Riemannian Geometry, Fall 2016/17
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The 2nd Test, given on January 18, 2017, consists of this problem

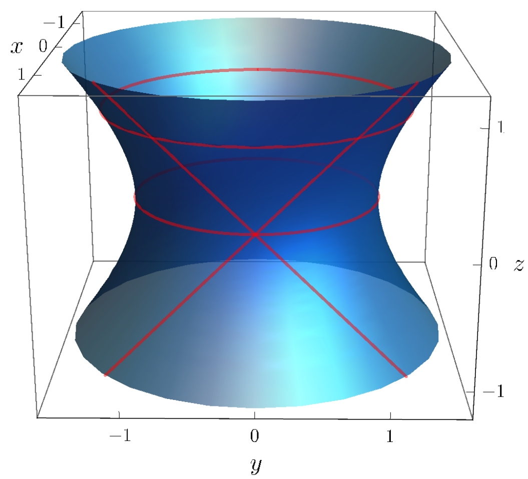
1. Consider the one-sheeted hyperboloid

$$M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\},$$

with coordinates

$$(x, y, z) = (\cosh\gamma\cos\theta, \cosh\gamma\sin\theta, \sinh\gamma),$$

and metric induced by the euclidean metric of $\mathbb{R}^3$.



a) The lines $x = 1 \wedge y = z$ e $x = 1 \wedge y = -z$ are contained in M. Are they geodesics of M?

b) Check that in coordinates (γ, θ) the metric is written as

$$ds^2 = \cosh(2\gamma)\, d\gamma^2 + \cosh^2\gamma\, d\theta^2.$$

c) Using the orthonormal frame

$$(E_\gamma, E_\theta) = \left(\frac{1}{\sqrt{\cosh(2\gamma)}}\frac{\partial}{\partial\gamma}, \frac{1}{\cosh\gamma}\frac{\partial}{\partial\theta}\right)$$

and Cartan's structure equations, show that M has curvature

$$K = -\frac{1}{\cosh^2(2\gamma)}.$$

Note: It might be useful to check that

$$\frac{d}{d\gamma}\frac{\sinh\gamma}{\sqrt{\cosh(2\gamma)}} = \frac{\cosh\gamma}{\sqrt{\cosh^3(2\gamma)}}, \qquad \frac{d}{d\gamma}\frac{\cosh\gamma}{\sqrt{\cosh(2\gamma)}} = -\frac{\sinh\gamma}{\sqrt{\cosh^3(2\gamma)}}.$$

d) Using the connection form ω_θ^γ, compute the geodesic curvature of the curve

$$c(s) = \left(\gamma_0, \frac{s}{\cosh\gamma_0}\right),$$

where $\gamma_0 \in \mathbb{R}_0^+$. Note that $\dot{c}(s) = (E_\theta)_{c(s)}$.

e) Obtain $\left(\tilde{\nabla}_{\dot{c}}\dot{c}\right)^\top$ by directly calculating $\tilde{\nabla}_{\dot{c}}\dot{c}$, and then projecting on E_γ and E_θ. Confirm your answer to the previous question.

f) Verify the equality of the Gauss-Bonnet Theorem applied to the portion of M with $0 \leq \gamma \leq \gamma_0$.

g) Compute $\int_M K$.

h) Compute the Gauss map, that associates to each point of M the unit normal n to M that points to the region in $\mathbb{R}^3$ in the interior of M.

i) Compute the matrix that represents the second fundamental form of M in the base (E_γ, E_θ). Confirm the result of **c)**.

j) Consider M only with its differential structure. Do there exist Riemannian metrics on M such that $\int_M K = 4\pi$? If so, give an example of one of them, writing it explicitly in the coordinates (γ, θ) above. If there do not exist metrics satisfying this condition, explain why.

1. Solution.

a) Yes, the two lines are images of geodesics of M. Indeed, consider two points on one of the lines. The segment joining them is the shortest path in $\mathbb{R}^3$ between the two points. Since this segment is contained in M, it is the shortest path among the family of curves in M joining the two points. Hence, the segment is the image of a geodesic in M.

b) As

$$\begin{aligned} dx &= \sinh\gamma\cos\theta\,d\gamma - \cosh\gamma\sin\theta\,d\theta, \\ dy &= \sinh\gamma\sin\theta\,d\gamma + \cosh\gamma\cos\theta\,d\theta, \\ dz &= \cosh\gamma\,d\theta, \end{aligned}$$

we have

$$ds^2 = dx^2 + dy^2 + dz^2 = \cosh(2\gamma)\,d\gamma^2 + \cosh^2\gamma\,d\theta^2.$$

c) The dual frame is

$$\omega^\gamma = \sqrt{\cosh(2\gamma)}\, d\gamma, \qquad \omega^\theta = \cosh\gamma\, d\theta.$$

Using the Cartran structure equations

$$\begin{aligned} d\omega^\gamma &= 0 = -\omega_\theta^\gamma \wedge \omega^\theta, \\ d\omega^\theta &= \sinh\gamma\, d\gamma \wedge d\theta = -\omega_\gamma^\theta \wedge \omega^\gamma = -\omega_\gamma^\theta \wedge \sqrt{\cosh(2\gamma)}\, d\gamma, \end{aligned}$$

we obtain the connection form

$$\omega_\gamma^\theta = \frac{\sinh\gamma}{\sqrt{\cosh(2\gamma)}}\, d\theta.$$

To get the curvature form, we compute

$$\begin{aligned} d\omega_\gamma^\theta &= \frac{\cosh\gamma}{\sqrt{\cosh^3(2\gamma)}}\, d\gamma \wedge d\theta = R_{\gamma\theta\gamma}{}^\theta\, \omega^\gamma \wedge \omega^\theta \\ &= R_{\gamma\theta\gamma\theta} \sqrt{\cosh(2\gamma)}\cosh\gamma\, d\gamma \wedge d\theta. \end{aligned}$$

So,

$$R_{\gamma\theta\gamma\theta} = \frac{1}{\cosh^2(2\gamma)}$$

and

$$K = -R_{\gamma\theta\gamma\theta} = -\frac{1}{\cosh^2(2\gamma)}.$$

d) The velocity is

$$\dot{c}(s) = \frac{1}{\cosh\gamma_0}\left(\frac{\partial}{\partial\theta}\right) = E_\theta.$$

Note that $(E_\theta, -E_\gamma)$ has positive orientation. The geodesic curvature is

$$\begin{aligned} k_g &= (\nabla_{E_\theta} E_\theta, -E_\gamma) = -\omega_\theta^\gamma(E_\theta) = \omega_\gamma^\theta(E_\theta) \\ &= \frac{\sinh\gamma}{\sqrt{\cosh(2\gamma)}}\, d\theta\left(\frac{1}{\cosh\gamma}\frac{\partial}{\partial\theta}\right) = \frac{\tanh\gamma}{\sqrt{\cosh(2\gamma)}}. \end{aligned}$$

e) The derivatives of c are

$$\begin{aligned} c(s) &= \left(\cosh\gamma_0 \cos\left(\frac{s}{\cosh\gamma_0}\right), \cosh\gamma_0 \sin\left(\frac{s}{\cosh\gamma_0}\right), \sinh\gamma_0\right), \\ \dot{c}(s) &= \left(-\sin\left(\frac{s}{\cosh\gamma_0}\right), \cos\left(\frac{s}{\cosh\gamma_0}\right), 0\right), \\ \tilde{\nabla}_{\dot{c}(s)}\dot{c}(s) &= \ddot{c}(s) = -\frac{1}{\cosh\gamma_0}\left(\cos\left(\frac{s}{\cosh\gamma_0}\right), \sin\left(\frac{s}{\cosh\gamma_0}\right), 0\right). \end{aligned}$$

The vectors E_γ and E_θ have coordinates in $\mathbb{R}^3$ equal to

$$\begin{aligned} E_\gamma &= \frac{1}{\sqrt{\cosh(2\gamma)}}(\sinh\gamma\cos\theta, \sinh\gamma\sin\theta, \cosh\gamma), \\ E_\theta &= \frac{1}{\cosh\gamma}(-\cosh\gamma\sin\theta, \cosh\gamma\cos\theta, 0). \end{aligned}$$

We compute the projections of $\tilde{\nabla}_{\dot{c}(s)}\dot{c}(s)$ on E_γ and E_θ:

$$\left(\tilde{\nabla}_{\dot{c}(s)}\dot{c}(s), E_\gamma\right) = -\frac{\tanh\gamma}{\sqrt{\cosh(2\gamma)}}, \qquad \left(\tilde{\nabla}_{\dot{c}(s)}\dot{c}(s), E_\theta\right) = 0.$$

We conclude that

$$\begin{aligned} \left(\nabla_{\dot{c}(s)}\dot{c}(s)\right)^\top &= \left(\tilde{\nabla}_{\dot{c}(s)}\dot{c}(s), -E_\gamma\right)(-E_\gamma) + \left(\tilde{\nabla}_{\dot{c}(s)}\dot{c}(s), E_\theta\right)E_\theta \\ &= \frac{\tanh\gamma}{\sqrt{\cosh(2\gamma)}}(-E_\gamma). \end{aligned}$$

This confirms that the geodesic curvature of c is $\frac{\tanh\gamma_0}{\sqrt{\cosh(2\gamma_0)}}$.

f) Let S be the portion of M with $0 \le \gamma \le \gamma_0$. According to the Gauss-Bonnet Theorem,

$$\int_S K + \int_{\partial S} k_g = 2\pi\chi = 0,$$

because the Euler characteristic of the S is 0 (as $\frac{\partial}{\partial\gamma}$ is a nonvanishing vector field). We verify this equality by noting that the two terms on the left-hand side cancel each other:

$$\begin{aligned} \int_S K &= \int_S -\frac{1}{\cosh^2(2\gamma)}\,\omega^\gamma\wedge\omega^\theta \\ &= -\int_S \frac{1}{\cosh^2(2\gamma)}\sqrt{\cosh(2\gamma)}\cosh\gamma\, d\gamma\wedge d\theta \\ &= -2\pi\int_0^{\gamma_0}\frac{\cosh\gamma}{\sqrt{\cosh^3(2\gamma)}}\,d\gamma \\ &= -2\pi\frac{\sinh\gamma_0}{\sqrt{\cosh(2\gamma_0)}}, \end{aligned}$$

$$\int_{\partial S} k_g = \int_{-\pi}^{\pi}\frac{\tanh\gamma_0}{\sqrt{\cosh(2\gamma_0)}}\cosh\gamma_0\,d\theta = 2\pi\frac{\sinh\gamma_0}{\sqrt{\cosh(2\gamma_0)}}.$$

The frame (E_γ, E_θ) is positively oriented. We remark that E_γ points out of S on the portion of ∂S where $\gamma = \gamma_0 > 0$, and so E_θ gives the positive orientation of the portion of ∂S where $\gamma = \gamma_0 > 0$. The orientation of $\gamma = 0$ is $-E_\theta$.

g) The integral of the curvature on M is

$$\int_M K = 2\lim_{\gamma_0\to\infty}\int_S K = -4\pi\lim_{\gamma_0\to\infty}\frac{\sinh\gamma_0}{\sqrt{\cosh(2\gamma_0)}} = -2\sqrt{2}\pi.$$

h) The normal to M which points to the region in $\mathbb{R}^3$ in the interior of M is

$$\begin{aligned} n &= -\frac{\nabla(x^2+y^2-z^2)}{\|\nabla(x^2+y^2-z^2)\|} = \frac{(-x,-y,z)}{\sqrt{x^2+y^2+z^2}} \\ &= \frac{1}{\sqrt{\cosh(2\gamma)}}(-\cosh\gamma\cos\theta,-\cosh\gamma\sin\theta,\sinh\gamma) \\ &= E_\gamma\times E_\theta. \end{aligned}$$

i) The derivative of n (considered as a function of (γ,θ)) is

$$(Dn)(\gamma,\theta)=\begin{bmatrix} \frac{\sinh\gamma}{\sqrt{\cosh^3(2\gamma)}}\cos\theta & \frac{\cosh\gamma}{\sqrt{\cosh(2\gamma)}}\sin\theta \\ \frac{\sinh\gamma}{\sqrt{\cosh^3(2\gamma)}}\sin\theta & -\frac{\cosh\gamma}{\sqrt{\cosh(2\gamma)}}\cos\theta \\ \frac{\cosh\gamma}{\sqrt{\cosh^3(2\gamma)}} & 0 \end{bmatrix}$$

The derivatives of n with respect to E_γ and E_θ are

$$\begin{aligned} \tilde{\nabla}_{E_\gamma}n &= \frac{1}{\sqrt{\cosh\gamma}}\frac{\partial}{\partial\gamma}n = \frac{1}{\cosh^2(2\gamma)}(\sinh\gamma\cos\theta,\sinh\gamma\sin\theta,\cosh\gamma), \\ \tilde{\nabla}_{E_\theta}n &= \frac{1}{\cosh\gamma}\frac{\partial}{\partial\theta}n = \frac{1}{\sqrt{\cosh(2\gamma)}}(\sin\theta,-\cos\theta,0). \end{aligned}$$

The components of the second fundamental form are

$$\begin{aligned} &\left(-\tilde{\nabla}_{E_\gamma}n,E_\gamma\right) = -\frac{1}{\sqrt{\cosh^3(2\gamma)}}, && \left(-\tilde{\nabla}_{E_\theta}n,E_\gamma\right)=0, \\ &\left(-\tilde{\nabla}_{E_\gamma}n,E_\theta\right)=0, && \left(-\tilde{\nabla}_{E_\theta}n,E_\theta\right)=\frac{1}{\sqrt{\cosh(2\gamma)}}. \end{aligned}$$

The second fundamental form is

$$\mathrm{II}(\gamma,\theta)=\begin{bmatrix} -\frac{1}{\sqrt{\cosh^3(2\gamma)}} & 0 \\ 0 & \frac{1}{\sqrt{\cosh(2\gamma)}} \end{bmatrix}.$$

This confirms the result of **c)** as

$$K=\det\mathrm{II}=-\frac{1}{\cosh^2(2\gamma)}.$$

j) We know that the integral of the curvature of S^2 is 4π. So, the integral of the curvature of S^2 without the North and South poles is also 4π. S^2 without the North and South poles is diffeomorphic to a cylinder. Thus, all we have to do is pull-back the metric of the S^2 without the North and South poles to the cylinder. The metric of S^2 is

$$ds^2 = d\sigma^2 + \sin^2\sigma\, d\theta^2.$$

We consider the diffeomorphism

$$(\gamma,\theta) \longrightarrow (\sigma,\theta) \qquad \text{with } \sigma = \frac{\pi}{2} - \arctan\gamma.$$

Note that

$$\begin{aligned} \gamma = +\infty &\ \Rightarrow\ \sigma = 0, \\ \gamma = -\infty &\ \Rightarrow\ \sigma = \pi. \end{aligned}$$

In order to compute the pull-back of the metric, note that

$$d\sigma = \frac{1}{1+\gamma^2}\, d\gamma$$

and

$$\begin{aligned} \sin\sigma &= \sin\left(\frac{\pi}{2} - \arctan\gamma\right) = \cos(\arctan\gamma) \\ &= \cos z \qquad \text{when } z = \arctan\gamma \Leftrightarrow \gamma = \tan z \\ &= \frac{1}{\sqrt{1+\gamma^2}}. \end{aligned}$$

Hence, the pull-back of the metric of the S^2 without the North and South poles to the cylinder is

$$ds^2 = \frac{1}{(1+\gamma^2)^2}\, d\gamma^2 + \frac{1}{1+\gamma^2}\, d\theta^2.$$