Riemannian Geometry

2^{nd} Test - January 17, 2018

LMAC and MMA

Duration: 120 minutes

Show your calculations

1. Consider $\mathbb{R}^3$ with the euclidean metric, here written in cylindrical coordinates,

$$ds^2 = dw^2 + dr^2 + r^2\, d\varphi^2.$$

Consider also the Flamm paraboloid $\mathcal{P}$, defined by

$$\left\{(w, r, \varphi) \in [0, +\infty[\times[1, +\infty[\times[0, 2\pi[: w = 2\sqrt{r-1}\right\}$$

with the induced metric.

a) Write the metric induced on $\mathcal{P}$. (2)

b) Consider the frame (2)

$$(E_r, E_\varphi) = \left(\sqrt{\frac{r-1}{r}}\partial_r, \frac{1}{r}\partial_\varphi\right)$$

and the dual coframe $(\omega^r, \omega^\varphi)$. Calculate the connection form ω_r^φ.

c) Compute the curvature form Ω_r^φ, $R_{r\varphi r\varphi}$ and the curvature K of $\mathcal{P}$. (2)

d) Consider the curve c parameterized by (2)

$$c(t) = \left(r, \frac{t}{r}\right).$$

Calculate the geodesic curvature of c.

e) Check the Gauss-Bonnet Theorem for the region $\mathcal{R} = [1, r_1] \times [0, 2\pi[$. (2)

f) Let $X = X^r E_r + X^\varphi E_\varphi$ and ∇ be the Levi-Civita connection for the metric induced on $\mathcal{P}$. Write $\nabla_{E_\varphi} X$ in terms of $E_\varphi \cdot X^r$, $E_\varphi \cdot X^\varphi$ and $\omega_r^\varphi(E_\varphi)$. Write the first order system satisfied by (X^r, X^φ) if X is parallel along the curve c. Solve the system if $X(c(0)) = X_0^r E_r + X_0^\varphi E_\varphi$. Write $X(c(2\pi r))$. (3)

g) Write E_r and E_φ in the frame $(\partial_r, \partial_\varphi, \partial_w)$. Write also the unit normal to $\mathcal{P}$ with positive ∂_w component in the frame $(\partial_r, \partial_\varphi, \partial_w)$. (2)

h) Obtain the equations for the geodesics of the euclidean metric in cylindrical coordinates and write the Christoffel symbols. (2)

i) Let $\tilde{\nabla}$ be the Levi-Civita connection for the euclidean metric on $\mathbb{R}^3$. Calculate $\tilde{\nabla}_{E_r} E_r$, $\tilde{\nabla}_{E_\varphi} E_r$ and $\tilde{\nabla}_{E_\varphi} E_\varphi$. Calculate the second fundamental form of $\mathcal{P}$. (3)