Riemannian Geometry

1^{st} Test - November 15, 2019

LMAC and MMA

Solutions

1.

a) Let $f : U(n) \to S^1$ be defined by $f(A) = \det A$. Indeed, for $A \in U(n)$,

$$1 = \det I = \det(A^*A) = \det(A^*)\det A = \overline{\det A}\det A = |\det A|^2,$$

so f has range in S^1. We show that 1 is a regular value of f. Note that the tangent space to S^1 at 1 is $i\,\mathbb{R}$. Let $A \in U(n)$ be such that $\det A = 1$ and $B \in T_AU(n)$ (i.e. $A^*B + B^*A = 0$). Then

$$Df(A)(B) = \frac{d}{dt}\det(A + tB)\bigg|_{t=0} = \det A\ \mathrm{tr}\,(A^{-1}B) = \mathrm{tr}\,(A^*B).$$

Given $iy \in T_1S^1$, for a $y \in \mathbb{R}$, let $B = \frac{iy}{n}A$. Note that $B \in T_AU(n)$. Since $Df(A)(B) = iy$, A is a regular point of f. As A is arbitrary, 1 is a regular value of f. Thus $SU(n) = f^{-1}(1)$ is a submanifold of $U(n)$ of dimension equal to dimension of $U(n)$ minus dimension of S^1, that is $n^2 - 1$. $\mathfrak{su}(n) = \{B \in \mathfrak{u}(n) : \mathrm{tr}\,B = 0\} = \{B \in M_{n\times n} : B^* = -B \text{ and } \mathrm{tr}\,B = 0\}$. Of course, the dimensions of $SU(n)$ and $\mathfrak{su}(n)$ coincide.

b) Calling

$$B = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

the left invariant vector field that is B at the identity is $X_A = DL_AB = AB$. Note that $X_A \in T_ASU(n)$ since $A^*X_A + X_A^*A = B + B^* = 0$ and $\mathrm{tr}\,(X_A^*A) = \mathrm{tr}\,B^* = 0$.

2.

a) The area of D is

$$\begin{aligned}
\int_D \omega &= \int_{-1}^{1}\int_{2-\sqrt{1-x^2}}^{2+\sqrt{1-x^2}} \frac{1}{y^2}\,dy\,dx \\
&= \int_{-1}^{1}\left(\frac{1}{2-\sqrt{1-x^2}} - \frac{1}{2+\sqrt{1-x^2}}\right)dx \\
&= 2\int_{-1}^{1}\frac{\sqrt{1-x^2}}{3+x^2}\,dx \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos^2\theta}{3+\sin^2\theta}\,d\theta \\
&= 2\left(\frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\tan\theta\right)-\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 2\pi\left(\frac{2}{\sqrt{3}}-1\right).
\end{aligned}$$

b) Using Stokes' Theorem, we have

$$\int_D \frac{dx\wedge dy}{y^2} = \int_D d\left(\frac{dx}{y}\right) = \int_{\partial D}\frac{dx}{y} = \int_{-\pi}^{\pi}\frac{-\sin\theta}{2+\sin\theta}\,d\theta.$$

We have used the parameterization that was suggested.

c) Clearly, $(E_1, E_1) = (E_2, E_2) = 1$ and $(E_1, E_2) = 0$. The vectors

$$(\cos\theta, 2+\sin\theta)' = (-\sin\theta, \cos\theta) = (2-y, x)$$

are tangent to ∂D, so E_1 is tangent to ∂D. The dual frame (ω^1, ω^2) is

$$\begin{aligned}
\omega^1 &= \tfrac{1}{\rho y}((2-y)\,dx + x\,dy), \\
\omega^2 &= \tfrac{1}{\rho y}(-x\,dx + (2-y)\,dy).
\end{aligned}$$

d) As an auxiliary computation,

$$d\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^3}(x\,dx + (y-2)\,dy).$$

Hence, we have that

$$\begin{aligned} d\omega^1 &= \left(-\frac{1}{\rho^3 y}(x^2+(y-2)^2)+\frac{2}{\rho y^2}+\frac{1}{\rho y}\right) dx\wedge dy \\ &= \frac{2}{\rho y^2}\, dx\wedge dy, \\ d\omega^2 &= \left(-\frac{1}{\rho^3 y}(x(2-y)+x(y-2))-\frac{x}{\rho y^2}+\frac{1}{\rho y}\right) dx\wedge dy \\ &= -\frac{x}{\rho y^2}\, dx\wedge dy. \end{aligned}$$

e) We wish to determine $\omega_1^2 = a\,dx + b\,dy$ such that

$$\begin{aligned} \frac{2}{\rho y^2}\, dx\wedge dy &= -\frac{1}{\rho y}(-x\,dx+(2-y)\,dy)\wedge(a\,dx+b\,dy), \\ -\frac{x}{\rho y^2}\, dx\wedge dy &= \frac{1}{\rho y}((2-y)\,dx+x\,dy)\wedge(a\,dx+b\,dy). \end{aligned}$$

So,

$$\begin{bmatrix} 2-y & x \\ -x & 2-y \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} +\frac{2}{y} \\ -\frac{x}{y} \end{bmatrix}.$$

The solution of this system is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\rho^2 y}\begin{bmatrix} 4-2y+x^2 \\ xy \end{bmatrix}.$$

Therefore,

$$\omega_1^2 = \tfrac{1}{\rho^2 y}((4-2y+x^2)\,dx + xy\,dy).$$

f) The geodesic curvature of ∂D is

$$k_g = \omega_1^2(E_1) = \frac{1}{\rho^3}((4-2y+x^2)(2-y)+(xy)x) = \frac{2}{\rho} = 2,$$

because $\rho = 1$ on ∂D.

g) If we parameterize ∂D as in **b)**, using $(x(\theta),y(\theta)) = (\cos\theta, 2+\sin\theta)$, $\theta \in]-\pi,\pi[$, then

$$\|(\dot x(\theta),\dot y(\theta))\| = \frac{1}{y} = \frac{1}{2+\sin\theta}.$$

Using the result of **b)**, we have that

$$\int_D K\omega + \int_{\partial D} k_g\, ds = \int_{-\pi}^{\pi}\frac{\sin\theta}{2+\sin\theta}\,d\theta + \int_{-\pi}^{\pi}\frac{2}{2+\sin\theta}\,d\theta = 2\pi.$$

3. Let ω be a closed form in $\Omega^1(S^2)$. Choose one point $x_0 \in S^2$, say the north pole. Define the function $\eta : S^2 \to \mathbb{R}$ by

$$\eta(x) = \int_\gamma \omega$$

where γ is a piece of a meridian connecting x_0 to x. Note that it does not matter which of the two pieces of meridian you choose, since (if x is not the south pole) their union forms a great circle, C, and, according to Stokes' Theorem,

$$\int_C \omega = \int_D d\omega = 0,$$

where D is a hemisphere. In fact, because of Stokes' Theorem and because ω is closed, the integral in the definition of η does not depend on γ, as long as it is a smooth curve connecting x_0 to x. Moreover, if we were to replace x_0 by $\tilde{x_0}$, then η would change by a constant to $\tilde{\eta}$, since $\tilde{\eta} = \eta + \int_{\tilde{x_0}}^{x_0} \omega$.

We claim that $d\eta = \omega$. To prove this, we pick $X \in \mathcal{X}(S^2)$ and compute $(d\eta)_x(X_x)$. Taking into account the previous paragraph, without loss of generality, we may assume that $x \neq x_0$ and X_x is tangent to the meridian that connects x_0 to x. Let $c : \mathbb{R} \to S^2$ parameterize such a meridian with $c(0) = x$ and $\dot{c}(0) = X_x$. Then, by the Fundamental Theorem of Calculus,

$$\begin{aligned} (d\eta)_x(X_x) &= \frac{d}{d\sigma}(\eta \circ c)(\sigma)\bigg|_{\sigma=0} = \frac{d}{d\sigma}\int_0^\sigma \omega_{c(\sigma)}(\dot{c}(\sigma))\, d\sigma \\ &= \omega_{c(0)}(\dot{c}(0)) = \omega_x(X_x). \end{aligned}$$

Since x and X_x are arbitrary, $d\eta = \omega$ and ω is exact.