1, \phi_0 \rangle & \langle \phi_1, \phi_1 \rangle \end{bmatrix} \begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} \langle f, \phi_0 \rangle \\ \langle f, \phi_1 \rangle \end{bmatrix}$$

$$\langle \phi, \psi \rangle = \int_0^1 \phi(x)\psi(x)\,dx, \qquad \forall \phi, \psi \in C([0,1])$$

$$\langle \phi_0, \phi_0 \rangle = \int_0^1 dx = 1$$

$$\langle \phi_0, \phi_1 \rangle = \int_0^1 x^3\,dx = \frac{1}{4}$$

$$\langle \phi_1, \phi_0 \rangle = \langle \phi_0, \phi_1 \rangle = \frac{1}{4}$$

$$\langle \phi_1, \phi_1 \rangle = \int_0^1 x^6\,dx = \frac{1}{7}$$

$$\langle f, \phi_0 \rangle = \int_0^1 f(x)\,dx$$

$$\langle f, \phi_1 \rangle = \int_0^1 x^3 f(x)\,dx$$

$$\begin{bmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} \int_0^1 f(x)\,dx \\ \int_0^1 x^3 f(x)\,dx \end{bmatrix}$$

(b)[25]

Fórmula dos trapézios composta ($f \in C([a,b])$):

$$I(f) = \int_a^b f(x)\,dx \quad \approx \quad I_1^{(M)}(f) = \frac{h_M}{2}\left[f(x_0) + f(x_M) + 2\sum_{j=1}^{M-1} f(x_j)\right]$$

$$h_M = \frac{b-a}{M}, \qquad x_j = a + jh_M, \quad j = 0, 1, \ldots, M$$

$M=4, \qquad a=0, \quad b=1, \qquad h_M=\frac{1}{4}, \qquad x_j=\frac{j}{4}, \quad j=0,1,2,3$

$f(x)=\exp(x^2)$

$\tilde{w}_1=\frac{1}{8}\ \{f(x_0)+f(x_4)+2[f(x_1)+f(x_2)+f(x_3)]\}=1.49068$

$g(x)=x^3 f(x)$

$\tilde{w}_2=\frac{1}{8}\ \{g(x_0)+g(x_4)+2[g(x_1)+g(x_2)+g(x_3)]\}=0.569173$

Erro da fórmula dos trapézios composta ($f\in C^2([a,b])$):

$$\left|E_1^{(M)}(f)\right|=\left|I(f)-I_1^{(M)}(f)\right|\leq\frac{b-a}{12}\,h_M^2\max_{x\in[a,b]}|f''(x)|$$

$|w_1-\tilde{w}_1|\leq\frac{1}{192}\max_{x\in[0,1]}|f''(x)|$

$|w_2-\tilde{w}_2|\leq\frac{1}{192}\max_{x\in[0,1]}|g''(x)|$

$\max_{x\in[0,1]}|f''(x)|=|f''(1)|=6e$

$\max_{x\in[0,1]}|g''(x)|=|g''(1)|=24e$

$|w_1-\tilde{w}_1|\leq\frac{e}{32}\overset{\approx}{<}0.0850$

$|w_2-\tilde{w}_2|\leq\frac{e}{8}\overset{\approx}{<}0.340$

(c)[20]

$$\frac{\|y-\tilde{y}\|_\infty}{\|y\|_\infty}\leq\frac{\mathrm{cond}_\infty(A)}{1-\frac{\|A-\tilde{A}\|_\infty}{\|A\|_\infty}\mathrm{cond}_\infty(A)}\left(\frac{\|A-\tilde{A}\|_\infty}{\|A\|_\infty}+\frac{\|w-\tilde{w}\|_\infty}{\|w\|_\infty}\right)$$

$$\frac{\|A-\tilde{A}\|_\infty}{\|A\|_\infty}\mathrm{cond}_\infty(A)<1$$

$\|w-\tilde{w}\|_\infty=\max\{|w_1-\tilde{w}_1|,|w_2-\tilde{w}_2|\}\overset{\approx}{<}0.340$

$w_1\in[\tilde{w}_1-|w_1-\tilde{w}_1|\,,\tilde{w}_1+|w_1-\tilde{w}_1|]\subset]1.40,1.58[$

$w_2\in[\tilde{w}_2-|w_2-\tilde{w}_2|\,,\tilde{w}_2+|w_2-\tilde{w}_2|]\subset]0.229,0.910[$

$\|w\|_\infty=\max\{|w_1|,|w_2|\}>1.40$

$\frac{\|w-\tilde{w}\|_\infty}{\|w\|_\infty}<0.243$

$$A - \tilde{A} = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & \frac{1}{7} - 0.143 \end{bmatrix} \quad \Rightarrow \quad \|A - \tilde{A}\|_\infty \stackrel{\approx}{<} 0.143 \times 10^{-3}$$

$$\|A\|_\infty = \frac{5}{4}$$

$$\frac{\|A - \tilde{A}\|_\infty}{\|A\|_\infty} \stackrel{\approx}{<} 0.115 \times 10^{-3}$$

$$A^{-1} = \frac{4}{9} \begin{bmatrix} 4 & -7 \\ -7 & 28 \end{bmatrix} \quad \Rightarrow \quad \|A^{-1}\|_\infty = \frac{140}{9}$$

$$\mathrm{cond}_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = \frac{175}{9} \stackrel{\approx}{<} 19.5$$

$$\frac{\|A - \tilde{A}\|_\infty}{\|A\|_\infty} \mathrm{cond}_\infty(A) \stackrel{\approx}{<} 0.224 \times 10^{-2} < 1$$

$$\frac{\|y - \tilde{y}\|_\infty}{\|y\|_\infty} < \frac{19.5}{1 - 0.224 \times 10^{-2}} (0.115 \times 10^{-3} + 0.243) < 4.74$$

[4]

(a)[20]

Fórmula de Quadratura de Gauss:

$$I_1(f) = w_0 f(x_0) + w_1 f(x_1)$$

$$I_1(x^m) = I(x^m), \qquad m = 0, 1, 2, 3$$

$$\begin{cases} w_0 + w_1 = I(1) = 4 \\ w_0 x_0 + w_1 x_1 = I(x) = 0 \\ w_0 x_0^2 + w_1 x_1^2 = I(x^2) = 8 \\ w_0 x_0^3 + w_1 x_1^3 = I(x^3) = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} w_0 = w_1 = 2 \\ -x_0 = \sqrt{2} = x_1 \end{cases}$$

$$I_1(f) = 2f(-\sqrt{2}) + 2f(\sqrt{2})$$

(b)[15]

A fórmula tem grau de precisão 3 pois, por construção, integra exactamente todos os polinómios de grau menor ou igual a 3 mas não integra nenhum polinómio de grau 4. Com efeito:

$$I_1(x^4) = 2(-\sqrt{2})^4 + 2(\sqrt{2})^4 = 16 \qquad \neq \qquad I(x^4) = \frac{64}{3}$$

[5] $f(x, y) = x \sin y$

(a)[10]

Método de Taylor de 2ª ordem (passo h):

$$y_1 = y_0 + hf(x_0, y_0) + \frac{h^2}{2}(df)(x_0, y_0)$$

$$(df)(x, y) = \left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)(x, y) = \sin y + x^2 \sin y \cos y$$

$$(x_0, y_0) = \left(0, \frac{\pi}{2}\right), \qquad f(x_0, y_0) = 0, \qquad (df)(x_0, y_0) = 1$$

$$y_1 = \frac{\pi}{2} + \frac{h^2}{2}$$

(b)[10]

Método de Runge-Kutta clássico de 2ª ordem (passo h):

$$\tilde{y}_1 = y_0 + \frac{h}{4}\left[f(x_0, y_0) + 3f\left(x_0 + \frac{2h}{3}, y_0 + \frac{2h}{3}f(x_0, y_0)\right)\right]$$

$$(x_0, y_0) = \left(0, \frac{\pi}{2}\right), \qquad f(x_0, y_0) = 0$$

$$\tilde{y}_1 = \frac{\pi}{2} + \frac{3h}{4}f\left(\frac{2h}{3}, \frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{3h}{4}\frac{2h}{3} = \frac{\pi}{2} + \frac{h^2}{2}$$

(c)[10]

Método de Adams-Moulton de 3ª ordem (passo h):

$$\hat{y}_2 = \hat{y}_1 + \frac{h}{12}\left[5f(x_2, \hat{y}_2) + 8f(x_1, \hat{y}_1) - f(x_0, \hat{y}_0)\right]$$

$$(x_0, \hat{y}_0) = \left(0, \frac{\pi}{2}\right), \qquad f(x_0, \hat{y}_0) = 0$$

$$(x_1, \hat{y}_1) = \left(h, \frac{\pi}{2} + \frac{h^2}{2}\right), \qquad f(x_1, \hat{y}_1) = h \sin\left(\frac{\pi}{2} + \frac{h^2}{2}\right) = h \cos\frac{h^2}{2}$$

$$(x_2, \hat{y}_2) = (2h, \hat{y}_2), \qquad f(x_2, \hat{y}_2) = 2h \sin \hat{y}_2$$

$$\hat{y}_2 = \frac{\pi}{2} + \frac{h^2}{2} + \frac{h}{12}\left(10h \sin \hat{y}_2 + 8h \cos\frac{h^2}{2}\right)$$

$$\hat{y}_2 = \frac{5h^2}{6}\sin \hat{y}_2 + \frac{\pi}{2} + \frac{h^2}{2} + \frac{2h^2}{3}\cos\frac{h^2}{2}$$