Riemannian Geometry

1^{st} Test - November 20, 2020

LMAC and MMA

Duration: 180 minutes

Show your calculations

1. Consider the hyperboloid

$$\mathcal{H} = \{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 - t^2 = -1,\ t > 0\},$$

parameterized by

$$(x, y, t) = \phi(u, \varphi) = (\sinh u \cos\varphi, \sinh u \sin\varphi, \cosh u).$$

a) Calculate the local representation $g = \phi^\star \tilde{g}$ in the (u, φ) half-space where $u > 0$ of Riemannian metric induced on the hyperboloid $\mathcal{H}$ by the (Lorentzian) metric $\tilde{g} = dx^2 + dy^2 - dt^2$ on $\mathbb{R}^3$. Check that the frame $(E_u, E_\varphi) = \left(\partial_u, \frac{1}{\sinh u}\partial_\varphi\right)$ is orthonormal and calculate its dual frame $(\omega^u, \omega^\varphi)$. Calculate the local representation ω of the Riemannian volume element on $\mathcal{H}$. (2)

b) Determine the one forms $\omega^u_\varphi = -\omega^\varphi_u$ that solve the system (3)

$$\begin{aligned} d\omega^u &= \omega^\varphi \wedge \omega^u_\varphi, \\ d\omega^\varphi &= \omega^u \wedge \omega^\varphi_u. \end{aligned}$$

Calculate $\Omega^u_\varphi = d\omega^u_\varphi$. Knowing that $\Omega^u_\varphi = K\omega$, determine the curvature K of the hyperboloid.

c) Consider the domain $D = \{(u, \varphi) \in (0, R) \times (0, 2\pi)\}$. Use Stokes' Theorem to calculate $\int_D K = \int_D K\omega = \int_D \Omega^u_\varphi$. (3)

d) Calculate curvature of $\{R\} \times (0, 2\pi)$ which is given by $k_g = -\omega^u_\varphi(E_\varphi)$. Parameterizing this curve by $r(\varphi) = (R, \varphi)$, for $\varphi \in (0, 2\pi)$, the line element is $ds = \sqrt{g(r'(\varphi), r'(\varphi))}\, d\varphi$. Calculate $\int_{\{R\}\times(0,2\pi)} k_g\, ds$. (2)

e) Consider (3)

$$X = \frac{\cos\varphi}{\cosh u}\partial_u - \frac{\sin\varphi}{\sinh u}\partial_\varphi.$$

Calculate $L_X\omega$ (using the fact that L_X commutes with the exterior derivative) and $\iota(X)\omega$. Verify Cartan's formula. What is the divergence of X?

2. Let G be a Lie group. Suppose $X \in \mathcal{X}(G)$ is left invariant and $Y \in \mathcal{X}(G)$ is a right invariant. Prove that $[X, Y] = 0$. (2)

3. Let $X \in \mathcal{X}(M)$ with flow ϕ_t.

a) Determine $(d\phi_t)X$. Justify. (2)

b) Let $Y \in \mathcal{X}(M)$. Determine the vector field (3)

$$\frac{d}{dt}(d\phi_{-t})_{\phi_t(p)} Y_{\phi_t(p)}\Big|_{t=t}$$

(note that the derivative is calculated at a general t).