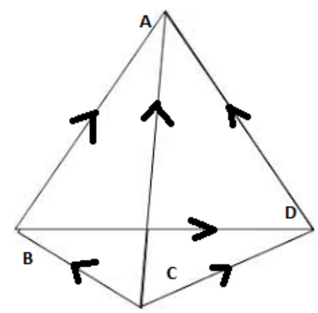

Figure 1: Graph for problem 2

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Test on Mathematical Modelling and Analysis, second module - 21/11/2019
Master Program on Mathematics and Applications
ANSWERS

1. Consider the following problem: given the coordinates (x, y, z) of an arbitrary point P in $\mathbb{R}^3$, we want to know the distance d from this point to an unknown point (x_0, y_0, z_0). For a certain set of points $S = \{(x_i, y_i, z_i), i = 1, 2, ..., n\}$ the distances d_i are known. We want to solve this problem using an artificial neural network.

   (a) Identify the input and output sets for this problem. **[1.0]**
   Answer: Input set: $(x, y, z) \in \mathbb{R}^3$. Output set:$d \in \mathbb{R}$

   (b) Describe a function h , depending on (x, y, z) and (x_0, y_0, z_0), which gives the needed value d. Answer:$d = h(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$ **[1.0]**

   (c) Propose a function F (empirical risk functional), depending on all the elements of the set S, which should be minimized, during the training process of the network. **[1.0]**
   Answer:$F(a, b, c) = \sum_{i=1}^{n} \left(\sqrt{(a - x_i)^2 + (b - y_i)^2 + (c - z_i)^2} - d_i \right)^2$.

   (d) Write a scheme of the neural network, indicating the input, output and hidden layer. **[1.0]**

   Answer: Intput layer: 3 neurons x, y, z; hidden layer: 3 neurons a, b, c ; output layer: 1 neuron d

2. Consider a graph with the form of a tetrahedron, where each vertex is a node and each edge has a certain direction (see fig. 1).

   (a) How many neurons and synapses are represented in the graph? **[1.0]**
   Answer: 4 neurons (vertices), 6 synapses (edges)

   (b) Is there a sink or a source in the given graph? If yes, identify them. **[1.0]**
   Answer: C - source; A - sink.

   (c) Looking at this graph as a simplicial complex, is there a cavity in it? If yes, what is its dimension? Justify. **[1.0]**
   Answer: There is a cavity (interior of the tetrahedron). The dimension is two (dimension of the faces as simplices).

(d) How many simplices are there in this graph: a) whith 0 dimensions? b) with 1 dimension? c) with 2 dimensions? **[1.0]**

Answer: 0 dimensions :$S_0 = 4$(vertices); 1 dimension: $S_1 = 6$ (edges); 2 dimensions: $S_2 = 4$ (faces)

(e) Compute the Euler characteristic of the graph. **[1.0]**

Answer: $\chi = S_0 - S_1 + S_2 = 2$.

3. Consider the discrete Fitzhugh-Nagumo equation:

$$\frac{dv(t)}{dt} = v(t+\tau) - 2v(t) + v(t-\tau) + F(v(t)), \quad t \in \mathbb{R}.$$

(a) Knowing that this equation models the propagation of electrical signals in a myelinated axon, what is the physical meaning of $v(t)$? **[1.0]**

Answer: $v(t)$ is the electrical potential in a Ranvier node at moment t.

(b) What is the physical meaning of τ and what do the terms $v(t+\tau)$ and $v(t-\tau)$ represent? **[1.0]**

Answer: τ is the time the electrical signal takes to go from a Ranvier node to the neighbor one. $v(t+\tau)$ and $v(t-\tau)$ represent the potential of at the two neighbor nodes.

(c) Suppose that $\tau = 1$ and the solution has the form $v(t) = 1 - e^{-2t}$, for $t \geq 2$. Moreover, assume the activation function has the form $F(v) = v(1-v)(v-a)$. Determine $v'(1)$, as a function of a. (You don't need to compute the numerical value, just obtain an expression to compute it). **[2.0]**

Answer: Writing the Fitzhugh-Nagumo equation with $t = 2$ it follows that:

$$v'(2) = v(3) - 2v(2) + v(1) + F(v(2)) \tag{1}$$

where, according to the conditions,

$$v(2) = 1 - e^{-4}. \tag{2}$$

Rewriting (1), we have

$$v(1) = v'(2) - v(3) + 2v(2) - F(v(2)) \tag{3}$$

where

$$v'(2) = 2e^{-4}. \tag{4}$$

Finally, differentiating both sides of (3),

$$v'(1) = v''(2) - v'(3) + 2v'(2) - F'(v(2))v'(2), \tag{5}$$

where

$$v''(2) = -4e^{-4},$$
$$v'(3) = 2e^{-6},$$
$$F'(v(2)) = -3v(2)^2 + 2(1+a)v(2) - a.$$

4. Consider the neural field equation

$$\frac{\partial V(x,t)}{\partial t} = -\alpha V(x,t) + \exp(-\alpha t) + \int_{-\infty}^{\infty} K(|x-y|)S(V(y,t))dy, \tag{6}$$

where $\alpha \in I\!R^{+}$,

$$K(x) = exp(-x^2),$$

$$S(v) = \begin{cases} 0, & \text{if} \quad v < 0.5 \\ 1, & \text{if} \quad v \geq 0.5. \end{cases}$$

(a) What is the physical meaning of K and S ? **[1.0]**

Answer: K is the connectivity function (represents the strength of the synapse between neurons at points x and y); S is the firing rate (the firing capacity of the neuron, as a function of its electric potential).

(b) Suppose the initial condition is $V(x,0) = \frac{1}{1+x^2}$. What is in this case the activation domain in the moment $t = 0$? **[1.0]**

Answer: Since the threshold of the firing rate function is 0.5, we have to solve the inequation:

$$\frac{1}{1+x^2} \geq 0.5;$$

this gives $-1 \leq x \leq 1$.

(c) Suppose now the initial condition is $V(x,0) = 0.2, \forall x \in I\!R$. What is in this case the value of the integral on the right-hand side of (1), when $t = 0$? **[1.0]**

Answer : Since $0.2 < 0.5$ we have $S = 0$ in the whole real axis and therefore the integral is 0.

(d) Under the assumption $V(x,0) = 0.2, \forall x \in I\!R$, show that the solution does not depend on x, for a certain time interval $t \in [0, t_0]$. Compute this solution, as a function of α. **[2.0]**

Suggestion: search for a solution in the form $V(t) = a\exp(-\alpha t) + bt\exp(-\alpha t)$.

Answer: As we have seen, since $V(x,0) = 0.2, \forall x \in I\!R$, the integral on the right-hand side of the equation is zero, and therefore equation (6) can be seen as an ordinary differential equation, so that the solution V does not depend on x, but only on t. This is will be true, while we have $V(x,t) < 0.5, \forall x \in I\!R$.

Equation (6) (without the integral term) is a non-homogeneous linear equation of first order with constant coefficients. The general solution of the associated homogeneous equation is of the form $V_0(t) = a\exp(-\lambda t)$, where λ is the root of the characteristic equation $\lambda - \alpha = 0$, that is $\lambda = \alpha$. Therefore $V_0(t) = a\exp(-\alpha t)$. Moreover, since the non-homogeneous term has the form $\exp(-\alpha t)$ and α is a root of the characteristic equation, the non-homogeneous equation has a particular solution of the form $V_1(t) = bt\exp(-\alpha t)$, for some constant b. By replacing $V_1(t)$ in equation (6), we easily find that $b = 1$. Therefore, the needed solution has the form

$$V(t) = V_0(t) + V_1(t) = a\exp(-\alpha t) + t\exp(-\alpha t),$$

for some constant a. Finally this constant can be found from the initial condition $V(0) = 0.2$, which gives us

$$a = 0.2.$$

Concluding we have

$$V(t) = (t + 0.2)\exp(-\alpha t).$$

(e) Based on the obtained solution, and assuming that $\alpha = 0.5$, answer to the question: is there a moment t_0 in which the solution V reaches the threshold 0.5 and the neurons become active? Answer to the same question in the case $\alpha = 1$.. **[2.0]**

Answer: It is easily observed that the solution $v(t)$ obtained in the previous item has a maximum at some $t* > 0$. To obtain the value of $t*$ we just have to solve the equation $V'(t*) = 0$:

$$\exp(-\alpha t*) - \alpha(t * + 0.2)\exp(-\alpha t*) = 0,$$

which gives $t* = \frac{1}{\alpha} - 0.2$.
Consider first the case $\alpha = 0.5$. In this case, we have $t* = 2 - 0.2 = 1.8$ and

$$\max_{t>0} V(t) = V(1.8) = 2\exp(-0.9) = 0.813.$$

Since $0.813 > 0.5$ we conclude that there is some point $t_0 < t*$, where the potential reaches the critical value 0.5 and the neurons become active.
Consider now the case $\alpha = 1$. In this case, we have $t* = 1 - 0.2 = 0.8$ and

$$\max_{t>0} V(t) = V(0.8) = \exp(-0.8) = 0.449.$$

Since $0.449 < 0.5$ we conclude that $V(t) < 0.5$, $\forall t > 0$, that is, the potential never reaches the threshold.