Riemannian Geometry

1^{st} Test - November 20, 2020

LMAC and MMA

Solutions

1.

a)

$$
\begin{aligned}
g &= du^2 + \sinh^2 u \, d\varphi^2, \\
(\omega^u, \omega^\varphi) &= (du, \sinh u \, d\varphi), \\
\omega &= \sinh u \, du \wedge d\varphi.
\end{aligned}
$$

b)

$$
\begin{aligned}
\omega^u_\varphi &= -\omega^\varphi_u = -\cosh u \, d\varphi, \\
\Omega^u_\varphi &= -\sinh u \, du \wedge d\varphi, \\
K &\equiv -1.
\end{aligned}
$$

c)

$$
\int_D K = \int_D \Omega^u_\varphi = \int_{\partial D} \omega^u_\varphi = \int_{\partial D} -\cosh u \, d\varphi = -2\pi \cosh R + 2\pi.
$$

d)

$$
\begin{aligned}
k_g &= \cosh u \, d\varphi \left(\frac{1}{\sinh u} \partial_\varphi \right) \Bigg|_{u=R} = \coth R, \\
ds &= \sqrt{g(\partial_\varphi, \partial_\varphi)} \, d\varphi = \sinh R \, d\varphi, \\
\int_{\{R\} \times (0,2\pi)} k_g \, ds &= 2\pi \cosh R.
\end{aligned}
$$

e)

$$\begin{aligned}
L_X\omega &= X\cdot \sinh u\, du\wedge d\varphi + \sinh u\, d\left(\frac{\cos\varphi}{\cosh u}\right)\wedge d\varphi \\
&\quad - \sinh u\, du \wedge d\left(\frac{\sin\varphi}{\sinh u}\right) \\
&= -\frac{\sinh^2 u\cos\varphi}{\cosh^2 u}\, du\wedge d\varphi \\
&= -\frac{\sinh u\cos\varphi}{\cosh^2 u}\,\omega, \\
\iota(X)\omega &= \tanh u\cos\varphi\, d\varphi + \sin\varphi\, du, \\
d\iota(X)\omega &= \frac{\cos\varphi}{\cosh^2 u}\, du\wedge d\varphi - \cos\varphi\, du\wedge d\varphi, \\
L_X\omega &= d\iota(X)\omega + \iota(X)d\omega = d\iota(X)\omega, \\
\operatorname{div} X &= -\frac{\sinh u\cos\varphi}{\cosh^2 u} = -\frac{x}{1+x^2+y^2}.
\end{aligned}$$

2. *First solution.* Let ϕ_t be the flow of X and ψ_t be the flow of Y. The assertion that $[X,Y]=0$ is equivalent to $\phi_t\circ\psi_s = \psi_s\circ\phi_t$ for all s and t. For all $g\in G$, since X is left invariant, we know that $(L_g)_\star X = X \Leftrightarrow \phi_t\circ L_g = L_g\circ\phi_t \Leftrightarrow \phi_t(g) = g\phi_t(e)$, and, since Y is right invariant, we know that $(R_g)_\star X = X \Leftrightarrow \psi_s\circ R_g = R_g\circ\psi_s \Leftrightarrow \psi_s(g) = \psi_s(e)g$. Hence, we have

$$\phi_t(\psi_s(g)) = \phi_t(\psi_s(e)g) = \psi_s(e)g\phi_t(e) = \psi_s(g\phi_t(e)) = \psi_s(\phi_t(g)).$$

As g is arbitrary, $\phi_t\circ\psi_s = \psi_s\circ\phi_t$.
Second solution.

$$[X,Y] = \frac{d}{dt}(\phi_{-t})_\star Y\bigg|_{t=0} = \frac{d}{dt}(R_{\phi_{-t}(e)})_\star Y\bigg|_{t=0} = \frac{d}{dt}Y\bigg|_{t=0} = 0.$$

3.

a) *First solution.* The vector field $(d\phi_t)X$ has flow $\phi_t(\phi_s(\,\cdot\,)) = \phi_{t+s}(\,\cdot\,)$ and so is X.
Second solution. Let $g: M\to N$ and let $f: N\to\mathbb{R}$. We recall that

$$Y\cdot(f\circ g) = ((dg)Y\cdot f)\circ g.$$

Thus, we have

$$
\begin{aligned}
((d\phi_t)X \cdot f)(p) &= ((d\phi_t)X \cdot f \circ \phi_t \circ \phi_{-t})(p) \\
&= (X \cdot (f \circ \phi_t) \circ \phi_{-t})(p) \\
&= (X \cdot (f \circ \phi_t) \circ \phi_s \circ \phi_{-t-s})(p) \\
&= \left(\frac{d}{ds}(f \circ \phi_t \circ \phi_s) \circ \phi_{-t-s}\right)(p) \\
&= \left(\frac{d}{d(s+t)}(f \circ \phi_{t+s}) \circ \phi_{-t-s}\right)(p) \\
&= (X \cdot f \circ \phi_{t+s} \circ \phi_{-t-s})(p) \\
&= (X \cdot f)(p).
\end{aligned}
$$

b) *First solution.*

$$
\begin{aligned}
\frac{d}{dt}(d\phi_{-t})_{\phi_t(p)}Y_{\phi_t(p)}\bigg|_{t=t} &= \frac{d}{ds}(d\phi_{-t-s})_{\phi_{t+s}(p)}Y_{\phi_{t+s}(p)}\bigg|_{s=0} \\
&= \frac{d}{ds}(d\phi_{-s})_{\phi_s(p)}(d\phi_{-t})_{\phi_t(\phi_s(p))}Y_{\phi_t(\phi_s(p))}\bigg|_{s=0} \\
&= \frac{d}{ds}(d\phi_{-s})_{\phi_s(p)}\left((d\phi_{-t})_{\phi_t(\,\cdot\,)}Y_{\phi_t(\,\cdot\,)}\right)_{\phi_s(p)}\bigg|_{s=0} \\
&= [X, (d\phi_{-t})Y](p)
\end{aligned}
$$

Second solution.

$$
\begin{aligned}
\frac{d}{dt}((d\phi_{-t})_{\phi_t(p)}Y_{\phi_t(p)}) \cdot f &= \frac{d}{dt}(((d\phi_{-t})Y) \cdot f)(p) \\
&= \frac{d}{dt}(((d\phi_{-t})Y) \cdot f \circ \phi_{-t} \circ \phi_t)(p) \\
&= \frac{d}{dt}(Y \cdot (f \circ \phi_{-t}) \circ \phi_t)(p) \\
&= [(-Y \cdot ((X \cdot f) \circ \phi_{-t}) \circ \phi_t) \\
&\quad +X \cdot Y \cdot (f \circ \phi_{-t}) \circ \phi_t](p) \\
&= (-(d\phi_{-t})Y \cdot X \cdot f + (d\phi_{-t})X \cdot (d\phi_{-t})Y \cdot f)(p) \\
&= ([X, (d\phi_{-t})Y] \cdot f)(p),
\end{aligned}
$$

as $(d\phi_{-t})X = X$.