Riemannian Geometry

1^{st} Test - November 14, 2017

LMAC and MMA

Solutions

1.

a) $O(1,2)$ is a subgroup of $GL(3)$. Indeed, suppose $A \in O(1,2)$ and $B \in O(1,2)$. Then $AB \in O(1,2)$ since

$$(AB)\Lambda(AB)^T = A(B\Lambda B^T)A^T = A\Lambda A^T = \Lambda,$$

and $A^{-1} \in O(1,2)$ as

$$A\Lambda A^T = \Lambda \;\Rightarrow\; A^{-1}A\Lambda A^T(A^T)^{-1} = A^{-1}\Lambda(A^T)^{-1} \;\Rightarrow\; \Lambda = A^{-1}\Lambda(A^{-1})^T.$$

b) We denote by $\mathcal{S}_{3\times 3}$ the space of symmetric 3×3 matrices. This is a 6-dimensional space. Let $f : \mathcal{M}_{3\times 3} \to \mathcal{S}_{3\times 3}$ be defined by

$$f(A) = A\Lambda A^T.$$

This function is smooth and

$$Df(A)(B) = A\Lambda B^T + B\Lambda A^T.$$

Suppose $A \in f^{-1}(\Lambda)$ and $S \in T_\Lambda \mathcal{S}_{3\times 3} \equiv \mathcal{S}_{3\times 3}$. Choosing $B = \frac{1}{2}S\Lambda A$, we get

$$Df(A)\left(\frac{1}{2}S\Lambda A\right) = \frac{1}{2}A\Lambda A^T\Lambda S + \frac{1}{2}S\Lambda A\Lambda A^T = S.$$

This shows that f is a submersion at A. Since A is arbitrary in $f^{-1}(\Lambda)$, Λ is a regular value of f. It follows that $O(1,2) = f^{-1}(\Lambda)$ is a submanifold of $\mathcal{M}_{3\times 3}$ of dimension $9 - 6 = 3$.

c)

$$T_I O(1,2) = \ker Df(I) = \{B \in \mathcal{M}_{3\times 3} : B\Lambda + \Lambda B^T = 0\}.$$

A basis for $T_I O(1,2)$ is $\{B_1, B_2, B_3\}$, where

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

One easily checks that $[B_1, B_2] = -B_3$, $[B_1, B_3] = -B_2$ and $[B_2, B_3] = B_1$.

d) The tangent space to $O(1,2)$ at Λ is

$$T_\Lambda O(1,2) = \ker Df(\Lambda) = \{B \in \mathcal{M}_{3\times 3} : B^T = -B\}.$$

e) The left-invariant vector field corresponding to B_1 is

$$X_Y^{B_1} := DL_Y\, B_1 = Y B_1.$$

As

$$B_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we obtain

$$\exp(tB_1) = \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.

a) Let $p \in S^2$, and v and w be two linearly independent vectors belonging to T_pS^2. Calling $n = x\partial_x + y\partial_y + z\partial_z$ the unit outer normal to S^2, we have

$$\omega(v,w) = \iota(n)\, dx \wedge dy \wedge dz(v,w) = dx \wedge dy \wedge dz(n,v,w) \neq 0,$$

because the volume parallelepiped with sides n, v and w is different from zero. We have

$$\begin{aligned} \iota(n)\, dx \wedge dy \wedge dz &= (\iota(n)\, dx) \wedge dy \wedge dz \\ &\quad - dx \wedge (\iota(n)\, dy) \wedge dz + dx \wedge dy \wedge (\iota(n)\, dz) \\ &= x\, dy \wedge dz - y\, dx \wedge dz + z\, dy \wedge dz. \end{aligned}$$

b) $\eta = r^*\omega = \sin\varphi\, d\varphi \wedge d\theta$.

c) $\int_{S^2} \omega = \int_0^{2\pi}\int_0^{\pi} r^*\omega = 4\pi$.

d)

$$\begin{aligned} L_X\eta = L_{\partial_\varphi}(\sin\varphi\, d\varphi \wedge d\theta) &= (L_{\partial_\varphi} \sin\varphi)\, d\varphi \wedge d\theta \\ &\quad + \sin\varphi\, d(L_{\partial_\varphi}\varphi) \wedge d\theta \\ &\quad + \sin\varphi\, d\varphi \wedge d(L_{\partial_\varphi}\theta) \\ &= \cos\varphi\, d\varphi \wedge d\theta. \end{aligned}$$

e) Let ϕ_t be the flow of X and ψ_t be the flow of r_*X. Then $r \circ \phi_t = \psi_t \circ r$. Hence

$$\begin{aligned} L_X\eta &= \frac{d}{dt}\phi_t^*\eta\Big|_{t=0} = \frac{d}{dt}\phi_t^* r^*\omega\Big|_{t=0} = \frac{d}{dt}(r\circ\phi_t)^*\omega\Big|_{t=0} \\ &= \frac{d}{dt}(\psi_t\circ r)^*\omega\Big|_{t=0} = \frac{d}{dt}r^*\psi_t^*\omega\Big|_{t=0} = r^*\frac{d}{dt}\psi_t^*\omega\Big|_{t=0} \\ &= r^*\left(L_{r_*X}\omega\right). \end{aligned}$$

f)

$$\begin{aligned} \int_\Omega L_{r_*X}\omega &= \int_{r^{-1}(\Omega)} r^*(L_{r_*X}\omega) = \int_{r^{-1}(\Omega)} L_X\eta \\ &= \int_0^{2\pi}\int_{\varphi_0}^{\varphi_1} \cos\varphi\, d\varphi\wedge d\theta = 2\pi(\sin\varphi_1 - \sin\varphi_0). \end{aligned}$$

g) Note that $r^*(\iota(r_*X)\omega) = \iota(X)\eta$ because

$$\begin{aligned} [r^*(\iota(r_*X)\omega)](v,w) &= (\iota(r_*X)\omega)(r_*v, r_*w) = \omega(r_*X, r_*v, r_*w) \\ &= r^*\omega(X,v,w) = [\iota(X)\eta](v,w). \end{aligned}$$

Moreover,

$$\iota(X)\eta = \iota(\partial_\varphi)\sin\varphi\, d\varphi\wedge d\theta = \sin\varphi\, d\theta.$$

Therefore,

$$\begin{aligned} \int_{\partial\Omega}\iota(r_*X)\omega &= \int_{r^{-1}(\partial\Omega)}\iota(X)\eta = \int_0^{2\pi}\sin\varphi_1\, d\theta - \int_0^{2\pi}\sin\varphi_0\, d\theta \\ &= 2\pi(\sin\varphi_1 - \sin\varphi_0). \end{aligned}$$

h) Using Cartan's formula, we get

$$L_{r_*X}\omega = \iota(r_*X)d\omega + d(\iota(r_*X)\omega) = d(\iota(r_*X)\omega).$$

According to Stokes' Theorem, we have

$$\int_\Omega L_{r_*X} = \int_\Omega d(\iota(r_*X)\omega) = \int_{\partial\Omega}\iota(r_*X)\omega.$$