Riemannian Geometry

2^{nd} Test - January 22, 2021

LMAC and MMA

Solutions

a) According to Cartan's structure equations, we have

$$
\begin{aligned}
\mathrm{d}\omega^\psi = 0 &= \omega^\alpha \wedge \omega_\alpha^\psi, \\
&= \omega^\varphi \wedge \omega_\varphi^\psi + \omega^\theta \wedge \omega_\theta^\psi, \\
&= \sinh\psi\, \mathrm{d}\varphi \wedge \omega_\varphi^\psi + \cosh\psi\, \mathrm{d}\theta \wedge \omega_\theta^\psi, \\
\mathrm{d}\omega^\varphi = \cosh\psi\, \mathrm{d}\psi \wedge \mathrm{d}\varphi &= \omega^\alpha \wedge \omega_\alpha^\varphi, \\
&= \omega^\psi \wedge \omega_\psi^\varphi + \omega^\theta \wedge \omega_\theta^\varphi, \\
&= -\mathrm{d}\psi \wedge \omega_\varphi^\psi + \cosh\psi\, \mathrm{d}\theta \wedge \omega_\theta^\varphi, \\
\mathrm{d}\omega^\theta = \sinh\psi\, \mathrm{d}\psi \wedge \mathrm{d}\theta &= \omega^\alpha \wedge \omega_\alpha^\theta, \\
&= \omega^\psi \wedge \omega_\psi^\theta + \omega^\varphi \wedge \omega_\varphi^\theta, \\
&= -\mathrm{d}\psi \wedge \omega_\theta^\psi - \sinh\psi\, \mathrm{d}\varphi \wedge \omega_\theta^\varphi.
\end{aligned}
$$

We readily conclude that the nonzero connection forms are

$$
\omega_\varphi^\psi = -\cosh\psi\, \mathrm{d}\varphi, \quad \omega_\theta^\psi = -\sinh\psi\, \mathrm{d}\theta, \quad \omega_\theta^\varphi = 0.
$$

b) According to Cartan's structure equations, we have

$$
\begin{aligned}
\mathrm{d}\omega_\varphi^\psi = -\sinh\psi\, \mathrm{d}\psi \wedge \mathrm{d}\varphi &= \Omega_\varphi^\psi + \omega_\varphi^\alpha \wedge \omega_\alpha^\psi, \\
&= \Omega_\varphi^\psi + \omega_\varphi^\theta \wedge \omega_\theta^\psi, \\
&= \Omega_\varphi^\psi, \\
\mathrm{d}\omega_\theta^\psi = -\cosh\psi\, \mathrm{d}\psi \wedge \mathrm{d}\theta &= \Omega_\theta^\psi + \omega_\theta^\alpha \wedge \omega_\alpha^\psi, \\
&= \Omega_\theta^\psi + \omega_\theta^\varphi \wedge \omega_\varphi^\psi, \\
&= \Omega_\theta^\psi, \\
\mathrm{d}\omega_\theta^\varphi = 0 &= \Omega_\theta^\varphi + \omega_\theta^\alpha \wedge \omega_\alpha^\varphi, \\
&= \Omega_\theta^\varphi + \omega_\theta^\psi \wedge \omega_\psi^\varphi, \\
&= \Omega_\theta^\varphi - \sinh\psi\cosh\psi\, \mathrm{d}\theta \wedge \mathrm{d}\varphi, \\
&= \Omega_\theta^\varphi + \sinh\psi\cosh\psi\, \mathrm{d}\varphi \wedge \mathrm{d}\theta.
\end{aligned}
$$

Hence, we have

$$
\Omega_\varphi^\psi = -\omega^\psi \wedge \omega^\varphi, \quad \Omega_\theta^\psi = -\omega^\psi \wedge \omega^\theta, \quad \Omega_\theta^\varphi = -\omega^\varphi \wedge \omega^\theta.
$$

c) From

$$\begin{aligned}
\Omega^{\psi}_{\varphi} &= -\,\omega^{\psi}\wedge\omega^{\varphi} = \sum_{\alpha<\beta} R_{\alpha\beta\varphi}{}^{\psi}\omega^{\alpha}\wedge\omega^{\beta},\\
\Omega^{\psi}_{\theta} &= -\,\omega^{\psi}\wedge\omega^{\theta} = \sum_{\alpha<\beta} R_{\alpha\beta\theta}{}^{\psi}\omega^{\alpha}\wedge\omega^{\beta},\\
\Omega^{\varphi}_{\theta} &= -\,\omega^{\varphi}\wedge\omega^{\theta} = \sum_{\alpha<\beta} R_{\alpha\beta\theta}{}^{\varphi}\omega^{\alpha}\wedge\omega^{\beta},
\end{aligned}$$

it follows that, in particular, we have

$$R_{\psi\varphi\psi\varphi} = 1, \quad R_{\psi\theta\psi\theta} = 1, \quad R_{\varphi\theta\varphi\theta} = 1.$$

The curvature tensor is

$$R = \omega^{\psi}\wedge\omega^{\varphi}\otimes\omega^{\psi}\wedge\omega^{\varphi} + \omega^{\psi}\wedge\omega^{\theta}\otimes\omega^{\psi}\wedge\omega^{\theta} + \omega^{\varphi}\wedge\omega^{\theta}\otimes\omega^{\varphi}\wedge\omega^{\theta}.$$

Therefore, the sectional curvatures are

$$\begin{aligned}
K(\Pi_{\theta}) &= -\frac{R_{\psi\varphi\psi\varphi}}{g_{\psi\psi}g_{\varphi\varphi}-(g_{\psi\varphi})^2} = -1,\\
K(\Pi_{\varphi}) &= -\frac{R_{\psi\theta\psi\theta}}{g_{\psi\psi}g_{\theta\theta}-(g_{\psi\theta})^2} = -1,\\
K(\Pi_{\psi}) &= -\frac{R_{\varphi\theta\varphi\theta}}{g_{\varphi\varphi}g_{\theta\theta}-(g_{\psi\theta})^2} = -1.
\end{aligned}$$

This implies that the sectional curvature of any 2-plane is equal to -1.

d) The curves $\varphi \mapsto (\psi_0, \varphi, \tanh\psi_0)$ and $\psi \mapsto (\psi, \varphi_0, \tanh\psi)$ lie on M. They have tangent vectors $\partial_{\varphi} = \sinh\psi\, X_{\varphi}$ and

$$\begin{aligned}
\partial_{\psi} + \operatorname{sech}^2\!\psi\,\partial_{\theta} &= X_{\psi} + \operatorname{sech}\psi\, X_{\theta}\\
&= \frac{\sqrt{1+\cosh^2\psi}}{\cosh\psi}\left(\frac{\cosh\psi\, X_{\psi} + X_{\theta}}{\sqrt{1+\cosh^2\psi}}\right)\\
&= \frac{\sqrt{1+\cosh^2\psi}}{\cosh\psi}\, Y,
\end{aligned}$$

respectively.

e)

$$\begin{aligned}
\nabla_{X_{\varphi}}X_{\varphi} &= \omega^{\psi}\left(\nabla_{X_{\varphi}}X_{\varphi}\right)X_{\psi} + \omega^{\varphi}\left(\nabla_{X_{\varphi}}X_{\varphi}\right)X_{\varphi} + \omega^{\theta}\left(\nabla_{X_{\varphi}}X_{\varphi}\right)X_{\theta}\\
&= \omega^{\psi}_{\varphi}(X_{\varphi})X_{\psi} + \omega^{\varphi}_{\varphi}(X_{\varphi})X_{\varphi} + \omega^{\theta}_{\varphi}(X_{\varphi})X_{\theta}\\
&= -\cosh\psi\,\mathrm{d}\varphi\left(\frac{1}{\sinh\psi}\partial_{\varphi}\right)X_{\psi}\\
&= -\coth\psi\, X_{\psi}.
\end{aligned}$$

f) The equation

$$\begin{aligned} -\coth\psi\, X_\psi &= \nabla_{X_\varphi}X_\varphi = \overline{\nabla}_{X_\varphi}X_\varphi + (\nabla_{X_\varphi}X_\varphi, N)N \\ &= \overline{\nabla}_{X_\varphi}X_\varphi + \frac{\coth\psi}{\sqrt{1+\cosh^2\psi}}\left(\frac{-X_\psi + \cosh\psi\, X_\theta}{\sqrt{1+\cosh^2\psi}}\right) \end{aligned}$$

implies that

$$\begin{aligned} \overline{\nabla}_{X_\varphi}X_\varphi &= -\frac{\cosh^3\psi}{\sinh\psi\,(1+\cosh^2\psi)}X_\varphi - \frac{\cosh^2\psi}{\sinh\psi\,(1+\cosh^2\psi)}X_\theta \\ &= -\frac{\cosh^2\psi}{\sinh\psi\,\sqrt{1+\cosh^2\psi}}Y. \end{aligned}$$

The normal curvature and the geodesic curvature of the integral curves of X_φ are $\frac{\coth\psi}{\sqrt{1+\cosh^2\psi}}$ and $-\frac{\cosh^2\psi}{\sinh\psi\,\sqrt{1+\cosh^2\psi}}$, respectively.

g) The matrix that represents the second fundamental form of M in the basis (X_φ, Y) is

$$\begin{bmatrix} (-\nabla_{X_\varphi}N, X_\varphi) & (-\nabla_Y N, X_\varphi) \\ (-\nabla_{X_\varphi}N, Y) & (-\nabla_Y N, Y) \end{bmatrix} = \begin{bmatrix} \frac{\coth\psi}{\sqrt{1+\cosh^2\psi}} & 0 \\ 0 & \frac{\tanh\psi}{\sqrt{(1+\cosh^2\psi)^3}} \end{bmatrix}.$$

The curvature of M is

$$\begin{aligned} K^M &= -1 + \frac{g(B(X_\varphi,X_\varphi),B(Y,Y)) - g(B(X_\varphi,Y),B(X_\varphi,Y))}{g(X_\varphi,X_\varphi)g(Y,Y) - (g(X_\varphi,Y))^2} \\ &= -1 + \frac{1}{(1+\cosh^2\psi)^2}. \end{aligned}$$

h) Since K is nonvanishing, we may write the normal to the hypersurface S as $N = fK$, for some function f. For X and Y orthogonal to K, we have

$$\begin{aligned} (\nabla_X N, Y) &= (\nabla_X(fK), Y) = f(\nabla_X K, Y) \\ &= -f(X, \nabla_Y K) = -(X, \nabla_Y(fK)) \\ &= -(X, \nabla_Y N). \end{aligned}$$

This shows that the second fundamental form of S is anti-symmetric. But the second fundamental form is always symmetric. We conclude that the second fundamental form of S is zero. Therefore, S is totally geodesic.

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The following are auxiliary calculations that justify the information given in **g)**.

$$\nabla_{X_\psi} X_\psi = \omega_\psi^\psi(X_\psi)X_\psi + \omega_\psi^\varphi(X_\psi)X_\varphi + \omega_\psi^\theta(X_\psi)X_\theta = 0,$$

$$\nabla_{X_\psi} X_\theta = \omega_\theta^\psi(X_\psi)X_\psi + \omega_\theta^\varphi(X_\psi)X_\varphi + \omega_\theta^\theta(X_\psi)X_\theta = 0,$$

$$\begin{aligned}\nabla_{X_\varphi} X_\psi &= \omega_\psi^\psi(X_\varphi)X_\psi + \omega_\psi^\varphi(X_\varphi)X_\varphi + \omega_\psi^\theta(X_\varphi)X_\theta \\ &= \cosh\psi\,\mathrm{d}\varphi\left(\frac{1}{\sinh\psi}\partial_\varphi\right)X_\varphi \\ &= \coth\psi\,X_\varphi,\end{aligned}$$

$$\nabla_{X_\varphi} X_\theta = \omega_\theta^\psi(X_\varphi)X_\psi + \omega_\theta^\varphi(X_\varphi)X_\varphi + \omega_\theta^\theta(X_\varphi)X_\theta = 0,$$

$$\begin{aligned}\nabla_{X_\theta} X_\psi &= \omega_\psi^\psi(X_\theta)X_\psi + \omega_\psi^\varphi(X_\theta)X_\varphi + \omega_\psi^\theta(X_\theta)X_\theta \\ &= \sinh\psi\,\mathrm{d}\theta\left(\frac{1}{\cosh\psi}\partial_\theta\right)X_\theta \\ &= \tanh\psi\,X_\theta,\end{aligned}$$

$$\nabla_{X_\theta} X_\varphi = \omega_\varphi^\psi(X_\theta)X_\psi + \omega_\varphi^\varphi(X_\theta)X_\varphi + \omega_\varphi^\theta(X_\theta)X_\theta = 0,$$

$$\begin{aligned}\nabla_{X_\theta} X_\theta &= \omega_\theta^\psi(X_\theta)X_\psi + \omega_\theta^\varphi(X_\theta)X_\varphi + \omega_\theta^\theta(X_\theta)X_\theta \\ &= -\sinh\psi\,\mathrm{d}\theta\left(\frac{1}{\cosh\psi}\partial_\theta\right)X_\psi \\ &= -\tanh\psi\,X_\psi.\end{aligned}$$

We are now ready to calculate the covariant derivatives of N with respect to X_φ and Y:

$$\begin{aligned}\nabla_{X_\varphi} N &= -\frac{1}{\sqrt{1+\cosh^2\psi}}\nabla_{X_\varphi}X_\psi + \frac{\cosh\psi}{\sqrt{1+\cosh^2\psi}}\nabla_{X_\varphi}X_\theta \\ &= -\frac{1}{\sqrt{1+\cosh^2\psi}}\coth\psi\,X_\varphi,\end{aligned}$$

$$
\begin{aligned}
\nabla_Y N &= \frac{\cosh\psi}{\sqrt{1+\cosh^2\psi}}\,\nabla_{X_\psi}\left(-\frac{1}{\sqrt{1+\cosh^2\psi}}\,X_\psi\right) \\
&\quad+\frac{\cosh\psi}{\sqrt{1+\cosh^2\psi}}\,\nabla_{X_\psi}\left(\frac{\cosh\psi}{\sqrt{1+\cosh^2\psi}}\,X_\theta\right) \\
&\quad+\frac{1}{\sqrt{1+\cosh^2\psi}}\,\nabla_{X_\theta}\left(-\frac{1}{\sqrt{1+\cosh^2\psi}}\,X_\psi\right) \\
&\quad+\frac{1}{\sqrt{1+\cosh^2\psi}}\,\nabla_{X_\theta}\left(\frac{\cosh\psi}{\sqrt{1+\cosh^2\psi}}\,X_\theta\right) \\
&= \frac{\cosh^2\psi\sinh\psi}{(1+\cosh^2\psi)^2}\,X_\psi+\frac{\cosh\psi\sinh\psi}{(1+\cosh^2\psi)^2}\,X_\theta \\
&\quad-\frac{\tanh\psi}{1+\cosh^2\psi}\,X_\theta-\frac{\sinh\psi}{1+\cosh^2\psi}\,X_\psi \\
&= -\frac{\tanh\psi}{\sqrt{(1+\cosh^2\psi)^3}}\left(\frac{\cosh\psi\,X_\psi+X_\theta}{\sqrt{1+\cosh^2\psi}}\right) \\
&= -\frac{\tanh\psi}{\sqrt{(1+\cosh^2\psi)^3}}Y.
\end{aligned}
$$