Riemannian Geometry

2[nd] Test - January 17, 2018

LMAC and MMA

Solutions

1.

a) The metric induced in $\mathcal{P}$ is

$$ds^2 = \left(\frac{1}{r-1} + 1\right) dr^2 + r^2\, d\varphi^2 = \frac{r}{r-1}\, dr^2 + r^2\, d\varphi^2.$$

b) The dual coframe is

$$(\omega^r, \omega^\varphi) = \left(\sqrt{\frac{r}{r-1}}\, dr, r\, d\varphi\right).$$

Using Cartan's structure equations, we get

$$\begin{aligned} d\omega^r &= 0 = \omega^\varphi \wedge \omega^r_\varphi, \\ d\omega^\varphi &= dr \wedge d\varphi = \omega^r \wedge \omega^\varphi_r = \sqrt{\frac{r}{r-1}}\, dr \wedge \omega^\varphi_r. \end{aligned}$$

This, together with $\omega^r_\varphi = -\omega^\varphi_r$, yields

$$\omega^\varphi_r = \sqrt{\frac{r-1}{r}}\, d\varphi = \frac{1}{r}\sqrt{\frac{r-1}{r}}\omega^\varphi.$$

c)

$$\begin{aligned} \Omega^\varphi_r = d\omega^\varphi_r &= \frac{1}{2}\sqrt{\frac{r}{r-1}}\frac{1}{r^2}\, dr \wedge d\varphi = R_{r\varphi r\varphi}\, \omega^r \wedge \omega^\varphi \\ &= R_{r\varphi r\varphi}\sqrt{\frac{r}{r-1}} r\, dr \wedge d\varphi. \end{aligned}$$

So

$$R_{r\varphi r\varphi} = \frac{1}{2r^3} \quad \text{and} \quad K = -\frac{1}{2r^3}.$$

d) We see that

$$\dot{c}(t) = \frac{1}{r}\partial_\varphi = E_\varphi.$$

Since $(E_\varphi, -E_r)$ is a positively oriented orthonormal frame,

$$\begin{aligned}\nabla_{\dot{c}(t)}\dot{c}(t) &= \nabla_{E_\varphi}E_\varphi = \left(\nabla_{E_\varphi}E_\varphi, -E_r\right)(-E_r) \\ &= -\omega^r_\varphi(E_\varphi)(-E_r) = \omega^\varphi_r(E_\varphi)(-E_r)\end{aligned}$$

Thus, the geodesic curvature of c is

$$k_g = \omega^\varphi_r(E_\varphi) = \frac{1}{r}\sqrt{\frac{r-1}{r}}.$$

e) According to the Gauss-Bonnet Theorem,

$$\int_{\mathcal{R}} K + \int_{\partial\mathcal{R}} k_g = 0 \tag{1}$$

because $\mathcal{R}$ is topologically a cylinder and so supports a nonzero vector field. Now,

$$\begin{aligned}\int_{\mathcal{R}} K &= \int_0^{2\pi}\int_1^{r_1} -\frac{1}{2r^3}\omega^r\wedge\omega^\varphi \\ &= \int_0^{2\pi}\int_1^{r_1} -\frac{1}{2r^2}\sqrt{\frac{r}{r-1}}\,drd\varphi \\ &= -2\pi\sqrt{\frac{r_1-1}{r_1}}.\end{aligned}$$

On the other hand,

$$\int_{\partial\mathcal{R}} k_g = \int_0^{2\pi}\frac{1}{r_1}\sqrt{\frac{r_1-1}{r_1}}\omega^\varphi = 2\pi\sqrt{\frac{r_1-1}{r_1}}.$$

So, (1) holds.

f)

$$\begin{aligned}\nabla_{E_\varphi}X &= \nabla_{E_\varphi}(X^rE_r + X^\varphi E_\varphi) \\ &= (E_\varphi\cdot X^r)E_r + X^r\nabla_{E_\varphi}E_r + (E_\varphi\cdot X^\varphi)E_\varphi + X^\varphi\nabla_{E_\varphi}E_\varphi \\ &= (E_\varphi\cdot X^r)E_r + X^r(\nabla_{E_\varphi}E_r, E_r)E_r + X^r(\nabla_{E_\varphi}E_r, E_\varphi)E_\varphi \\ &\quad +(E_\varphi\cdot X^\varphi)E_\varphi + X^\varphi(\nabla_{E_\varphi}E_\varphi, E_r)E_r + X^\varphi(\nabla_{E_\varphi}E_\varphi, E_\varphi)E_\varphi \\ &= (E_\varphi\cdot X^r)E_r + X^r\omega^\varphi_r(E_\varphi)E_\varphi + (E_\varphi\cdot X^\varphi)E_\varphi + X^\varphi\omega^r_\varphi(E_\varphi)E_r.\end{aligned}$$

If X is parallel along c, then

$$\begin{cases}\dot{X}^r - \frac{1}{r}\sqrt{\frac{r-1}{r}}X^\varphi = 0, \\ \dot{X}^\varphi + \frac{1}{r}\sqrt{\frac{r-1}{r}}X^r = 0.\end{cases}$$

If $X(c(0)) = X_0^r E_r + X_0^\varphi E_\varphi$, then

$$\begin{cases} X^r(c(t)) &= \cos\left(\frac{1}{r}\sqrt{\frac{r-1}{r}}t\right) X_0^r + \sin\left(\frac{1}{r}\sqrt{\frac{r-1}{r}}t\right) X_0^\varphi, \\ X^\varphi(c(t)) &= -\sin\left(\frac{1}{r}\sqrt{\frac{r-1}{r}}t\right) X_0^r + \cos\left(\frac{1}{r}\sqrt{\frac{r-1}{r}}t\right) X_0^\varphi. \end{cases}$$

In particular,

$$\begin{cases} X^r(c(2\pi r)) &= \cos\left(2\pi\sqrt{\frac{r-1}{r}}\right) X_0^r + \sin\left(2\pi\sqrt{\frac{r-1}{r}}\right) X_0^\varphi, \\ X^\varphi(c(2\pi r)) &= -\sin\left(2\pi\sqrt{\frac{r-1}{r}}\right) X_0^r + \cos\left(2\pi\sqrt{\frac{r-1}{r}}\varphi\right) X_0^\varphi. \end{cases}$$

g) $\mathcal{P}$ is parameterized by

$$p(r,\varphi) = (r\cos\varphi, r\sin\varphi, 2\sqrt{r-1}).$$

Tangent vectors to $\mathcal{P}$ are

$$\begin{aligned} \frac{\partial p}{\partial r} &= \cos\varphi\,\partial_x + \sin\varphi\,\partial_y + \frac{1}{\sqrt{r-1}}\,\partial_w \;=\; \partial_r + \frac{1}{\sqrt{r-1}}\,\partial_w, \\ \frac{\partial p}{\partial \varphi} &= -r\sin\varphi\,\partial_x + r\cos\varphi\,\partial_y \;=\; \partial_\varphi. \end{aligned}$$

$$E_r = \frac{\frac{\partial p}{\partial r}}{\left\|\frac{\partial p}{\partial r}\right\|} = \sqrt{\frac{r-1}{r}}\,\partial_r + \frac{1}{\sqrt{r}}\,\partial_w, \quad E_\varphi = \frac{\frac{\partial p}{\partial \varphi}}{\left\|\frac{\partial p}{\partial \varphi}\right\|} = \frac{1}{r}\,\partial_\varphi.$$

Note that $\partial_r \times E_\varphi = \partial_w$ and $\partial_w \times E_\varphi = -\,\partial_r$. The unit normal to $\mathcal{P}$ with positive ∂_w coordinate is

$$n = E_r \times E_\varphi = -\frac{1}{\sqrt{r}}\,\partial_r + \sqrt{\frac{r-1}{r}}\,\partial_w.$$

h) The action is

$$I(r,\varphi,w) = \frac{1}{2}\int (\dot{w}^2 + \dot{r}^2 + r^2\dot{\varphi}^2)\,dt.$$

The derivative of I in the direction of (ρ,θ,y) is

$$\begin{aligned} DI(r,\varphi,w)(\rho,\theta,y) &= \int (\dot{w}\dot{y} + \dot{r}\dot{\rho} + r\rho\dot{\varphi}^2 + r^2\dot{\varphi}\dot{\theta})\,dt \\ &= \int (-\ddot{w}y - \ddot{r}\rho + r\rho\dot{\varphi}^2 - 2r\dot{r}\dot{\varphi}\theta - r^2\ddot{\varphi}\theta)\,dt. \end{aligned}$$

Therefore, the equations for the geodesics are

$$\begin{cases} \ddot{w} = 0, \\ \ddot{r} - r\dot{\varphi}^2 = 0, \\ \ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0. \end{cases}$$

It follows that the nonzero Christoffel symbols are

$$\Gamma^r_{\varphi\varphi} = -r, \quad \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}.$$

i)

$$\begin{aligned}
\tilde{\nabla}_{E_r} E_r &= \sqrt{\frac{r-1}{r}}\tilde{\nabla}_{\partial_r}\left(\sqrt{\frac{r-1}{r}}\,\partial_r + \frac{1}{\sqrt{r}}\,\partial_w\right) \\
&\quad + \frac{1}{\sqrt{r}}\tilde{\nabla}_{\partial_w}\left(\sqrt{\frac{r-1}{r}}\,\partial_r + \frac{1}{\sqrt{r}}\,\partial_w\right) \\
&= \frac{1}{2r^2}\,\partial_r - \frac{\sqrt{r-1}}{2r^2}\,\partial_w, \\
\tilde{\nabla}_{E_\varphi} E_r &= \frac{1}{r}\tilde{\nabla}_{\partial_\varphi}\left(\sqrt{\frac{r-1}{r}}\,\partial_r + \frac{1}{\sqrt{r}}\,\partial_w\right) \\
&= \frac{1}{r^2}\sqrt{\frac{r-1}{r}}\,\partial_\varphi, \\
\tilde{\nabla}_{E_\varphi} E_\varphi &= \frac{1}{r}\tilde{\nabla}_{\partial_\varphi}\left(\frac{1}{r}\partial_\varphi\right) \\
&= -\frac{1}{r}\,\partial_r.
\end{aligned}$$

Recalling that $n = -\frac{1}{\sqrt{r}}\,\partial_r + \sqrt{\frac{r-1}{r}}\,\partial_w$, we get

$$\begin{aligned}
\left(n, \tilde{\nabla}_{E_r} E_r\right) &= -\frac{1}{2r^{3/2}}, \\
\left(n, \tilde{\nabla}_{E_\varphi} E_r\right) &= 0, \\
\left(n, \tilde{\nabla}_{E_\varphi} E_\varphi\right) &= \frac{1}{r^{3/2}}.
\end{aligned}$$

The second fundamental form of $\mathcal{P}$ is

$$\mathrm{II}(r,\varphi) = \begin{bmatrix} -\frac{1}{2r^{3/2}} & 0 \\ 0 & \frac{1}{r^{3/2}} \end{bmatrix}.$$