# Riemannian Geometry

2nd Test - January 22, 2021

LMAC and MMA

Duration: 90 minutes

**Show your calculations**

Consider $\mathbb{R}^2 \times S^1$ with metric

$$g = \mathrm{d}\psi^2 + \sinh^2\psi\,\mathrm{d}\varphi^2 + \cosh^2\psi\,\mathrm{d}\theta^2.$$

The frames

$$X_\psi = \partial_\psi, \quad X_\varphi = \frac{1}{\sinh\psi}\partial_\varphi, \quad X_\theta = \frac{1}{\cosh\psi}\partial_\theta,$$

and

$$\omega^\psi = \mathrm{d}\psi, \quad \omega^\varphi = \sinh\psi\,\mathrm{d}\varphi, \quad \omega^\theta = \cosh\psi\,\mathrm{d}\theta$$

are dual and orthonormal.

**a)** Check that the connection forms are given by (2)

$$\omega^\psi_\varphi = -\cosh\psi\,\mathrm{d}\varphi, \quad \omega^\psi_\theta = -\sinh\psi\,\mathrm{d}\theta, \quad \omega^\varphi_\theta = 0.$$

**b)** Show that the curvature forms are given by (2)

$$\Omega^\psi_\varphi = -\,\omega^\psi \wedge \omega^\varphi, \quad \Omega^\psi_\theta = -\omega^\psi \wedge \omega^\theta, \quad \Omega^\varphi_\theta = -\omega^\varphi \wedge \omega^\theta.$$

**c)** Show that the sectional curvatures of the planes spanned by any two of $\{X_\psi, X_\varphi, X_\theta\}$ are all equal. This, of course, implies that the manifold has constant sectional curvature. Write the curvature tensor. (2)

**d)** Consider the two dimensional manifold $M$ described by $\theta = \tanh\psi$, for $\psi > 0$. The frame (2)

$$(X_\varphi, Y, N) := \left( X_\varphi, \frac{\cosh\psi\,X_\psi + X_\theta}{\sqrt{1+\cosh^2\psi}}, \frac{-X_\psi + \cosh\psi\,X_\theta}{\sqrt{1+\cosh^2\psi}} \right)$$

is orthonormal. Show that $X_\varphi$ and $Y$ are tangent to $M$.

**e)** Using the connection forms of the metric $g$, calculate $\nabla_{X_\varphi} X_\varphi$, where $\nabla$ denotes the Levi-Civita connection of $g$. (3)

**f)** Calculate $\overline{\nabla}_{X_\varphi} X_\varphi$, where $\overline{\nabla}$ is the Levi-Civita connection on $M$. What are the normal curvature and the geodesic curvature of the integral curves of $X_\varphi$? (4)

**g)** Knowing that (3)

$$\nabla_{X_\varphi} N = -\frac{\coth\psi}{\sqrt{1+\cosh^2\psi}} X_\varphi, \quad \nabla_Y N = -\frac{\tanh\psi}{\sqrt{(1+\cosh^2\psi)^3}} Y,$$

calculate the matrix that represents the second fundamental form of $M$ in the basis $(X_\varphi, Y)$. Use it to calculate the curvature of $M$.

**h)** Recall that a Killing vector field $K$ is a vector field satisfying (2)

$$(\nabla_X K, Y) + (\nabla_Y K, X) = 0, \quad \text{for all vector fields } X \text{ and } Y$$

Suppose that $S$ is a hypersurface orthogonal to a nonvanishing Killing vector field $K$ (as $\varphi =$ constant is orthogonal to $\partial_\varphi$, or as $\theta =$ constant is orthogonal to $\partial_\theta$, for example). Show that $S$ is totally geodesic.