Riemannian Geometry

1^{st} Test - November 15, 2019

LMAC and MMA

Duration: 180 minutes

Show your calculations

1. Recall that the $\mathfrak{u}(n)$ is the space of anti-hermitean matrices and that

$$SU(n) = \{M \in U(n) : \det M = 1\}.$$

a) Prove that $SU(n)$ is a submanifold of $U(n)$ by showing that it is the inverse image of a regular value of a map $f : U(n) \to S^1$. What is $\mathfrak{su}(n)$ and what is its dimension? (4)

b) Compute the left invariant vector field corresponding to (1)

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \in \mathfrak{su}(2).$$

Check directly that the vector field you construct belongs to $\mathcal{X}(SU(2))$.

2. Consider the upper half plane, H, with the area form

$$\omega = \frac{dx \wedge dy}{y^2}$$

and the disc

$$D = \{(x, y) \in H : x^2 + (y-2)^2 < 1\}.$$

a) Compute the area of D. Suggestion: integrate first in y and use $\int \frac{\cos^2\theta}{a+\sin^2\theta}\, d\theta = \sqrt{\frac{1+a}{a}} \arctan\left(\sqrt{\frac{1+a}{a}} \tan\theta\right) - \theta$. (2)

b) Use Stokes' Theorem to write $\int_D \omega$ as an integral of a one form on ∂D, and write this integral using the parameterization $(x, y) = (\cos\theta, 2 + \sin\theta)$. (2)

c) Let $\rho := \sqrt{x^2 + (y-2)^2}$. Check that the frame (2)

$$\begin{aligned} E_1 &= \tfrac{y}{\rho}((2-y)\partial_x + x\partial_y), \\ E_2 &= \tfrac{y}{\rho}(-x\partial_x + (2-y)\partial_y), \end{aligned}$$

is orthonormal for the metric with line element

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

and that E_1 is tangent to the boundary of D. Write the dual frame (ω^1, ω^2).

d) Check that (2)

$$
\begin{aligned}
d\omega^1 &= \frac{2}{\rho y^2}\, dx \wedge dy, \\
d\omega^2 &= -\frac{x}{\rho y^2}\, dx \wedge dy.
\end{aligned}
$$

e) Compute the one form ω_1^2 such that (2)

$$
\begin{aligned}
d\omega^1 &= -\omega^2 \wedge \omega_1^2, \\
d\omega^2 &= \omega^1 \wedge \omega_1^2.
\end{aligned}
$$

f) The geodesic curvature of the boundary of D is given by $k_g = \omega_1^2(E_1)$. Check that $k_g = 2$. (1)

g) The Gaussian curvature, K, of H is identically equal to -1. Calculate (2)

$$\int_D K\omega + \int_{\partial D} k_g\, ds.$$

3. Prove that every closed form in $\Omega^1(S^2)$ is exact. (2)