Riemannian Geometry, Fall 2016/17
Instituto Superior Técnico, Pedro Girão

The 1st Test, given on November 16, 2016, consists of Problem 1 and part of Problem 2.

1. Let M be the Lie group $SL(2) = \{A \in M_{2\times 2} : \det A = 1\}$.

a) Show that M is diffeomorphic to $\mathbb{R}^2 \times S^1$. Suggestion: Write $A \in M$ as

$$A = A(p,q,r,s) = \begin{bmatrix} p+q & r+s \\ r-s & p-q \end{bmatrix},$$

and use the map

$$\varphi(q,r,\theta) = (p,q,r,s) = \left(\sqrt{1+q^2+r^2}\cos\theta, q, r, \sqrt{1+q^2+r^2}\sin\theta\right)$$

defined on $\mathbb{R}^2 \times S^1$.

b) Compute the matrices

$$B := A_*\varphi_*\left(\frac{\partial}{\partial q}\right)_{(0,0,0)} \qquad \text{and} \qquad C := A_*\varphi_*\left(\frac{\partial}{\partial \theta}\right)_{(0,0,0)}.$$

(Note that we have $A_* = A$, because A is linear.)

c) Compute the Lie algebra sl(2) and T_AM.

d) Compute X^B and X^C, the left invariant vector fields corresponding to B and C respectively, i.e. such that $(X^B)_I = B$ and $(X^C)_I = C$.

e) Compute the integral curves of X^B and X^C through I. Suggestion: Calculate C^2.

f) Compute $[X^B, X^C]_I$.

g) Is it true that both $X^B = A_*\varphi_*\frac{\partial}{\partial q}$ and $X^C = A_*\varphi_*\frac{\partial}{\partial \theta}$? Explain.

2. Consider the region

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2-z^2 < 1 \ \wedge \ 0 < z < \sinh 1\},$$

and consider the piece of hyperboloid

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2-z^2 = 1 \ \wedge \ 0 < z < \sinh 1\}$$

with the parametrization

$$\varphi(\gamma,\theta) = (\cosh\gamma\cos\theta, -\cosh\gamma\sin\theta, \sinh\gamma)$$

(defined on an appropriate set). Moreover, in $\mathbb{R}^3$ define ω to be the 2-form

$$\omega = \iota(X)\, dx \wedge dy \wedge dz,$$

where the vector field X is given by

$$X(x, y, z) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

a) Compute ω and $\varphi^*\omega$.
b) Compute $\int_S \omega$, where S has the orientation induced by φ.
c) Compute $d\omega$.
d) Compute $\int_\Omega d\omega$, where Ω has the canonical orientation of $\mathbb{R}^3$.
e) Use Stokes' Theorem to relate the results of **b)** and **d)**.
f) Compute the normal n to S, which is exterior to Ω, and extend it to a smooth vector field defined in a neighborhood of S. Note: S is contained in the level set of a smooth function.
g) Argue that $(\iota(n)\, dx \wedge dy \wedge dz)|_{TS\times TS}$ is a volume form on S.
h) Compute $\iota(n)\, dx \wedge dy \wedge dz$ and $\varphi^*(\iota(n)\, dx \wedge dy \wedge dz)$.
i) Compute the Euclidean inner product $\langle X, n\rangle$ and $\varphi^*\langle X, n\rangle$.
j) Argue that

$$\omega|_{TS\times TS} = \langle X, n\rangle\, (\iota(n)\, dx \wedge dy \wedge dz)|_{TS\times TS}\,.$$

k) Compute $L_X(dx \wedge dy \wedge dz)$ using

i) The definition of the Lie derivative using the flow of X.

ii) The formula about the Lie derivative of the tensor product and the fact that the Lie derivative commutes with the exterior derivative.

iii) Cartan's formula.

l) Combine the conclusions of the previous items to obtain the Divergence Theorem.

1. Solution.

a) The condition that the $\det A = 1$ is equivalent to

$$p^2 + s^2 - (q^2 + r^2) = 1.$$

The map φ is a bijection between $\mathbb{R}^2 \times S^1$ and $M := \{(p, q, r, s) \in \mathbb{R}^4 : p^2 + s^2 - (q^2 + r^2) = 1\}$.
We can use the map φ (defined in appropriate subsets of $\mathbb{R}^3$) to write admissible parametrizations of subsets of M because φ is smooth and $D\varphi$ is injective.

Consider the parametrizations $(q,r,\theta)\mapsto(q,r,e^{i\theta})$ of a neighborhood of a point $(q_0,r_0,e^{i\theta_0})\in\mathbb{R}^2\times S^1$, and $(q,r,\theta)\mapsto\varphi(q,r,\theta)$ of a neighborhood of a point $\varphi(q_0,r_0,\theta_0)\in M$. The map $\varphi:\mathbb{R}^2\times S^1\to M$ is the identity when seen through the charts, from a neighborhood of (q_0,r_0,θ_0) in $\mathbb{R}^3$ to a neighborhood of (q_0,r_0,θ_0) in $\mathbb{R}^3$. So, φ is a diffeomorphism from $\mathbb{R}^2\times S^1$ to M.
Finally, the map A is a diffeomorphism between $\mathbb{R}^4$ and $\mathbb{R}^4$. Its inverse is

$$A^{-1}(\alpha,\beta,\gamma,\delta)=\frac{1}{2}(\alpha+\delta,\alpha-\delta,\beta+\gamma,\beta-\gamma).$$

So, A is a diffeomorphism between M and $SL(2)$.

b) We have

$$\varphi_*\left(\frac{\partial}{\partial q}\right)_{(0,0,0)}=(0,1,0,0),\qquad \varphi_*\left(\frac{\partial}{\partial \theta}\right)_{(0,0,0)}=(0,0,0,1)$$

$$B=A_*(0,1,0,0)=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad C=A_*(0,0,0,1)=\begin{bmatrix}0&1\\-1&0\end{bmatrix}.$$

c) Let $c(t)$ be a curve in $SL(2)$ such that $c(0)=A$ and $\dot c(0)=B$. Since $\det c(t)=1$, differentiating with respect to t and setting t equal to 0,

$$c(0)\,\mathrm{tr}\,(c(0)^{-1}\dot c(0))=0,\quad \text{i.e.}\quad \mathrm{tr}\,(A^{-1}B)=0.$$

This shows that

$$T_AM=\{B\in M_{2\times2}:\mathrm{tr}\,(A^{-1}B)=0\}$$

and

$$\mathrm{sl}(2)=\{B\in M_{2\times2}:\mathrm{tr}\,B=0\}.$$

d) The left invariant vector fields corresponding to B and C, are

$$(X^B)_X=XB=\begin{bmatrix}x^{11}&x^{12}\\x^{21}&x^{22}\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\begin{bmatrix}x^{11}&-x^{12}\\x^{21}&-x^{22}\end{bmatrix},$$

$$(X^C)_X=XC=\begin{bmatrix}x^{11}&x^{12}\\x^{21}&x^{22}\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}=\begin{bmatrix}-x^{12}&x^{11}\\-x^{22}&x^{21}\end{bmatrix}.$$

e) $C^2=-I$. The integral curves of X^B and X^C through I are

$$e^{Bt}=\begin{bmatrix}e^t&0\\0&e^{-t}\end{bmatrix}$$

and

$$e^{Ct}=I+Ct-I\frac{t^2}{2!}-C\frac{t^3}{3!}+I\frac{t^4}{4!}+\ldots=\begin{bmatrix}\cos t&\sin t\\-\sin t&\cos t\end{bmatrix}.$$

f)

$$[X^B, X^C]_I = [B, C] = BC - CB = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

g) It is not true that $X^B = A_*\varphi_*\frac{\partial}{\partial q}$ and $X^C = A_*\varphi_*\frac{\partial}{\partial \theta}$, otherwise

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = [X^B, X^C]_I = A_*\varphi_* \left[\frac{\partial}{\partial q}, \frac{\partial}{\partial \theta}\right]_{(0,0,0)} = 0,$$

as $A \circ \varphi(0,0,0) = I$.

2. Solution.

a) We know that, for ω with degree k,

$$\iota(X)\, \omega \wedge \eta = (\iota(X)\omega) \wedge \eta + (-1)^k \omega \wedge (\iota(X)\eta).$$

This implies

$$\begin{aligned} \omega &= \iota(X)\, dx \wedge dy \wedge dz \\ &= (\iota(X)dx) \wedge dy \wedge dz - dx \wedge (\iota(X)dy) \wedge dz + dx \wedge dy \wedge (\iota(X)dz) \\ &= x\, dy \wedge dz - y\, dx \wedge dz. \end{aligned}$$

The pull-back of ω by φ is

$$\begin{aligned} \varphi^*\omega &= \cosh\gamma\cos\theta \begin{vmatrix} -\sinh\gamma\sin\theta & -\cosh\gamma\cos\theta \\ \cosh\gamma & 0 \end{vmatrix} d\gamma \wedge d\theta \\ &\quad + \cosh\gamma\sin\theta \begin{vmatrix} \sinh\gamma\cos\theta & -\cosh\gamma\sin\theta \\ \cosh\gamma & 0 \end{vmatrix} d\gamma \wedge d\theta \\ &= \cosh^3\gamma\, d\gamma \wedge d\theta. \end{aligned}$$

b)

$$\begin{aligned} \int_S \omega &= \int_{-\pi}^{\pi}\int_0^1 \cosh^3\gamma\, d\gamma d\theta \\ &= 2\pi \int_0^1 (1 + \sinh^2\gamma)\cosh\gamma\, d\gamma \\ &= 2\pi\left(\sinh 1 + \frac{1}{3}\sinh^3 1\right). \end{aligned}$$

c) Clearly, $d\omega = 2\, dx \wedge dy \wedge dz$.

d)

$$\begin{aligned}\int_\Omega d\omega &= 2\int_\Omega dx\wedge dy\wedge dz \\ &= 2\int_{-\pi}^{\pi}\int_0^{\sinh 1}\int_0^{\sqrt{1+z^2}} r\,drdzd\theta \\ &= 4\pi\int_0^{\sinh 1}\frac{1}{2}(1+z^2)\,dz \\ &= 2\pi\left(\sinh 1+\frac{1}{3}\sinh^3 1\right).\end{aligned}$$

e) By Stokes' Theorem

$$\int_\Omega d\omega = \int_S \omega + \int_B \omega + \int_T \omega,$$

where

$$B = \{(x,y,z)\in\mathbb{R}^3 : z = 0 \text{ and } x^2+y^2<1\}$$

and

$$T = \{(x,y,z)\in\mathbb{R}^3 : z = \sinh 1 \text{ and } x^2+y^2<\cosh^2 1\}.$$

But $\int_B \omega = \int_T \omega = 0$ because both terms in the expression for ω contain a dz, and both in B and in T the value of z is constant. Therefore,

$$\int_\Omega d\omega = \int_S \omega.$$

f) The normal to S, which is exterior to Ω, is

$$n = \frac{\nabla(x^2+y^2-z^2)}{\|\nabla(x^2+y^2-z^2)\|} = \frac{(x,y,-z)}{\sqrt{x^2+y^2+z^2}}.$$

g) Let $p\in S$, and u and v be two linearly independent vectors belonging to T_pS. Then

$$\begin{aligned}(\iota(n)\,dx\wedge dy\wedge dz)(u,v) &= dx\wedge dy\wedge dz(n,u,v) \\ &= \text{volume of the parallelogram} \\ &\ \text{defined by } n,\ u \text{ and } v \\ &\neq 0.\end{aligned}$$

This shows that $(\iota(n)\,dx\wedge dy\wedge dz)|_{TS\times TS}$ is a volume form on S.

h)

$$\begin{aligned}
\iota(n)\,dx\wedge dy\wedge dz &= (\iota(n)dx)\wedge dy\wedge dz - dx\wedge(\iota(n)dy)\wedge dz\\
&\quad + dx\wedge dy\wedge(\iota(n)dz)\\
&= \frac{x\,dy\wedge dz - y\,dx\wedge dz - z\,dx\wedge dy}{\sqrt{x^2+y^2+z^2}}.
\end{aligned}$$

$$\begin{aligned}
&\varphi^*(\iota(n)\,dx\wedge dy\wedge dz)\\
&\quad = \frac{\cosh\gamma\cos\theta}{\sqrt{\cosh(2\gamma)}}\begin{vmatrix} -\sinh\gamma\sin\theta & -\cosh\gamma\cos\theta \\ \cosh\gamma & 0 \end{vmatrix} d\gamma\wedge d\theta\\
&\qquad + \frac{\cosh\gamma\sin\theta}{\sqrt{\cosh(2\gamma)}}\begin{vmatrix} \sinh\gamma\cos\theta & -\cosh\gamma\sin\theta \\ \cosh\gamma & 0 \end{vmatrix} d\gamma\wedge d\theta\\
&\qquad - \frac{\sinh\gamma}{\sqrt{\cosh(2\gamma)}}\begin{vmatrix} \sinh\gamma\cos\theta & -\cosh\gamma\sin\theta \\ -\sinh\gamma\sin\theta & -\cosh\gamma\cos\theta \end{vmatrix} d\gamma\wedge d\theta\\
&\quad = \cosh\gamma\sqrt{\cosh(2\gamma)}\,d\gamma\wedge d\theta.
\end{aligned}$$

i)

$$\begin{aligned}
\langle X, n\rangle &= (x,y,0)\cdot\frac{(x,y,-z)}{\sqrt{x^2+y^2+z^2}}\\
&= \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}},
\end{aligned}$$

$$\varphi^*\langle X,n\rangle = \frac{\cosh^2\gamma}{\sqrt{\cosh(2\gamma)}}.$$

j) Denote by $X^\top$ the projection of X on the tangent space of X.

$$\begin{aligned}
\omega|_{TS\times TS} &= \iota(X)\,dx\wedge dy\wedge dz|_{TS\times TS}\\
&= \iota(\langle X,n\rangle n + X^\top)\,dx\wedge dy\wedge dz\big|_{TS\times TS}\\
&= \langle X,n\rangle\iota(n)\,dx\wedge dy\wedge dz|_{TS\times TS} + \iota(X^\top)\,dx\wedge dy\wedge dz\big|_{TS\times TS}\\
&= \langle X,n\rangle\,(\iota(n)\,dx\wedge dy\wedge dz)|_{TS\times TS}\,,
\end{aligned}$$

because, for Y and Z in TS,

$$\begin{aligned}
\iota(X^\top)\,dx\wedge dy\wedge dz(Y,Z) &= dx\wedge dy\wedge dz(X^\top,Y,Z)\\
&= 0
\end{aligned}$$

as $X^\top$, Y and Z are linearly dependent. This can be checked directly using the results of **a)**, **h)** and **i)**:

$$\omega|_{TS\times TS} = \cosh^3\gamma\, d\gamma\wedge d\theta,$$

$$\varphi^*(\iota(n)\, dx\wedge dy\wedge dz) = \cosh\gamma\sqrt{\cosh(2\gamma)}\, d\gamma\wedge d\theta,$$

$$\varphi^*\langle X, n\rangle = \frac{\cosh^2\gamma}{\sqrt{\cosh(2\gamma)}}.$$

k)

i) The flow of the vector field X at time t is

$$\phi_t(x,y,z) = (e^t x, e^t y, z)$$

According to the definition of Lie derivative,

$$\begin{aligned} L_X(dx\wedge dy\wedge dz) &= \left.\frac{d}{dt}\phi_t^*(dx\wedge dy\wedge dz)\right|_{t=0} \\ &= \left.\frac{d}{dt}(e^{2t}\, dx\wedge dy\wedge dz)\right|_{t=0} \\ &= 2\, dx\wedge dy\wedge dz. \end{aligned}$$

ii) Using the formula about the Lie derivative of the tensor product and the fact that the Lie derivative commutes with the exterior derivative, we have

$$\begin{aligned} L_X(dx\wedge dy\wedge dz) &= (L_X\, dx)\wedge dy\wedge dz + dx\wedge(L_X\, dy)\wedge dz \\ &\quad + dx\wedge dy\wedge(L_X\, dz) \\ &= d(L_X x)\wedge dy\wedge dz + dx\wedge d(L_X y)\wedge dz \\ &\quad + dx\wedge dy\wedge d(L_X z) \\ &= dx\wedge dy\wedge dz + dx\wedge dy\wedge dz + 0 \\ &= 2\, dx\wedge dy\wedge dz. \end{aligned}$$

iii) According to Cartan's formula, we have

$$\begin{aligned} L_X(dx\wedge dy\wedge dz) &= d(\iota(X)\, dx\wedge dy\wedge dz) + \iota(X)\, d(dx\wedge dy\wedge dz) \\ &= d(x\, dy\wedge dz - y\, dx\wedge dz) \\ &= 2\, dx\wedge dy\wedge dz. \end{aligned}$$

l) Combining what we saw above, we deduce the Divergence Theorem:

$$\begin{aligned}
\int_\Omega \operatorname{div} X \, dx\wedge dy\wedge dz &= \int_\Omega L_X(dx\wedge dy\wedge dz)\\
&= \int_\Omega d(\iota(X)\, dx\wedge dy\wedge dz)\\
&= \int_{\partial\Omega} \iota(X)\, dx\wedge dy\wedge dz\\
&= \int_{\partial\Omega} \langle X, n\rangle\, \iota(n)\, dx\wedge dy\wedge dz.
\end{aligned}$$