Riemannian Geometry

2^{nd} Test - January 21, 2020

LMAC and MMA

Duration: 180 minutes

Show your calculations

Consider $\mathbb{R}^3$ with the Euclidean metric, and consider the hyperbolic paraboloid $\mathcal{M}$, parameterized by $\phi : \mathbb{R}^2 \to \mathbb{R}^3$, defined by

$$\phi(x, y) = (x, y, xy),$$

with the induced metric g.

a) Write the metric g in the local coordinates (x, y). (1)

b) Consider the frame (1)

$$(E_1, E_2) = \left(\frac{x\partial_x - y\partial_y}{r}, \frac{y\partial_x + x\partial_y}{r\rho} \right),$$

where

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \rho = \sqrt{1 + x^2 + y^2}.$$

Check that it is orthonormal and that its dual frame is

$$(\omega^1, \omega^2) = \left(\frac{x\,dx - y\,dy}{r}, \frac{\rho\,(y\,dx + x\,dy)}{r} \right).$$

Calculate the volume form $\omega^1 \wedge \omega^2$.

c) Check that (2)

$$d\left(\frac{1}{r}\right) = -\frac{x}{r^3}\,dx - \frac{y}{r^3}\,dy \quad \text{and} \quad d\rho = \frac{x}{\rho}\,dx + \frac{y}{\rho}\,dy.$$

Compute $d\omega^1$. Another simple computation shows that

$$d\omega^2 = \frac{1}{r^3\rho}(y^2 - x^2)\,dx \wedge dy.$$

d) Writing the connection form ω^1_2 as $\omega^1_2 = a\,dx + b\,dy$, arrive at a system of linear equations for (a, b), and check that its solution is (2)

$$(a, b) = \frac{1}{r^2\rho}(-y, x).$$

e) Check that the curvature form Ω_1^2 is given by (2)

$$\Omega_1^2 \;=\; \frac{1}{\rho^3}\, dx \wedge dy.$$

What is the curvature of $\mathcal{M}$?

f) Let $\alpha \in \mathbb{R} \setminus \{0\}$. Show that E_1 is tangent to the hyperbola $\{(x, y) \in \mathbb{R}^2 : xy = \alpha\}$. Compute its geodesic curvature. (2)

g) Compute $\tilde{E}_1 = \phi_* E_1$ and $\tilde{E}_2 = \phi_* E_2$. What are the norms of $\tilde{E}_1$ and $\tilde{E}_2$? Why? Calculate $N := \tilde{E}_1 \times \tilde{E}_2$. (2)

h) Denote by $\tilde{\nabla}$ the Levi-Civita connection of $\mathbb{R}^3$ with the Euclidean metric. Calculate $\tilde{\nabla}_{\tilde{E}_1} N$ and $(\tilde{\nabla}_{\tilde{E}_1} N, \tilde{E}_1)$. (3)

i) What is the matrix representation S of the second fundamental form of $\mathcal{M}$ with respect to the basis $(\tilde{E}_1, \tilde{E}_2)$? Explain. You don't have to compute the entries of S in terms of the coordinates x and y. (2)

j) Let $a > 0$. Use the Gauss-Bonnet Theorem and the geometric meaning of the geodesic curvature to determine (3)

$$I(a) := \iint_{[0,a]^2} K\, \omega^1 \wedge \omega^2.$$

What is $\lim_{a \to +\infty} I(a)$?
Suggestion: What are $\phi(a, y)$ and $\phi(x, a)$?