Riemannian Geometry

2nd Test - January 21, 2020

LMAC and MMA

Solutions

a) The metric g in coordinates (x, y) is

$$\begin{aligned} g &= \phi^*(dx^2 + dy^2 + dz^2) = dx^2 + dy^2 + (d(xy))^2 \\ &= (1+y^2)\,dx^2 + (1+x^2)\,dy^2 + 2xy\,dxdy. \end{aligned}$$

b) The frame (E_1, E_2) is orthonormal since

$$\begin{aligned} g(E_1, E_1) &= (1+y^2)\left(\frac{x}{r}\right)^2 + (1+x^2)\left(\frac{-y}{r}\right)^2 + 2xy\left(-\frac{xy}{r^2}\right) \\ &= 1, \\ g(E_1, E_2) &= (1+y^2)\left(\frac{x}{r}\right)\left(\frac{y}{r\rho}\right) + (1+x^2)\left(\frac{-y}{r}\right)\left(\frac{x}{r\rho}\right) \\ &\quad + xy\left(\frac{x^2}{r^2\rho}\right) + xy\left(-\frac{y^2}{r^2\rho}\right) \\ &= 0, \\ g(E_2, E_2) &= (1+y^2)\left(\frac{y}{r\rho}\right)^2 + (1+x^2)\left(\frac{x}{r\rho}\right)^2 + 2xy\left(\frac{xy}{r^2\rho^2}\right) \\ &= 1. \end{aligned}$$

The dual of (E_1, E_2) is (ω^1, ω^2) since

$$\omega^1(E_1) = 1, \quad \omega^1(E_2) = 0, \quad \omega^2(E_1) = 0, \quad \omega^2(E_2) = 1.$$

The volume form is

$$\omega^1 \wedge \omega^2 = \rho\, dx \wedge dy.$$

c) Clearly,

$$d\left(\frac{1}{r}\right) = -\frac{x}{r^3}\,dx - \frac{y}{r^3}\,dy \quad \text{and} \quad d\rho = \frac{x}{\rho}\,dx + \frac{y}{\rho}\,dy.$$

Thus, we have that

$$\begin{aligned} d\omega^1 &= \left(-\frac{x}{r^3}\,dx - \frac{y}{r^3}\,dy\right) \wedge (x\,dx - y\,dy) \\ &= \frac{2xy}{r^3}\,dx \wedge dy. \end{aligned}$$

d) Writing $\omega^1_2 = a\,dx + b\,dy$, according to Cartan's structure equations,

$$\begin{aligned}
d\omega^1 &= \frac{2xy}{r^3}\,dx\wedge dy \;=\; \omega^2\wedge\omega^1_2\\
&= \frac{\rho}{r}(y\,dx + x\,dy)\wedge(a\,dx + b\,dy)\\
&= \frac{\rho}{r}(-ax + by)\,dx\wedge dy,\\
d\omega^2 &= \frac{1}{r^3\rho}(y^2 - x^2)\,dx\wedge dy \;=\; \omega^1\wedge(-\omega^1_2),\\
&= \frac{\rho}{r}(x\,dx - y\,dy)\wedge(-a\,dx - b\,dy)\\
&= \frac{1}{r}(-ay - bx)\,dx\wedge dy.
\end{aligned}$$

Thus, (a,b) solves the system

$$\begin{bmatrix} -x & y \\ -y & -x \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{r^2\rho}\begin{bmatrix} 2xy \\ y^2 - x^2 \end{bmatrix}$$

whose solution is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{r^2\rho}\begin{bmatrix} -y \\ x \end{bmatrix}.$$

We conclude that

$$\omega^1_2 = \frac{1}{r^2\rho}(-y\,dx + x\,dy).$$

e) According to Cartan's structure equations, we have

$$\begin{aligned}
\Omega^1_2 \;=\; d\omega^1_2 &= \frac{2}{r^2\rho}\,dx\wedge dy\\
&\quad - \frac{2}{r^4\rho}(x\,dx + y\,dy)\wedge(-y\,dx + x\,dy)\\
&\quad - \frac{1}{r^2\rho^3}(x\,dx + y\,dy)\wedge(-y\,dx + x\,dy)\\
&= \frac{1}{r^2\rho^3}(2\rho^2 - 2\rho^2 - r^2)\,dx\wedge dy\\
&= -\frac{1}{\rho^3}\,dx\wedge dy.
\end{aligned}$$

This shows that

$$\Omega^2_1 \;=\; \frac{1}{\rho^3}\,dx\wedge dy.$$

Since we have that

$$\Omega^2_1 = R_{121}{}^2\,\omega^1\wedge\omega^2,$$

it follows that

$$R_{121}{}^2 = \frac{1}{\rho^4}.$$

Finally, as the frame is orthonormal, $R_{121}{}^2 = R_{1212}$ and the curvature of $\mathcal{M}$ is

$$K = -R_{1212} = -\frac{1}{\rho^4}.$$

f) The hyperbola $\{(x, y) \in \mathbb{R}^2 : xy = \alpha\}$ can be parameterized by $r : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, defined by

$$r(x) = \left(x, \frac{\alpha}{x}\right).$$

Since

$$r'(x) = \left(1, -\frac{\alpha}{x^2}\right) = \frac{1}{x}\left(x, -\frac{\alpha}{x}\right),$$

$r'(x)$ is a multiple (the multiple $\frac{1}{x}$) of $(r^1(x), -r^2(x))$. This shows that E_1 is tangent to the hyperbola.
The geodesic curvature of the hyperbola is

$$k_g = \omega_1^2(E_1) = \frac{1}{r^2\rho}(y\,dx - x\,dy)\frac{(x\partial_x - y\partial_y)}{r} = \frac{2xy}{r^3\rho} = \frac{2\alpha}{r^3\rho}.$$

g) We have

$$\begin{aligned}
\tilde{E}_1 &= \phi_* E_1 = d\phi(E_1) = \frac{x}{r}\partial_x - \frac{y}{r}\partial_y + d\phi^3(E_1)\partial_z \\
&= \frac{x}{r}\partial_x - \frac{y}{r}\partial_y + (y\,dx + x\,dy)\left(\frac{x}{r}\partial_x - \frac{y}{r}\partial_y\right)\partial_z \\
&= \frac{x}{r}\partial_x - \frac{y}{r}\partial_y, \\
\tilde{E}_2 &= \phi_* E_2 = d\phi(E_2) = \frac{y}{r\rho}\partial_x + \frac{x}{r\rho}\partial_y + d\phi^3(E_2)\partial_z \\
&= \frac{y}{r\rho}\partial_x + \frac{x}{r\rho}\partial_y + (y\,dx + x\,dy)\left(\frac{y}{r\rho}\partial_x + \frac{x}{r\rho}\partial_y\right)\partial_z \\
&= \frac{1}{\rho}\left(\frac{y}{r}\partial_x + \frac{x}{r}\partial_y + r\partial_z\right).
\end{aligned}$$

These vectors have norm equal to one since E_1 and E_2 have norm equal to one. Moreover, they are orthogonal. Indeed, denoting the Euclidean norm in $\mathbb{R}^3$ by $\tilde{g}$, we have that

$$g(E_i, E_j) = \phi^*\tilde{g}(E_i, E_j) = \tilde{g}(\phi_* E_i, \phi_* E_j) = \tilde{g}(\tilde{E}_i, \tilde{E}_j).$$

The unit normal to the surface is given by

$$N := \tilde{E}_1 \times \tilde{E}_2 = \frac{1}{\rho}(-y\partial_x - x\partial_y + \partial_z).$$

h) Since the Christoffel symbols are zero, we have that

$$\begin{aligned}
\tilde{\nabla}_{\tilde{E}_1} N &= \frac{x}{r}\left(-\frac{1}{\rho}\partial_y\right) + \frac{x}{r}\left(-\frac{x}{\rho^3}\right)(-y\partial_x - x\partial_y + \partial_z) \\
&\quad -\frac{y}{r}\left(-\frac{1}{\rho}\partial_x\right) - \frac{y}{r}\left(-\frac{y}{\rho^3}\right)(-y\partial_x - x\partial_y + \partial_z) \\
&= \frac{1}{r\rho^3}\left(y(x^2 - y^2 + \rho^2)\partial_x - x(-x^2 + y^2 + \rho^2)\partial_y + (y^2 - x^2)\partial_z\right) \\
&= \frac{1}{r\rho^3}\left(y(1 + 2x^2)\partial_x - x(1 + 2y^2)\partial_y + (y^2 - x^2)\partial_z\right).
\end{aligned}$$

Although it was not asked, we also have that

$$\begin{aligned}
\tilde{\nabla}_{\tilde{E}_2} N &= \frac{y}{r\rho}\left(-\frac{1}{\rho}\partial_y\right) + \frac{y}{r\rho}\left(-\frac{x}{\rho^3}\right)(-y\partial_x - x\partial_y + \partial_z) \\
&\quad +\frac{x}{r\rho}\left(-\frac{1}{\rho}\partial_x\right) + \frac{x}{r\rho}\left(-\frac{y}{\rho^3}\right)(-y\partial_x - x\partial_y + \partial_z) \\
&= \frac{1}{r\rho^4}\left(-x(\rho^2 - 2y^2)\partial_x - y(\rho^2 - 2x^2)\partial_y - 2xy\partial_z\right) \\
&= -\frac{1}{r\rho^4}\left(x(1 + x^2 - y^2)\partial_x + y(1 - x^2 + y^2)\partial_y + 2xy\partial_z\right).
\end{aligned}$$

A trivial computation shows that

$$(\tilde{\nabla}_{\tilde{E}_1} N, \tilde{E}_1) = \frac{2xy}{r^2\rho}.$$

Moreover, we also have the following non required results:

$$\begin{aligned}
(\tilde{\nabla}_{\tilde{E}_1} N, \tilde{E}_2) = (\tilde{\nabla}_{\tilde{E}_2} N, \tilde{E}_1) &= \frac{1}{r^2\rho^4}(y^4 + y^2 - x^2 - x^4), \\
(\tilde{\nabla}_{\tilde{E}_2} N, \tilde{E}_2) &= -\frac{2xy}{r^2\rho^3}.
\end{aligned}$$

i) The matrix representation S of the second fundamental form of $\mathcal{M}$ with respect to the basis $(\tilde{E}_1, \tilde{E}_2)$ is

$$\begin{bmatrix} -(\tilde{\nabla}_{\tilde{E}_1} N, \tilde{E}_1) & -(\tilde{\nabla}_{\tilde{E}_2} N, \tilde{E}_1) \\ -(\tilde{\nabla}_{\tilde{E}_1} N, \tilde{E}_2) & -(\tilde{\nabla}_{\tilde{E}_2} N, \tilde{E}_2) \end{bmatrix}.$$

In fact, if we multiply this matrix on the right by the column vector

$$\begin{bmatrix} X^1 \\ X^2 \end{bmatrix},$$

corresponding to the vector $X = X^1\tilde{E}_1 + X^2\tilde{E}_2$, we obtain

$$\begin{aligned} \begin{bmatrix} -(X^1\tilde{\nabla}_{\tilde{E}_1}N, \tilde{E}_1) - (X^2\tilde{\nabla}_{\tilde{E}_2}N, \tilde{E}_1) \\ -(X^1\tilde{\nabla}_{\tilde{E}_1}N, \tilde{E}_2) - (X^2\tilde{\nabla}_{\tilde{E}_2}N, \tilde{E}_2) \end{bmatrix} &= \begin{bmatrix} -(\tilde{\nabla}_{X^1\tilde{E}_1+X^2\tilde{E}_2}N, \tilde{E}_1) \\ -(\tilde{\nabla}_{X^1\tilde{E}_1+X^2\tilde{E}_2}N, \tilde{E}_2) \end{bmatrix} \\ &= \begin{bmatrix} -(\tilde{\nabla}_X N, \tilde{E}_1) \\ -(\tilde{\nabla}_X N, \tilde{E}_2) \end{bmatrix} \\ &= \begin{bmatrix} (-\tilde{\nabla}_X N)^1 \\ (-\tilde{\nabla}_X N)^2 \end{bmatrix}, \end{aligned}$$

the components of $-\tilde{\nabla}_X N$ in the base $(\tilde{E}_1, \tilde{E}_2)$, since this base is orthonormal.
Although it was not required, we mention that in local coordinates (x, y) the matrix S is written as

$$\begin{bmatrix} -\frac{2xy}{r^2\rho} & -\frac{1}{r^2\rho^4}(y^4 + y^2 - x^2 - x^4) \\ -\frac{1}{r^2\rho^4}(y^4 + y^2 - x^2 - x^4) & \frac{2xy}{r^2\rho^3} \end{bmatrix}.$$

Of course, the determinant of this matrix is equal to the curvature of $\mathcal{M}$, that is $-\frac{1}{\rho^4}$.

j) According to the Gauss-Bonnet Theorem,

$$\iint_{[0,a]^2} K\,\omega^1 \wedge \omega^2 + \int_{\partial([0,a]^2)} k_g\,ds = 2\pi\chi([0,a]^2) = 2\pi.$$

But we have to be careful, this is only true for regular domains. To apply the theorem to a square, we have to approximate the square by smooth domains and then pass to the limit. We know that the integral of the geodesic curvature along a curve measures the change in direction of the tangent to the curve with respect to a vector field which is parallel along the curve. Now, the restriction of $\mathcal{M}$ to the boundary of the domain of integration consists of the curves parameterized by $\phi(\,\cdot\,, 0)$, $\phi(a, \,\cdot\,)$, $\phi(\,\cdot\,, a)$ and $\phi(0, \,\cdot\,)$, and these are segments in $\mathbb{R}^3$. Thus, they are images of geodesics of $\mathbb{R}^3$, and hence they are images of geodesics of $\mathcal{M}$. Therefore, the tangents to the boundaries of the domain of integration are parallel along these curves.
We conclude that, in the present case, the integral of the geodesic curvature of the boundary of the region of integration measures the

sum of the supplementary angles of the angles at the four vertices of the square, which we will call $\mathcal{R}$. We proceed to compute these angles.
The vector $(1,0)$ is unitary and tangent to the base of the square $\mathcal{R}$.
The vector $\left(0, \frac{1}{\sqrt{1+a^2}}\right)$ is unitary and tangent to the right side of the square $\mathcal{R}$.
The vector $\left(-\frac{1}{\sqrt{1+a^2}}, 0\right)$ is unitary and tangent to the top of the square $\mathcal{R}$.
The vector$(-1,0)$ is unitary and tangent to the left side of the square $\mathcal{R}$.
Hence, the angles of the vertices at $(0,0)$, $(a,0)$ and $(0,a)$ are right angles. However, we have that

$$g\left(\left(0, \frac{1}{\sqrt{1+a^2}}\right), \left(-\frac{1}{\sqrt{1+a^2}}, 0\right)\right) = -\frac{a^2}{1+a^2}$$

So, the angle β which measures the change of direction of the tangent to the boundary of $\mathcal{R}$ at (a,a) is such that

$$\beta = \arccos\left(-\frac{a^2}{1+a^2}\right)$$

(and the angle of the vertex at (a,a) is $\pi - \beta$).
We conclude that

$$\iint_{[0,a]^2} K\,\omega^1 \wedge \omega^2 + \frac{3\pi}{2} + \arccos\left(-\frac{a^2}{1+a^2}\right) = 2\pi.$$

This of course is equivalent to

$$I(a) = \iint_{[0,a]^2} K\,\omega^1 \wedge \omega^2 = \frac{\pi}{2} - \arccos\left(-\frac{a^2}{1+a^2}\right).$$

Clearly, it follows that

$$\lim_{a\to+\infty} I(a) = -\frac{\pi}{2}.$$