NOISE INDUCED CHANGES IN THE DYNAMICAL BEHAVIOUR OF NEURAL FIELDS

Pedro M. Lima[1], Evelyn Buckwar [2]

[1]Instituto Superior Técnico, University of Lisbon, PORTUGAL [2]Institute of Stochastics, Johannes Kepler University , AUSTRIA

FORMULATION OF THE PROBLEM

Neural Field Equations (NFE) are a powerful tool for analysing the dynamical behaviour of populations of neurons.
Literature on computational methods for NFE: [1], [2], [3] (deterministic case); [4] (stochastic models).
The main goal of the present work is to analyse the effect of noise in certain neural fields . The stochastic neural field equation with delay has the form:

$$dU_t(x) = \left(I(x,t) - \frac{1}{c}U_t(x) + \int_\Omega K(|x-y|)S(U_{t-\tau}(y))dy \right) dt + \epsilon dW_t(x), \qquad (1)$$

where $t \in [0,T]$, $x \in \Omega = [-l,l] \subset R$, $S(x)$ is the Firing rate function; K is the Connectivity kernel; I is the External input; τ is a delay, depending on the distance $|x-y|$; W_t is a Q-Wiener process. Initial condition:

$$U_t(x) = U_0(x,t), \qquad t \in [-\tau_{max}, 0], \quad x \in \Omega, \qquad (2)$$

where $U_0(x,t)$ is some given stochastic process, τ_{max} is the maximum value of the delay.

SPACE DISCRETIZATION

We apply a numerical scheme which uses the Galerkin method. Consider the following expansion of the solution:

$$U_t(x) = \sum_{k=0}^{\infty} u_t^k v_k(x), \qquad (3)$$

where v_k - eigenfunctions of the covariance operator of the noise in (1); we define

$$v_k(x) = \exp(ikx), \qquad k = 0,1,..,N. \qquad (4)$$

Take the inner product of equation (1) with the basis functions v_i:

$$(dU_t, v_i) = \left[(I(x,t), v_i) - \frac{1}{c}(U_t, v_i) + \left(\int_\Omega K(|x-y|)S(U_{t-\tau}(y))dy, v_i \right) \right] dt + \epsilon(dW_t, v_i). \qquad (5)$$

We expand dW_t as

$$dW_t(x) = \sum_{k=0}^{\infty} v_k(x)\lambda_k d\beta_t^k, \qquad (6)$$

where β_t^k - independent white noises in time ; λ_k are the eigenvalues of the covariance operator of the noise.
We define an approximate solution (3)

$$U_t^N(x) = \sum_{k=0}^{N-1} u_t^{k,N} v_k(x). \qquad (7)$$

The coefficients $u_t^{k,N}$ satisfy the following nonlinear system of stochastic delay differential equations:

$$du_t^{i,N} = \left[(I(x,t), v_i) - \frac{1}{c}u_t^{i,N} + (KS)^{i,N}(\bar{u}_{t-\tau}) \right] dt + \epsilon\lambda_i d\beta_t^i, \qquad (8)$$

where $(KS)^{i,N}(\bar{u}_{t-\tau})$ is given by

$$(KS)^{i,N}(\bar{u}_{t-\tau}) = h^2 \sum_{j=1}^{N} v_i(x) \left(\sum_{l=1}^{N} K(|x_l - y_j|) S \left(\sum_{k=1}^{N} u_{t-\tau}^k v_k(y_j) \right) \right) \qquad (9)$$

$i = 0, ..., N-1$. In this case we are introducing in $[-l, l]$ a set of N equidistant gridpoints $x_j = -l + j * h$, $j = 1, ..., N$, where $h = 2l/N$, and using the rectangular rule to evaluate the integrals.

TIME DISCRETIZATION

we can apply the Euler-Maruyama method to the solution of the system (8). Let $t_j = jh_t$, $j = 0, 1, ..., n$;

$$u_j^{k,N} \approx u_{t_j}^{k,N}.$$

In these notations, the Euler-Maruyama method may be written as

$$u_{j+1}^{i,N} = \frac{u_j^{i,N} + h_t \left[(I(x_i, t_j), v_i) + (KS)^{i,N}(\bar{u}_{t_j - \tau}) \right] + \sqrt{h_t}\epsilon\lambda_i w_i}{1 + \frac{1}{c}h_t}, \qquad (10)$$

COMPUTATIONAL COMPLEXITY

The straightforward computation of each integral in (9) requires about N^3 evaluations of the integrand function. However two of the sums in (9) can be evaluated efficiently by the Fast Fourier Transform (FFT). This reduces the number of function evaluations to $O(N(\log N)^2)$.

References

[1] G. Faye and O. Faugeras, Some theoretical and numerical results for delayed neural field equations, Physica D 239 (2010) 561–578.

[2] A. Hutt and N. Rougier, Activity spread and breathers induced by finite transmission speeds in two-dimensional neuronal fields, Physical Review,E 82 (2010) 055701.

[3] P.M. Lima and E. Buckwar, Numerical solution of the neural field equations in the two-dimensional case, SIAM J. Sci. Comput. (to appear).

[4] C. Kühn and M.G. Riedler, Large deviations for nonlocal stochastic neural fields, J. Math. Neurosci. 4:1 (2014) 1–33.

[5] Flora Ferreira, Multi-bump solutions in dynamic neural fields: analysis and applications, PhD thesis, University of Minho, 2014 (http://hdl.handle.net/1822/34416).

NUMERICAL EXAMPLE

We investigate the effect of noise in the formation of spatio-temporal patterns in dynamic neural fields. The stability analysis of these patterns in the deterministic case was carried out in [5], p.37. This numerical example does not include delays.
Firing rate function $S(x)$ - Heaviside function;
Connectivity kernel $K(x) = A\exp(-kx)\,(K\sin(\alpha x) + \cos(\alpha x))$;
External input $I(x) = -I_0 + B\exp\left(-\frac{x^2}{2\sigma^2}\right)$.

Deterministic Case

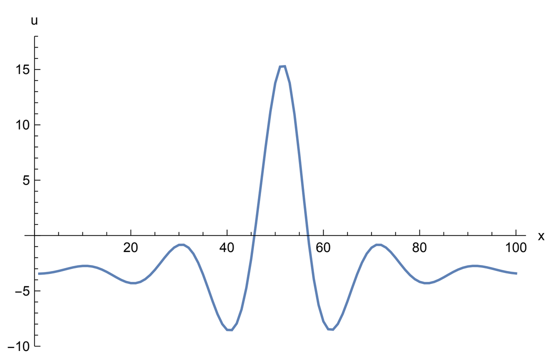

Fig. 1. Stationary solution

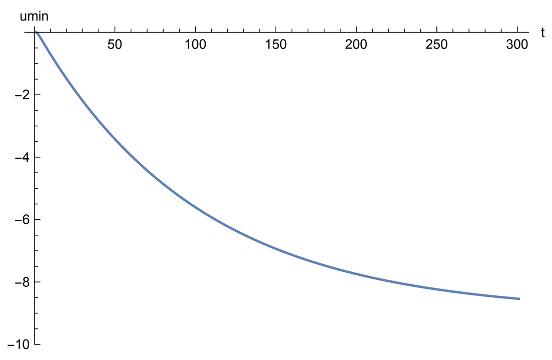

Fig.2. Evolution of $u_{min}(t)$.

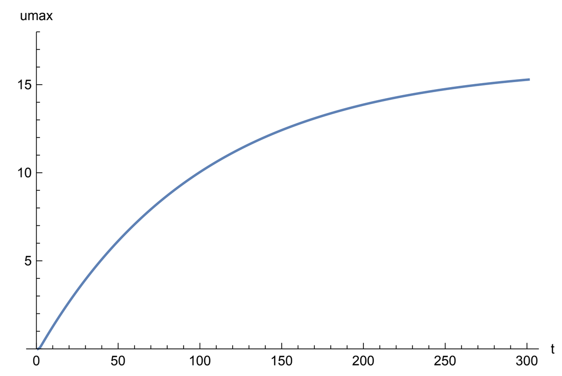

Fig.3. Evolution of $u_{max}(t)$

$u_{min}(t)$ and $umax(t)$ are the minimum and the maximum of the solution (on the whole domain) at time t

Case $\epsilon = 0.01$

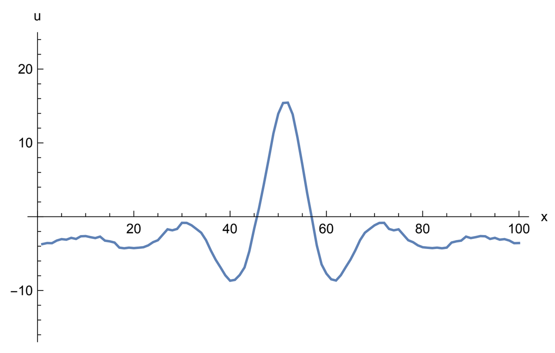

Fig.4. One-bump stationary solution.

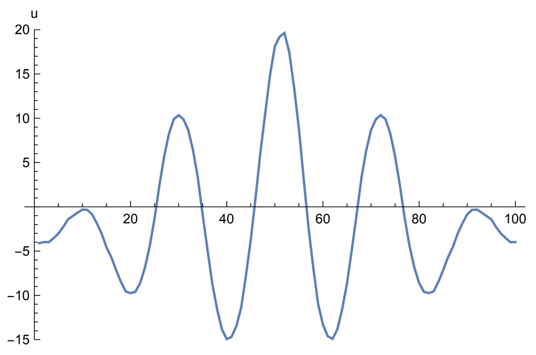

Fig.5. Three-bump stationary solution

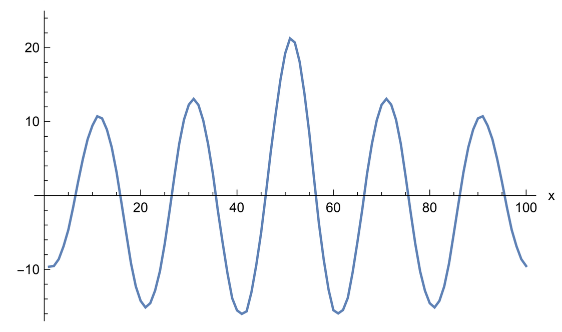

Fig.6. Five-bump stationary solution

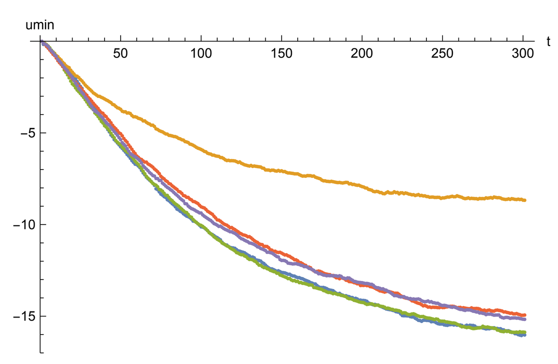

Fig.7. Evolution of $u_{min}(t)$.

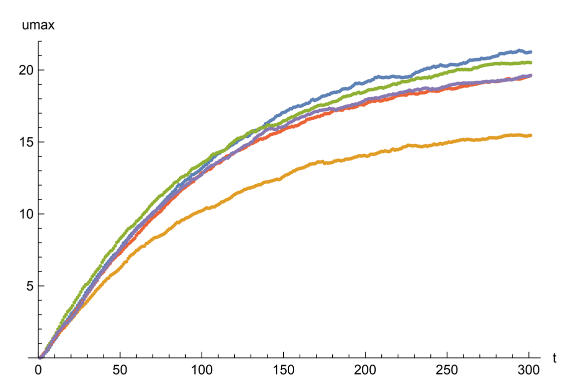

Fig. 8. Evolution of $u_{max}(t)$

Case $\epsilon = 0.1$

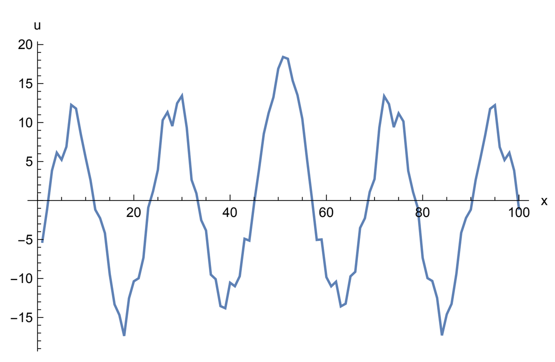

Fig. 9. First trajectory.

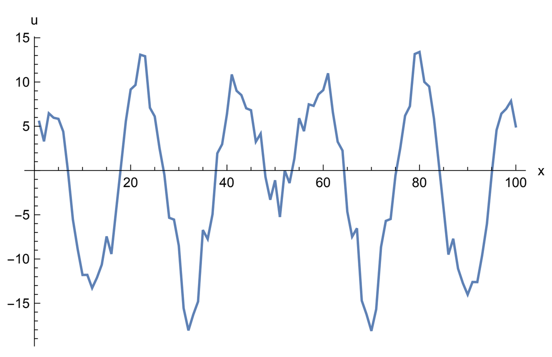

Fig. 10. Second trajectory

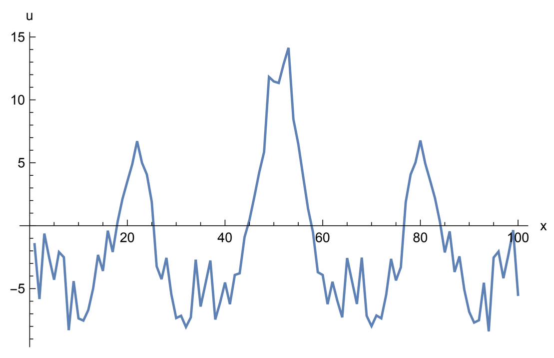

Fig. 11. Third trajectory

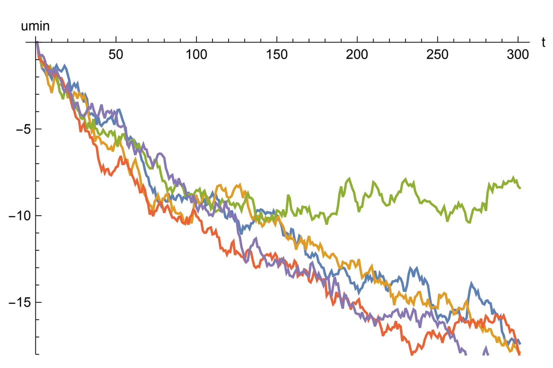

Fig. 12. Evolution of $u_{min}(t)$.

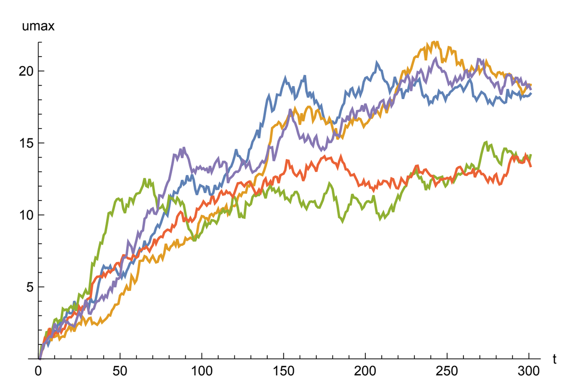

Fig. 13. Evolution of $u_{max}(t)$

Summary of Numerical Results

- In the case $\epsilon = 0.01$ the trajectories of the stochastic equation converge to some stationary states, which are similar to the stationary states of the deterministic equation. From 10 trajectories, 6 have converged to the five-bump stationary solution (fig. 6), 3 have converged to the three-bump stationary solution (fig. 5) and one has converged to the one-bump stationary solution (fig. 4). In fig. 7 (fig. 8) the graphs of $u_{min}(t)(u_{max}(t))$ are plotted for five different trajectories.
- In the case $\epsilon = 0.1$ the trajectories of the stochastic equation stabilize after a certain time, but the resulting patterns (see Figs. 9-11) are rather different from the stationary solutions of the deterministic equation. In fig. 12 (fig. 13) the graphs of $u_{min}(t)(u_{max}(t))$ are plotted for five different trajectories.

CONCLUSIONS AND FUTURE WORK

- The Galerkin approximation combined with the Euler-Maruyama method provide an effective computational method for the numerical solution of stochastic neural field equations.
- The efficiency of the algorithm is guaranteed by the use of the Fast Fourier Transform for the evaluation of integrals.
- The numerical simulations carried out so far suggest that for sufficiently small noise the stochastic Neural Field Equations have stable stationary solutions, close to the ones of the deterministic case.
- Using this algorithm we plan to analyse other types of dynamic neural fields, including the case of finite propagation speed (delay equations).
- This algorithm can be extended to the case of multidimensional neural fields.